EXPLICIT UPPER BOUNDS FOR $\prod_{p \leq \omega(n)} p/(p - 1)$

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Abstract. Let $\omega(n)$ be the number of distinct prime divisors of $n$, $\phi(n)$ the Euler totient function, $\sigma(n)$ the sum of divisors of $n$, $p_n$ the $n^{th}$ prime and $\gamma$ the Euler constant. We consider the arithmetic function

$$f(n) = \prod_{\substack{p \leq \omega(n) \text{ }\text{ } p \text{ prime}}} \frac{p}{p - 1}$$

and show that

$$\lim_{n \to \infty} \frac{f(n)}{e^{\gamma} \log \log n} = 1.$$

Next we describe an algorithm that, for a given $0 < \epsilon < 1$, determines all the exceptions to the inequality

$$f(n) < e^\gamma (1 + \epsilon) \log \log n.$$

Finally, by employing this algorithm, we establish some explicit upper bounds for $n/\phi(n)$ and $\sigma(n)/n$. More specifically, we prove that

$$\frac{\sigma(n)}{n \log \log n} \leq \frac{\sigma(180)}{180 \log \log 180} = (1.0338 \ldots) e^\gamma \text{ for } n \geq 121.$$

1. INTRODUCTION

Let $\sigma(n)$ denote the sum of divisors function and $\phi(n)$ the Euler totient function, so that $\sigma(n)\phi(n) < n^2$. Nicolas [4] proved that $n/\phi(n) > e^\gamma \log \log n$ infinitely often, where $\gamma$ is the Euler constant. Also for the smaller quantity $\sigma(n)/n$, Robin showed that $\sigma(n)/n > e^\gamma \log \log n$ infinitely often provided the Riemann Hypothesis is false. More precisely, let $g(n) = \sigma(n)/n \log \log n$. Then in [5], Robin proved the following theorem.

Theorem 1.1 (Robin). The Riemann Hypothesis is true if and only if

$$g(n) < e^\gamma \text{ for } n \geq 5041.$$

As a consequence of this theorem, one can show that under the assumption of the Riemann Hypothesis, the only values of $n$ that fail $g(n) < e^\gamma$ are

$$n \in \{2, 3, 4, 5, 6, 8, 9, 10, 12, 16, 18, 20, 24, 30, 36, 48, 60, 72, 84, 120, 180, 240, 360, 720, 840, 2520, 5040\}$$

Received by the editors August 31, 2006, and in revised form April 25, 2007.
2000 Mathematics Subject Classification. 11Y70, 11A25.
Key words and phrases. Arithmetic functions, Euler totient function, sum of divisors function.
Research of the first and second authors was partially supported by NSERC.

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Table 1 shows the values of $g(n)$ in decreasing order for the above exceptions (other than $n = 2$) to the inequality $g(n) < e^\gamma$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$g(n)$</th>
<th>$n$</th>
<th>$g(n)$</th>
<th>$n$</th>
<th>$g(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>14.177...</td>
<td>60</td>
<td>1.986...</td>
<td>9</td>
<td>1.834...</td>
</tr>
<tr>
<td>4</td>
<td>5.357...</td>
<td>36</td>
<td>1.980...</td>
<td>360</td>
<td>1.833...</td>
</tr>
<tr>
<td>6</td>
<td>3.429...</td>
<td>30</td>
<td>1.960...</td>
<td>240</td>
<td>1.822...</td>
</tr>
<tr>
<td>12</td>
<td>2.563...</td>
<td>120</td>
<td>1.915...</td>
<td>2520</td>
<td>1.804...</td>
</tr>
<tr>
<td>8</td>
<td>2.561...</td>
<td>20</td>
<td>1.913...</td>
<td>840</td>
<td>1.797...</td>
</tr>
<tr>
<td>5</td>
<td>2.521...</td>
<td>48</td>
<td>1.908...</td>
<td>84</td>
<td>1.791...</td>
</tr>
<tr>
<td>24</td>
<td>2.162...</td>
<td>16</td>
<td>1.899...</td>
<td>5040</td>
<td>1.790...</td>
</tr>
<tr>
<td>10</td>
<td>2.158...</td>
<td>72</td>
<td>1.863...</td>
<td>840</td>
<td>1.791...</td>
</tr>
<tr>
<td>18</td>
<td>2.041...</td>
<td>180</td>
<td>1.841...</td>
<td>5040</td>
<td>1.790...</td>
</tr>
</tbody>
</table>

Table 1. Values of $g(n)$.

The following assertion is a direct corollary of Robin’s theorem together with the values recorded in Table 1.

**Corollary 1.2.** *Under the assumption of the Riemann Hypothesis, we have the following inequalities:*

1. $g(n) \leq g(3) = (7.959914266\ldots)e^\gamma$ for $n \geq 3$,
2. $g(n) \leq g(4) = (3.008117079\ldots)e^\gamma$ for $n \geq 4$,
3. $g(n) \leq g(6) = (1.925450381\ldots)e^\gamma$ for $n \geq 5$,
4. $g(n) \leq g(12) = (1.439267874\ldots)e^\gamma$ for $n \geq 7$,
5. $g(n) \leq g(24) = (1.213946496\ldots)e^\gamma$ for $n \geq 13$,
6. $g(n) \leq g(60) = (1.115266133\ldots)e^\gamma$ for $n \geq 25$,
7. $g(n) \leq g(120) = (1.075588326\ldots)e^\gamma$ for $n \geq 61$,
8. $g(n) \leq g(180) = (1.033867784\ldots)e^\gamma$ for $n \geq 121$,
9. $g(n) \leq g(360) = (1.029419689\ldots)e^\gamma$ for $n \geq 181$,
10. $g(n) \leq g(2520) = (1.013215898\ldots)e^\gamma$ for $n \geq 361$,
11. $g(n) \leq g(5040) = (1.005558981\ldots)e^\gamma$ for $n \geq 2521$.

One of our goals in this paper is to prove some of the above inequalities unconditionally. Note that establishing an inequality in one line establishes all the inequalities in the previous lines. The inequality (1.4) is proved by Robin ([5], Proposition 2). This inequality gives an improvement of a previous result of Ivić [3]. Here we will prove the following.

**Theorem 1.3.** $g(n) \leq g(180) = (1.0338\ldots)e^\gamma$ for $n \geq 121$. More precisely, $g(n) < 1.03e^\gamma$ for $n \geq 121$ except for $n = 180$. 
The methodology of the proof is as follows. Let

\[ f(n) = \prod_{p \leq p_\omega(n)} \frac{p}{p - 1}, \]

where \( \omega(n) \) is the number of distinct prime divisors of \( n \) and \( p_n \) is the \( n \)th prime. We develop an algorithm that generates all the exceptions to the inequality

\[ f(n) < 1.03 e^\gamma \log \log n. \]

Since \( \sigma(n)/n < f(n) \), it is clear that the exceptions to

\[ g(n) \leq g(180) = (1.0338 \ldots) e^\gamma \]

are among the exceptions to

\[ f(n) < 1.03 e^\gamma \log \log n. \]

So, numerically checking the exceptions generated by the algorithm against the inequality

\[ g(n) < 1.03 e^\gamma \]

will establish the result.

The structure of the paper is as follows. In Section 2, we study some properties of the arithmetic function \( f(n) \). We describe the algorithm in Section 3. In the last section, we employ our algorithm to establish an explicit upper bound for \( n/\phi(n) \) and prove Theorem 1.3.

2. The Arithmetic Function \( f(n) \)

Let

\[ f(n) = \prod_{p \leq p_\omega(n)} \frac{p}{p - 1}. \]

We have

\[ \frac{\sigma(n)}{n} < \frac{n}{\phi(n)} \leq f(n). \]

The right-hand side inequality is trivial and the left-hand side one is Theorem 329 of [2].

For a given \( n \), let \( p_1, \ldots, p_{\omega(n)-1} \) be the primes less than or equal to \( \log n \) and \( p_{\omega(n)-1+1}, \ldots, p_{\omega(n)} \) be those that exceed \( \log n \). One can show that

\[ l < \frac{\log n}{\log \log n} \]

and

\[ f(n) < \left(1 - \frac{1}{\log n}\right)^{-\log n/\log \log n} \prod_{p \leq \log n} \frac{p}{p - 1}. \]
Proposition 2.1. \[
\lim_{n \to \infty} \frac{f(n)}{e^{\gamma \log \log n}} = 1.
\]

Proof. First of all, recall the Mertens theorem:
\[
\prod_{p \leq x} \frac{p}{p-1} \sim e^{\gamma \log x}
\]
as \(x \to \infty\) ([2], Theorem 429). Now, an application of the Mertens theorem to (2.1) implies
\[
\lim_{n \to \infty} \frac{f(n)}{e^{\gamma \log \log n}} \leq 1.
\]
Let \(n_x = \prod_{p \leq x} p\). By the Prime Number Theorem, we know that \(\log \log n_x \sim \log x\) as \(x \to \infty\). So, by the Mertens theorem,
\[
\lim_{x \to \infty} \frac{f(n_x)}{e^{\gamma \log \log n_x}} = 1.
\]
This completes the proof. \(\Box\)

Corollary 2.2. For any \(\epsilon > 0\), there exists a number \(N_\epsilon\) such that
\[
(2.2) \quad f(n) < e^{\gamma}(1 + \epsilon) \log \log n
\]
for all \(n > N_\epsilon\).

Our next goal is to establish explicit versions of (2.2). More precisely, for a given \(\epsilon > 0\), we want to find an algorithm that finds the smallest value for \(N_\epsilon\). In the next section, we describe an algorithm that, for a given \(\epsilon > 0\), generates all the exceptions to (2.2).

3. Explicit Upper Bounds for \(f(n)\)

For \(\epsilon > 0\), let
\[
M_\epsilon = \exp\left(\exp\left(\sqrt{\frac{2.50637}{\epsilon \gamma}}\right)\right).
\]
Lemma 3.1. Equation (2.2) holds for \(n > M_\epsilon\) with \(\omega(n) \geq 3\).

Proof. Let \(Q_{\omega(n)} = p_1 \cdots p_{\omega(n)}\). Then by Theorem 15 of [6], we have
\[
f(n) = \prod_{p | Q_{\omega(n)}} \frac{p}{p-1} = \frac{Q_{\omega(n)}}{\phi(Q_{\omega(n)})} < e^{\gamma \log \log Q_{\omega(n)}} + \frac{2.50637}{\log \log Q_{\omega(n)}}
\]
for \(n\) with \(\omega(n) > 1\). Note that \(Q_{\omega(n)} \leq n\) and the function
\[
g(t) = e^{\gamma t} + \frac{2.50637}{t}
\]
is increasing for \(t \geq 1.2\). So for any \(n\) with \(\omega(n) \geq 3\), we have
\[
f(n) < e^{\gamma \log n} + \frac{2.50637}{\log \log n}.
\]
From here, it is clear that (2.2) holds for any \(n > M_\epsilon\) with \(\omega(n) \geq 3\). \(\Box\)
For an integer $\beta \geq 1$, let
\[ n_\beta = \exp \left( \exp \left( \frac{1}{(1+\epsilon)e^{\gamma}} \prod_{p \leq p_\beta} \frac{p}{p-1} \right) \right). \]
Note that $n_\beta$ depends on $\epsilon$. For simplicity, we use $n_\beta$ instead of $n_{\beta,\epsilon}$.

**Lemma 3.2.** Equation (2.2) holds for all $n > n_\beta$ with $\omega(n) \leq \beta$.

**Proof.** Let $n > n_\beta$ with $\omega(n) \leq \beta$. Then,
\[ f(n) = \prod_{p \leq p_\beta} \frac{p}{p-1} < e^{\gamma}(1+\epsilon)\log \log n. \]

Since $n_2 < M_\epsilon$ for $0 < \epsilon < 1$, the following is a direct corollary of Lemmas 3.1 and 3.2.

**Corollary 3.3.** Let $0 < \epsilon < 1$. Then for $n > M_\epsilon$, (2.2) holds.

Next, we give a description of all exceptions to the inequality
\[ f(n) < e^{\gamma}(1+\epsilon)\log \log n \]
with $\omega(n) \leq \beta$.

**Lemma 3.4.** Let $\epsilon > 0$ be real and $\beta \geq 2$ be an integer.

1. If $\prod_{p \leq p_\beta} p > n_\beta$, then (2.2) holds for all $n_{\beta-1} < n \leq n_\beta$.
2. If $\prod_{p \leq p_\beta} p \leq n_\beta$, then integers with $\beta$ distinct prime divisors not exceeding $n_\beta$ do not satisfy (2.2).

**Proof.** (1) If $\prod_{p \leq p_\beta} p > n_\beta$, then for any $n \leq n_\beta$ we have $\omega(n) \leq \beta - 1$. So, by Lemma 3.2, inequality (2.2) holds for $n_{\beta-1} < n \leq n_\beta$.

(2) If $n$ is an integer with exactly $\beta$ prime factors and $n \leq n_\beta$, then
\[ f(n) = \prod_{p \leq p_\beta} \frac{p}{p-1} \geq e^{\gamma}(1+\epsilon)\log \log n. \]

We are now ready to describe our algorithm.

**Algorithm 3.5** Finds exceptions to $f(n) < e^{\gamma}(1+\epsilon)\log \log n$.

**Input:** $0 < \epsilon < 1$.

**Output:** All exceptions to the inequality.

1. Calculate $M_\epsilon$.
2. Find the largest $\beta$ such that
\[ \prod_{p \leq p_\beta} p \leq M_\epsilon. \]
3. **while** $\prod_{p \leq p_\beta} p > n_\beta$ **do**
4. \( \beta \leftarrow \beta - 1. \)
5. \textbf{end while}
6. Calculate \( n_\alpha \) for \( 1 \leq \alpha \leq \beta. \)
7. For any \( 1 \leq \alpha \leq \beta, \) find all integers with \( \alpha \) distinct prime divisors not exceeding \( n_\alpha \) and write them in a file.

The correctness of this algorithm is a direct consequence of the previous lemmas. From now on, we call an exception to the inequality \( f(n) < e^\gamma (1 + \varepsilon) \log \log n \) simply an \textit{exception}.

**Example 3.6.** For \( \varepsilon = 0.07 \) we have

\[
[M_{0.07}] = 288657528452597095122710571703443536840.
\]

Since 26 is the greatest number such that \( \prod_{p \leq p_\beta} p \leq M_{0.07}, \) we have 26 as the initial value of \( \beta \). After the while-loop executes, the algorithm will give 18 as the final value of \( \beta \). Then for any \( \alpha \) (\( 1 \leq \alpha \leq \beta \)), Step 7 of the algorithm constructs all the exceptions. Table 2 records the number of exceptions for each \( \alpha \).

For example, there are 212 exceptions with \( \omega(n) = 18 \). In total, there are 3,281,014 exceptions to the inequality \( f(n) \leq 1.07e^\gamma \log \log n. \)

<table>
<thead>
<tr>
<th>( \alpha = \omega(n) )</th>
<th># of Exceptions</th>
<th>( \alpha = \omega(n) )</th>
<th># of Exceptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>11</td>
<td>10</td>
<td>542533</td>
</tr>
<tr>
<td>2</td>
<td>69</td>
<td>11</td>
<td>740427</td>
</tr>
<tr>
<td>3</td>
<td>373</td>
<td>12</td>
<td>564065</td>
</tr>
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<td>7418</td>
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<td>210907</td>
</tr>
<tr>
<td>6</td>
<td>27134</td>
<td>15</td>
<td>106791</td>
</tr>
<tr>
<td>7</td>
<td>66268</td>
<td>16</td>
<td>27963</td>
</tr>
<tr>
<td>8</td>
<td>197450</td>
<td>17</td>
<td>3043</td>
</tr>
<tr>
<td>9</td>
<td>454229</td>
<td>18</td>
<td>212</td>
</tr>
</tbody>
</table>

Table 2. Number of exceptions for each \( \alpha \).

4. Applications

We recall that

\[
\frac{\sigma(n)}{n} < \frac{n}{\phi(n)} \leq f(n).
\]

Moreover,

\[
\lim_{n \to \infty} \frac{\sigma(n)}{ne^\gamma \log \log n} = \lim_{n \to \infty} \frac{n}{\phi(n)e^\gamma \log \log n} = \lim_{n \to \infty} \frac{f(n)}{e^\gamma \log \log n} = 1
\]

(see [2], p. 353 and Proposition 2.1). We can use the above inequality, together with our algorithm, to establish explicit upper bounds for \( \sigma(n)/n \) and \( n/\phi(n) \). For example, by numerically checking the exceptions generated by
the algorithm for $\epsilon = 0.07$ against the inequality $n/\phi(n) < 1.07e^\gamma \log \log n$, we find that there are 6,569 exceptions to $n/\phi(n) < 1.07e^\gamma \log \log n$, the largest being 234576762718813941966540. This gives us the following proposition.

**Proposition 4.1.**

$$\frac{n}{\phi(n)} < 1.07e^\gamma \log \log n \text{ for } n > 234576762718813941966540.$$  

Using the same method, we deduce that there are only 14 exceptions to the inequality $\sigma(n)/n < 1.07e^\gamma \log \log n$. The largest of these is 120. Hence,  

$$\frac{\sigma(n)}{n} < 1.07e^\gamma \log \log n \text{ for } n \geq 121.$$  

As expected, the number of exceptions increases dramatically as $\epsilon$ gets smaller. For example, for $\epsilon = 0.06$ there are 32,707,736 exceptions and for $\epsilon = 0.05$ there are 798,101,126 exceptions. If we are only interested in upper bounds for $\sigma(n)$, we can reduce the number of possible exceptions to the inequality $\sigma(n) < e^\gamma (1 + \epsilon) \log \log n$ significantly by the following two observations.

First of all, by a result of Robin ([5], Theorem 2), for $n \geq 3$, we have  

$$\frac{\sigma(n)}{n} \leq e^\gamma \log \log n + \frac{0.6483}{\log \log n}.$$  

So, by an argument similar to Lemma 3.1,  

\begin{equation}
\frac{\sigma(n)}{n} < e^\gamma (1 + \epsilon) \log \log n
\end{equation}

for  

$$n > M_\epsilon = \exp \left( \exp \left( \frac{0.6483}{\epsilon e^\gamma} \right) \right)$$  

with $\omega(n) \geq 2$. Note that $M_\epsilon$ is much smaller than $M_\epsilon$. Thus we need only check (4.1) for the exceptions not exceeding $M_\epsilon$.

Secondly, by a recent result of Choie, Lichiardopol, Moree and Solé [1], we know that if $n \geq 5041$ does not satisfy Robin’s inequality $\sigma(n) > e^\gamma n \log \log n$, then $n$ is even, is neither squarefree nor squarefull, and is divisible by a fifth power greater than 1. In other words, all $n \geq 5041$ that are odd, squarefree, squarefull, or are not divisible by a fifth power greater than 1 satisfy (4.1).

Let $S_1$ be the set of natural numbers not exceeding $M_\epsilon$. Let $S_2$ be those numbers in $S_1$ that are even, neither squarefree nor squarefull, and are divisible by a fifth power greater than 1. Table 3 gives the number of exceptions, the number of exceptions in $S_1$ and the number of exceptions in $S_2$ for different values of $\epsilon$.

Notice that in order to prove Theorem 1.3, we need to check (4.1) in the case $\epsilon = 0.03$ for about 55,000,000 values of $n$.  

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**Explicit Upper Bounds for $\Pi_{p \leq \omega(n)/p/(p-1)$**
Table 3. Number of exceptions total, in $S_1$, and in $S_2$.

<table>
<thead>
<tr>
<th>$\epsilon$</th>
<th># of exceptions</th>
<th># of exceptions in $S_1$</th>
<th># of exceptions in $S_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.07</td>
<td>3,281,014</td>
<td>10,190</td>
<td>135</td>
</tr>
<tr>
<td>0.06</td>
<td>32,707,734</td>
<td>63,076</td>
<td>850</td>
</tr>
<tr>
<td>0.05</td>
<td>798,101,116</td>
<td>418,627</td>
<td>5,672</td>
</tr>
<tr>
<td>0.04</td>
<td></td>
<td>23,472,726</td>
<td>323,069</td>
</tr>
<tr>
<td>0.03</td>
<td></td>
<td>4,420,980,851</td>
<td>55,258,878</td>
</tr>
<tr>
<td>0.029</td>
<td></td>
<td>15,910,840,055</td>
<td>183,084,959</td>
</tr>
</tbody>
</table>

Proof of Theorem 1.3. By running our algorithm for $\epsilon = 0.03$, we generate all the exceptions to the inequality $f(n) < 1.03e^\gamma \log \log n$ in $S_2$. By checking these exceptions against the inequality $g(n) < 1.03e^\gamma$, we find that 180 is the largest $n$ that does not satisfy $g(n) < 1.03e^\gamma$. \qed

We end with a note on the computations. The algorithm is implemented in C++. The checking of exceptions against the inequality $g(n) < (1 + \epsilon)e^\gamma$ is done by Maple 10. The running time for checking in the case $\epsilon = 0.03$ is 40 hours with a 3 GHz Intel Pentium 3 having 1 GB of memory.

References