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ON THE CHROMATIC NUMBER OF GENERALIZED KNESER GRAPHS

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Abstract. For integers n, k, and i, the generalized Kneser graph $K(n, k, i)$, is a graph whose vertices are subsets of size k of the set $\{1, 2, ..., n\}$ and two vertices F and F' are connected if and only if their intersection has less than i elements. In this paper we study the chromatic number of this graph. Some new bounds and properties for this chromatic number are derived.

1. Introduction

First let us fix some notation. For a finite set A, $|A|$ denotes the cardinality of A. We use $\binom{A}{k}$ for the set of subsets of size k in A, we sometimes use k-subset for its elements. Suppose that X is a finite set, $|X| = n$. For integers k and i, the generalized Kneser graph $K(n, k, i)$ is defined by

$$V(K(n,k,i)) = \binom{X}{k},$$
$$E(K(n,k,i)) = \{(F, F') : F, F' \in \binom{X}{k}, |F \cap F'| < i\}.$$

We use $\chi(n, k, i)$ for the chromatic number of $K(n, k, i)$. Note that when $0 \leq k \leq n$ is not satisfied then $V(K(n, k, i)) = \emptyset$, and in such cases the convention $\chi(n, k, i) = 0$ is used. If $i \geq k$, this graph is a complete graph and $\chi(n, k, i) = \binom{n}{k}$. Furthermore if $i \leq 0$ or $n \leq 2k - i$ then $K(n, k, i)$ is a discrete graph with no edges and if $0 \leq k \leq n$ then $\chi(n, k, i) = 1$.

Kneser [6] conjectured that when $n \geq 2k$, the chromatic number of $K(n, k, 1)$ is equal to $n - 2k + 2$. Lovász [8] and Bárány [2] proved this conjecture. Later, Tort [10] proved that $\chi(n, 3, 2) = \lfloor ((n-1)/2)^2 \rfloor$ for $n \geq 6$. Frankl [4] showed that for $n > 10k^3e^k$, $\chi(n, k, 2) = (k-1)\binom{s}{2} + rs$, where $n = (k-1)s + r$, $0 \leq r < k - 1$. In this paper we introduce some upper and lower bounds for $\chi(n, k, i)$ and give an identity to compute them in some new cases.

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2. Upper bounds

In this section we present some upper bounds for computing $\chi(n,k,i)$.

2.1. **The first upper bound.**

Lemma 2.1. *For all integers n, k, and i, we have:*

$$\chi(n,k,i) \leq \chi(n-1,k,i) + \chi(n-1,k-1,i-1).$$

Proof. If $n < k$, then all the terms in the inequality are zero. If $k \leq i$, then the inequality becomes the Pascal identity. If $i \leq 0$, then all the terms in the equality are 1. So we can assume that $0 < i < k \leq n$.

Let V_1 be the set of vertices of $K(n,k,i)$ corresponding to the k-subsets containing 1. Obviously, it is possible to color the vertices of V_1 with $\chi(n-1,k-1,i-1)$ colors. This is consistent with the above convention $\chi(n,k,0) = 1$, since in case $i = 1$, V_1 is a discrete graph which has chromatic number 1. Now $V \setminus V_1$ is the set of vertices corresponding to the k-subsets not containing 1, so we can color them with $\chi(n-1,k,i)$ new colors. So, we can color all the vertices with $\chi(n-1,k,i)+\chi(n-1,k-1,i-1)$ colors and the lemma is proved. □

Using Lemma 2.1, we have:

Theorem 2.2. *For all integers $0 < i \leq k \leq n$, if $n - 2k + 2i < i$, then $\chi(n,k,i) = 1$ and otherwise,*

$$\chi(n,k,i) \leq \binom{n-2k+2i}{i}.$$

Proof. If $n-2k+2i < i$, then $n < 2k-i$ and any two k-subsets of $\{1,\dots,n\}$ have at least i elements in common and hence $K(n,k,i)$ is a discrete graph with chromatic number 1. Now we prove the theorem by induction on n. The case $n = 1$ is trivial. If $n-2k+2i = i$, then $n = 2k-i$ and one sees that $K(n,k,i)$ is discrete and the chromatic number is 1 and $\binom{n-2k+2i}{i} = 1$. Now suppose $n-2k+2i > i$, then $(n-1)-2k+2i \geq i$ and $(n-1)-2(k-1)+2(i-1) \geq i$ so by the inductive hypothesis $\chi(n-1,k,i) \leq \binom{n-2k+2i-1}{i}$ and $\chi(n-1,k-1,i-1) \leq \binom{n-2k+2i-1}{i-1}$. Notice that these inequalities also hold when $n = k$ or $i = 1$. Now the theorem follows from the previous lemma and the Pascal identity. □

The following proposition gives a generalization of Lemma 2.1.

Proposition 2.3. *For any integer $1 \leq m < i$,*

$$\chi(n,k,i) \leq \sum_{j=0}^{m} \chi(n-m,k-j,i-j)\binom{m}{j}.$$

Proof. Let $Y \subset X$ be a proper subset of X and $|Y| = m$. For $j = 0,\dots,m$, let $V_j \subset V$ be the set of vertices such that their corresponding k-subsets have exactly j elements of Y. For each subset of size j inside Y, the vertices of V_j which contain that subset can be colored with $\chi(n-m, k-j, i-j)$ colors. Since, we have $\binom{m}{j}$ subsets of size j inside Y, the vertices of V_j can be colored with $\chi(n-m, k-j, i-j)\binom{m}{j}$ colors. Adding these, we have a method for coloring all the vertices and hence, the proposition is proved. □

Using Lemma 2.1, we compute an upper bound for $\chi(n, 4, 3)$.

Lemma 2.4. *For $n > 4$, we have $\chi(n, 4, 3) \leq n^3/12 + O(n^2)$.*

Proof. Lemma 2.1 implies the following inequalities

$$\begin{aligned} \chi(n,4,3) &\leq \chi(n-1,4,3) + \chi(n-1,3,2), \\ \chi(n-1,4,3) &\leq \chi(n-2,4,3) + \chi(n-2,3,2), \\ &\vdots \\ \chi(6,4,3) &\leq \chi(5,4,3) + \chi(5,3,2). \end{aligned}$$

We sum both sides of the inequalities, so we have

$$\chi(n,4,3) \leq \chi(5,4,3) + \chi(5,3,2) + \sum_{m=6}^{n-1} \chi(m,3,2).$$

According to [10] we know that $\chi(m, 3, 2) = \lfloor ((m-1)/2)^2 \rfloor$ for $m \geq 6$. We also know that $\chi(5, 4, 3) = 1$ and $\chi(5, 3, 2) = 3$, so

$$\chi(n,4,3) \leq 4 + \sum_{m=6}^{n-1} \left\lfloor \left(\frac{m-1}{2}\right)^2 \right\rfloor.$$

This yields, if n is even,

$$\chi(n,4,3) \leq \frac{(n-2)(n-1)(n)}{12} - \frac{n(n-2)}{8} - 3,$$

and if n is odd,

$$\chi(n,4,3) \leq \frac{(n-1)(n-2)(n-3)}{12} + \frac{(n-1)(n-3)}{8} - 3.$$

□

By generalizing Lemma 2.4 we have the following Proposition.

Proposition 2.5. *For all integers $2 < k+1 < n$,*

$$\chi(n,k+1,k) \leq \frac{n^k}{2.k!} + O(n^{k-1}).$$

Proof. The proof is by induction on k. We know that $\chi(n, 3, 2) = n^2/4 + O(n)$. Suppose that $\chi(n, k+1, k) \leq n^k/(2.k!) + O(n^{k-1})$, we prove that $\chi(n, k+2, k+1) = n^{k+1}/(2.(k+1)!) + O(n^k)$.

According to Lemma 2.1, we have the following inequalities,

$$\begin{aligned}\chi(n,k+2,k+1) &\leq \chi(n-1,k+2,k+1)+\chi(n-1,k+1,k),\\ \chi(n-1,k+2,k+1) &\leq \chi(n-2,k+2,k+1)+\chi(n-2,k+1,k),\\ &\vdots\\ \chi(k+3,k+2,k+1) &\leq \chi(k+2,k+2,k+1)+\chi(k+2,k+1,k).\end{aligned}$$

So, if we sum these inequalities, we get

$$\chi(n,k+2,k+1) \leq \frac{n^{k+1}}{2.(k+1)!}+O(n^k).$$

□

2.2. **The relation between Turán problem and $\chi(n,k,i)$.** Let $t(n,k,i)$ be the minimum number of i-element subsets of X, $|X|=n$, such that any k-subset of X contains at least one of them. The determination of $t(n,k,i)$ is known as Turán's problem. It is easily seen that $\chi(n,k,i) \leq t(n,k,i)$. Frankl [4] has conjectured that for $i \geq 2$ and $n \geq n_0(k,i)$, one has

$$\chi(n,k,i)=t(n,k,i).$$

He has also claimed the following:

$$\chi(n,k,i)=(1+o(1))t(n,k,i).$$

Turán's problem [11] is well known and there are many results about it. Turán himself calculated $t(n,k,2)$ to be $\chi(n,k,2)$ for $n > n_0(k)$. Chung and Lu [3] showed that

$$\lim_{n\to\infty} \frac{t(n,4,3)}{\binom{n}{3}} \leq \frac{3+\sqrt{17}}{12}=0.593592\ldots.$$

Also, the asymptotic value of $t(n,4,3)$ is conjectured in [11] as follows:

$$\lim_{n\to\infty} \frac{t(n,4,3)}{\binom{n}{3}}=\frac{5}{9}.$$

Kostochka [7] gave several different constructions which achieve the conjectured asymptotic value for $t(n,4,3)$. According to Frankl, $\chi(n,4,3)=(1+o(1))t(n,4,3)$. So, for the family of examples given by Kostochka [7] we should have, $\chi(n,4,3)=(1+o(1))\binom{n}{3}5/9$. On the other hand by Proposition 2.4, we have $\chi(n,4,3) \leq n^3/12+O(n^2)$. This is in contradiction to Frankl's claim.

2.3. **The second upper bound.** In this subsection we give another upper bound for $\chi(n,k,i)$ for all $1 < i < k < n$, which is a generalization of the method used in [4].

Proposition 2.6. *Let* $q = \lceil k/(i-1) \rceil - 1$ *and* $n = qs + r$, *where* $0 \le r < q$, *then*

$$\chi(n,k,i) \le (q-r)\binom{s}{i} + r\binom{s+1}{i}.$$

Proof. Suppose that we partition the elements of X into $q-r$ subsets of size s and r subsets of size $s+1$. Let I be the set of all subsets of size i from each of these q subsets. Then, $|I| = (q-r)\binom{s}{i} + r\binom{s+1}{i}$. We consider a color for each of the elements of I. We claim that any k-subset A of X contains at least one of the elements of I. Because otherwise A contains at most $i-1$ elements of each of these q partitions, so it has at most $q(i-1)$ elements, but since $q = \lceil k/(i-1) \rceil - 1$, this number is less than k which is a contradiction. Therefore, we can color the vertices of $K(n,k,i)$ with one of the elements of I that they contain. □

3. An Inequality

We also have the following inequality which in some cases gives us better upper bounds.

Theorem 3.1. *For any* $0 < i < k < n$,

$$\chi(n+2, k+1, i) \le \chi(n,k,i).$$

Proof. We give a coloring of $K(n+2,k+1,i)$ with $\chi(n,k,i)$ colors. First, to any $(k+1)$-subset A of $\{1,..,n+2\}$, we associate a k-subset A' of $\{1,2,..,n\}$ as follows: If A does not contain both of $n+1$ and $n+2$, then A' is obtained by deleting the largest element of A. Otherwise, A' is obtained by deleting both $n+1$ and $n+2$ and adding the largest element from $\{1,2,..,n\}$ that does not belong to A. We claim that for any pair of $(k+1)$-subsets A and B of $\{1,2,...,n+2\}$, $|A' \cap B'| \le |A \cap B|$. Then by giving A the color of A' we give an example of coloring $K(n+2,k+1,i)$ with $\chi(n,k,i)$ colors. To prove our claim, if neither of A and B contain both $n+1$ and $n+2$, then since $A' \subseteq A$ and $B' \subseteq B$, the claim is obvious. If both A and B contain both $n+1$ and $n+2$ then since we delete two common elements and add only one element, the claim follows. Now suppose A contains $n+1$ and $n+2$ and B contains exactly one of $n+1$ and $n+2$. Since one common element is deleted and only one new element is added the claim follows. Finally, we consider the case where A contain both $n+1$ and $n+2$ and B contains neither of them. If the largest element in B is inside A, since that common element is deleted and only one element is added in A', the claim follows. Otherwise, the largest element in B is not in A and therefore the new element added in A' will not be in B', so the last case for proving our claim is settled. □

Using this inequality we have the following upper bounds:

$$\chi(2k,k,2) \le \chi(6,3,2) = 6.$$

This upper bound is the same as the one obtained from Theorem 2.2.

$$\chi(2k+1,k,2) \le \chi(7,3,2) = 9.$$

Note that if we use Theorem 2.2 the obtained upper bound will be 10. Finally if we use the computation of $\chi(n,3,2)$ in [10] and this theorem, we have:

$$\chi(2k+i,k,2) \leq \chi(i+6,3,2) = \left\lfloor \left(\frac{i+5}{2}\right)^2 \right\rfloor.$$

4. Duality

In this section we present an equality which helps us to find the exact value or improve the upper bounds for some special cases of $\chi(n,k,i)$.

Theorem 4.1. *For $0 < i < k < n$,*

$$\chi(n,k,i) = \chi(n,n-k,n-2k+i).$$

Proof. First, we recall the following easy observation about k-subsets of X. If A and B are two k-subsets of X such that $|A \cap B| < i$, then $|A^c \cap B^c| < n-2k+i$, where A^c and B^c are the complements of A and B in X. Therefore, sending a k-subset A of X to the $(n-k)$-subset A^c of X will provide an isomorphism between $K(n,k,i)$ and $K(n,n-k,n-2k+i)$ and hence, they have the same chromatic numbers. □

Using this equality we can improve some upper bounds and also compute the exact chromatic number of some special cases, for example:

$$\chi(n,n-3,n-4) = \chi(n,3,2) = \left\lfloor (\frac{n-1}{2})^2 \right\rfloor.$$

This yields $\chi(7,4,3) = 9$ and also by Theorem 3.1 for $i \geq 0$

$$\chi(7+2i,4+i,3) \leq 9.$$

5. Lower bounds

5.1. **A lower bound from topology.** In this section we present a lower bound for $\chi(n,k,i)$. Our proof is similar to the topological method used by Greene in [5, 9].

Theorem 5.1. *For $0 < i < k < n$*

$$n-2k+2i \leq \chi(n,k,i).$$

Proof. Let $d = n-2k+2i-1$ and take n generic points on the d-dimensional sphere S^d. This means that no $d+1$ of these points lie on a hyperplane that contains the origin. Assume we can color all the k-subsets of these n points with d colors. For $1 \leq j \leq d$, let U_j be the open set of points $x \in S^d$ such that there is a k-subset of color j such that at least $k-i+1$ elements of it lie in the open hemisphere with center x. Notice that U_j can not contain two antipodal points. Otherwise we would have two k-subsets of color j such that each of them has at least $k-i+1$ elements in one of two disjoint hemispheres, so their intersection has at most $i-1$ elements which is impossible. If we define $U_{d+1} = S^d \setminus (\cup_{j=1}^{d} U_j)$ then $S^d = U_1 \cup U_2 \cup ... \cup U_{d+1}$

and according to Borsuk-Ulam theorem [1] one of the U_j's should contain two antipodal points. This has to be U_{d+1} which means there are two disjoint open hemispheres each of which contain at most $k-i$ points and therefore the remaining at least $n-2(k-i)=d+1$ points should lie on a hyperplane that contains the origin. This contradicts the genericity of our chosen points. □

5.2. **Lower bound using independence number.** The independence number of a graph G, $\alpha(G)$, is defined to be the maximum number of vertices of G that are pairwise disconnected. Obviously, for coloring G, at least we need $n/\alpha(G)$ colors where n is the number of vertices of G, i.e.

$$\left\lceil \frac{n}{\alpha(G)} \right\rceil \leq \chi(G).$$

Using this, we compute the independence number of $K(n,k,i)$ to give a lower bound for $\chi(n,k,i)$. According to [**?**], we have

Theorem 5.2. *The independence number of $K(4m,2m,2)$ is*

$$\frac{1}{2}\left(\binom{4m}{2m}-\binom{2m}{m}^2\right).$$

By Theorem 5.1 and Theorem 2.2 we have,

$$4 \leq \chi(2k,k,2) \leq 6.$$

While using Theorem 5.2 we have,

$$5 \leq \chi(8,4,2) \leq 6.$$

This gives a better bound than Theorem 5.1. Using a computer, we have computed the exact value of $\chi(8,4,2)$ to be 6. Stirling's formula implies that the independence number of $K(4m,2m,2)$ is asymptotic to $\binom{4m}{2m}/2$, so the lower bound will become 3 which is worse than the bound found using topology. We have the following conjecture for $\chi(2k,k,2)$,

Conjecture. *For any $k \geq 3$ we have, $\chi(2k,k,2)=6$.*

6. Conclusion and open problems

In this paper we have presented some new results on the chromatic number of generalized Kneser graphs. We have given some new upper and lower bounds. Also, we have given some new inequalities and also some other equalities which help us to improve the bounds and compute the exact value for some special cases. We conjecture that some of these inequalities can become equalities.

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