ON THE SPECTRUM OF OCTAGON QUADRANGLE SYSTEMS OF ANY INDEX

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Abstract. An octagon quadrangle is the graph consisting of a length 8 cycle \((x_1, x_2, \ldots, x_8)\) and two chords, \(\{x_1, x_4\}\) and \(\{x_5, x_8\}\). An octagon quadrangle system of order \(v\) and index \(\lambda\) is a pair \((X, B)\), where \(X\) is a finite set of \(v\) vertices and \(B\) is a collection of octagon quadrangles (called blocks) which partition the edge set of \(\lambda K_v\), with \(X\) as the vertex set. In this paper we completely determine the spectrum of octagon quadrangle systems for any index \(\lambda\), with the only possible exception of \(v = 20\) for \(\lambda = 1\).

1. Introduction

Let \(G = (X, E)\) be the graph having \(X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}\) and \(E = \{\{x_i, x_{i+1}\}, \{x_1, x_4\}, \{x_5, x_8\} \mid i \in \mathbb{Z}_8\}\). A graph of this type will be denoted \(\{(x_1), x_2, x_3, (x_4), (x_5), x_6, x_7, (x_8)\}\). It is called octagon quadrangle (briefly OQ).

A \(G\)-design of order \(v\) and index \(\lambda\) is a couple \(\Sigma = (X, B)\), where \(X\) is a finite set of \(v\) elements and \(B\) is a family of graphs all isomorphic to \(G\) such that for any \(x, y \in X\), with \(x \neq y\), there exist \(\lambda\) graphs \(G \in B\) having \(\{x, y\}\) as an edge. A \(G\)-design is also called a \(G\)-decomposition of \(\lambda K_v\) \([11, 14]\).

An octagon quadrangle system of order \(v\) and index \(\lambda\) will be denoted by \(OQS(v)\). Concepts and definitions of octagon quadrangle and octagon quadrangle systems have been introduced in \([1, 2, 4]\), where the authors studied perfect \(OQSs\), determining their spectrum. Similar questions have been studied in all the other papers cited in the references (see, e.g., \([5, 3, 6, 7]\)).

If a block \(\{(x_1), x_2, x_3, (x_4), (x_5), x_6, x_7, (x_8)\}\) is repeated \(k\) times in an \(OQS\), we use the notation \(\{(x_1), x_2, x_3, (x_4), (x_5), x_6, x_7, (x_8)\}(k)\).

A technique used in the constructions in the main results of the paper is the difference method. Given \(\mathbb{Z}_n\), for some \(n \in \mathbb{N}\), and given any two \(a, b \in \mathbb{Z}_n\), \(a \neq b\), there exists precisely one \(x \in \{1, \ldots, [n/2]\}\) such that either \(a = x + b\) or \(b = x + a\). In this case we say that the edge \(\{a, b\}\) has difference \(x\).

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Let $n$ be odd. Given an edge $\{a, b\}$ of difference $x \in \{1, \ldots, |n/2|\}$, any edge of the same difference $x$ is of type $\{a + i, b + i\}$ for exactly one $i \in \mathbb{Z}_n$.

Let $n$ even. Given an edge $\{a, b\}$ of difference $x \in \{1, \ldots, (n/2) - 1\}$, any edge of same difference $x$ is of type $\{a + i, b + i\}$ for exactly one $i \in \mathbb{Z}_n$; given an edge $\{a, b\}$ of difference $n/2$, any edge of same difference $x$ is of type $\{a + i, b + i\}$ for exactly one $i \in \{0, \ldots, (n/2) - 1\}$. So in this paper, often blocks in an $OQS$ are given by the translated forms of a base block. Other techniques used in these type of problems can also be found in [6, 7].

In this paper we will determine the spectrum of all $OQS(v)$ for any $\lambda$, with the exception of $\lambda = 1$ for $v = 20$.

2. Index $\lambda = 1$

In the following theorem we will give necessary conditions for the existence of an $OQS(v)$ of fixed index $\lambda$.

**Theorem 2.1.** Let $\Sigma = (X, \mathcal{B})$ be an $OQS(v)$ of index $\lambda \geq 1$. Then:

1. if $\lambda \equiv 0 \mod 10$, then $v \in \mathbb{N}$, with $v \geq 8$,
2. if $\lambda \equiv 1, 3, 7, 9 \mod 10$, then $v \equiv 0, 1, 5, 16 \mod 20$, with $v \geq 16$,
3. if $\lambda \equiv 2, 4, 6, 8 \mod 10$, then $v \equiv 0, 1 \mod 5$, with $v \geq 10$,
4. if $\lambda \equiv 5 \mod 10$, then $v \equiv 0, 1 \mod 4$, with $v \geq 8$.

**Proof.** Since $\Sigma = (X, \mathcal{B})$ is an $OQS(v)$ of index $\lambda$, we have:

$$|\mathcal{B}| = \frac{\lambda v(v - 1)}{20}.$$ 

\[\square\]

In the following theorem we get the spectrum for $OQS(v)$ of index 1 with a possible exception.

**Theorem 2.2.** For $\lambda = 1$ and for every $v \equiv 0, 1, 5, 16 \mod 20$, with $v \neq 20$, there exists an $OQS(v)$ of index 1.

**Proof.** Let $v = 20k + 1$, for some $k \geq 1$. In this case we use the difference method. Let us consider $\Sigma = (\mathbb{Z}_{20k+1}, \mathcal{B})$ whose blocks are:

$$(20k + 8 - 10i), 0, 20k + 10 - 10i, (1), (20k + 6 - 10i), 3, 20k + 4 - 10i, (2)$$

for $i = 1, \ldots, k$ and all their translated forms. Then $\Sigma$ is an $OQS(v)$ of index 1.

Let $v = 20k + 5$, for some $k \geq 1$. Let us consider $\Sigma = (\mathbb{Z}_{20k+4} \cup \{\infty\}, \mathcal{D})$, with $\infty \notin \mathbb{Z}_{20k+4}$, whose blocks are:

1. $A_i = [(2i + 1), \infty, 2i, (2i + 3), (2i + 6), 2i + 8, 2i + 4, (2i + 5)]$ for $i \in \{0, \ldots, 10k + 1\}$,
2. $B_i = [(2i), 2i + 10k + 1, 2i + 10k + 6, (2i + 5), (2i + 10k + 7), 2i + 4, 2i + 20k + 3, (2i + 10k + 2)]$ for $i \in \{0, \ldots, 5k\}$,
3. $C_{ij} = [(2i + 5j + 8), 2i, 2i + 5j + 10, (2i + 1), (2i + 5j + 11), 2i + 3, 2i + 5j + 9, (2i + 2)]$ for $i \in \{0, \ldots, 10k + 1\}$ and $j \in \{0, \ldots, 2k - 2\}$. 

Then $\Sigma$ is an $OQS(v)$ of index 1. Indeed, in this case we are using the difference method in an appropriate way, since $20k + 4$ is even. So in the blocks $A_i$ we have the differences:

- 1, given by the edges $\{2i + 4, 2i + 5\}$ and $\{2i + 5, 2i + 6\}$ for $i \in \{0, \ldots, 10k + 1\}$,
- 2, given by the edges $\{2i + 1, 2i + 3\}$ and $\{2i + 6, 2i + 8\}$ for $i \in \{0, \ldots, 10k + 1\}$,
- 3, given by the edges $\{2i, 2i + 3\}$ and $\{2i + 3, 2i + 6\}$ for $i \in \{0, \ldots, 10k + 1\}$,
- 4, given by the edges $\{2i + 1, 2i + 5\}$ and $\{2i + 4, 2i + 8\}$ for $i \in \{0, \ldots, 10k + 1\}$.

In the blocks $B_i$ we have the differences:

- 5, given by the edges $\{2i, 2i + 5\}$, $\{2i + 10k + 2, 2i + 10k + 7\}$, $\{2i + 10k + 1, 2i + 10k + 6\}$ and $\{2i + 10k + 3, 2i + 4\}$ for $i \in \{0, \ldots, 5k\}$,
- $10k + 1$, given by the edges $\{2i, 2i + 10k + 1\}$, $\{2i + 10k + 2, 2i + 10k + 3\}$, $\{2i + 5, 2i + 10k + 6\}$ and $\{2i + 10k + 7, 2i + 4\}$ for $i \in \{0, \ldots, 5k\}$,
- $10k + 2$, given by the edges $\{2i, 2i + 10k + 2\}$ and $\{2i + 5, 2i + 10k + 7\}$ for $i \in \{0, \ldots, 5k\}$.

In the blocks $C_{ij}$ we have the differences:

- $5j + 6$, given by the differences $\{2i + 3, 2i + 5j + 9\}$ and $\{2i + 2, 2i + 5j + 8\}$ for $i \in \{0, \ldots, 10k + 1\}$,
- $5j + 7$, given by the differences $\{2i + 2, 2i + 5j + 9\}$ and $\{2i + 1, 2i + 5j + 8\}$ for $i \in \{0, \ldots, 10k + 1\}$,
- $5j + 8$, given by the differences $\{2i + 3, 2i + 5j + 11\}$ and $\{2i, 2i + 5j + 8\}$ for $i \in \{0, \ldots, 10k + 1\}$,
- $5j + 9$, given by the differences $\{2i + 1, 2i + 5j + 10\}$ and $\{2i + 2, 2i + 5j + 9\}$ for $i \in \{0, \ldots, 10k + 1\}$,
- $5j + 10$, given by the differences $\{2i, 2i + 5j + 10\}$ and $\{2i + 1, 2i + 5j + 11\}$ for $i \in \{0, \ldots, 10k + 1\}$,

with $j \in \{0, \ldots, 2k - 2\}$.

Let $v = 16$. Let us consider $\Sigma = (\mathbb{Z}_{16}, \mathcal{B})$ whose blocks are:

1. $A_i = [(2i), 2i + 4, 2i + 11, (2i + 5), (2i + 13), 2i + 3, 2i + 12, (2i + 8)]$
   for $i \in \{0, 1, 2, 3\}$,
2. $B_i = [(2i + 1), 2i + 5, 2i + 3, (2i + 6), (2i + 7), 2i + 4, 2i + 10, (2i + 8)]$
   for $i \in \{0, 1, \ldots, 7\}$.

Then $\Sigma$ is an $OQS(v)$ of index 1. Indeed, we use again the difference method in a way similar to the previous one and we get:

- the differences 1, 2 and 3 in the blocks $B_i$,
- the differences 4, 5, 6 and 7 in the blocks $A_i$ and $B_i$,
- the difference 8 in the blocks $A_i$.

Let $v = 20k + 16$, for some $k \geq 1$. Let us consider $\Sigma = (\mathbb{Z}_{20k + 16}, \mathcal{B})$ whose blocks are:
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(1) \(A_i = [(20k+23-10i), 0, 20k+25-10i, (1), (20k+21-10i), 3, 20k+19-10i, (2)] \) for \(i \in \{1, \ldots, k\} \) and all their translated forms,

(2) \(B_i = [(2i), 2i+10k+4, 2i+20k+11, (2i+10k+5), (2k+20k+13), 2i+10k+3, 2i+20k+12, (2i+10k+8)] \) for \(i \in \{0, 1, \ldots, 5k+3\} \),

(3) \(C_i = [(2i), 2i+10k+1, 2i+3, 2i+10k+2, (2i+1), 2i+10k+4, 2i+20k+10, (2i+10k+7)] \) for \(i \in \{0, 1, \ldots, 10k+7\} \).

Then \(\Sigma\) is an OQS\((v)\) of index 1. In fact, using the previous method we get:

- the differences 1, 2, \ldots, 10k in the blocks \(A_i\) and their translated forms,
- the differences 10k + 1, 10k + 2 and 10k + 3 in the blocks \(C_i\),
- the differences 10k + 4, 10k + 5, 10k + 6 and 10k + 7 in the blocks \(B_i\) and \(C_i\),
- the difference 10k + 8 in the blocks \(B_i\).

Let \(v = 40\). Let us consider \(\Sigma = (\mathbb{Z}_{13} \times \mathbb{Z}_3 \cup \{\infty\}, \mathcal{B})\), where \(\infty \notin \mathbb{Z}_{13} \times \mathbb{Z}_3\) and whose blocks are:

- (1) \([(i, 1)), (i+1, 2), (i, 0), (\infty), ((i, 2)), (i+1, 0), (i-1, 2), ((i+1, 1))] \) for any \(i \in \mathbb{Z}_{13}\),
- (2) \([(i+2, 0)), (i, 0), (i+1, 0), ((i+5, 0)), ((i+1, 2)), (i, 2), (i+2, 2), ((i+5, 2))] \) for any \(i \in \mathbb{Z}_{13}\),
- (3) \([(i+5, 1)), (i+2, 1), (i, 1), ((i, 0)), ((i, 2)), (i+11, 1), (i+4, 1), ((i+9, 1))] \) for any \(i \in \mathbb{Z}_{13}\),
- (4) \([(i+6, 0)), (i, 0), (i+5, 0), ((i+12, 1)), ((i+5, 2)), (i, 2), (i+6, 2), ((i+10, 1))] \) for any \(i \in \mathbb{Z}_{13}\),
- (5) \([(i+12, 1)), (i+6, 2), (i+9, 1), ((i, 0)), ((i+2, 1)), (i+7, 0), (i+4, 1), ((i+1, 0))] \) for any \(i \in \mathbb{Z}_{13}\),
- (6) \([(i, 2)), (i+11, 0), (i+5, 2), ((i, 1)), ((i+3, 2)), (i+6, 0), (i+12, 2), (i+8, 0))] \) for any \(i \in \mathbb{Z}_{13}\).

Then \(\Sigma\) is an OQS\((v)\) of index 1.

Let \(v = 60\). Let us consider \(\Sigma' = (X, \mathcal{B}')\), an OQS\((45)\) of index 1, with \(X = \{a_i \mid i = 0, \ldots, 44\}\). Given \(\mathbb{Z}_{15}\), consider:

- (1) \(C_1 = \{(i+5), i+1, i, (a_{42}), (i+10), i+4, i+12, (i+7) \mid i = 0, \ldots, 4\}\),
- (2) \(C_2 = \{(i+5), i+1, i, (a_{43}), (i+10), i+4, i+12, (i+7) \mid i = 5, \ldots, 9\}\),
- (3) \(C_3 = \{(i+5), i+1, i, (a_{44}), (i+10), i+4, i+12, (i+7) \mid i = 10, \ldots, 14\}\),
- (4) \(C_4 = \{(i+1), a_{2i}, i, (a_{2i-1}), (i+2), a_{2i-3}, i+3, (a_{2i-2}) \mid i = 0, \ldots, 20\}\), where \(i, i+1, i+2, i+3\) are taken modulo 15 and the indices of the \(a_j\) are taken modulo 42,
- (5) \(C_5 = \{(i+6), a_{2i}, i+5, (a_{2i-1}), (i+7), a_{2i-3}, i+8, (a_{2i-2}) \mid i = 0, \ldots, 20\}\), where \(i+5, i+6, i+7, i+8\) are taken modulo 15 and the indices of the \(a_j\) are taken modulo 42,
- (6) \(C_6 = \{(i+11), a_{2i}, i+10, (a_{2i-1}), (i+12), a_{2i-3}, i+13, (a_{2i-2}) \mid i = 0, \ldots, 20\}\), where \(i+10, i+11, i+12, i+13\) are taken modulo 15 and the indices of the \(a_j\) are taken modulo 42.

Then \(\Sigma = (X \cup \mathbb{Z}_{15}, \mathcal{B}' \cup \bigcup_{i=1}^{6} C_i)\) is an OQS\((v)\) of index 1.
Let \( \Sigma' = (X', B') \) be an \( OQS(v) \) of index 1, for some \( v \equiv 0 \mod 20 \), \( v \neq 20 \), with \( X' = \{a_i \mid i = 0, \ldots, v-1\} \), and let \( \Sigma'' = (X'', B'') \) be an \( OQS(40) \), with \( X'' = \{b_i \mid i = 0, \ldots, 39\} \). Let us consider:

\[
\mathcal{C} = \{ [b_{i+1+10j}, a_i, b_{i+10j}, (a_{i-2}), b_{i+2+10j}, a_{i-6}, b_{i+3+10j}, (a_{i-4})] \mid i = 0, \ldots, v-1, j = 0, 1, 2, 3 \},
\]

where the indices are taken modulo \( v \) and modulo 40. Then, given \( X = X' \cup X'' \) and \( B = B' \cup B'' \cup \mathcal{C}, \Sigma = (X, B) \) is an \( OQS(v + 40) \) of index 1. This proves that for any \( v \equiv 0 \mod 20 \), \( v \geq 40 \), there exists an \( OQS(v) \) of index 1. \( \square \)

3. Index \( \lambda = 2 \)

**Theorem 3.1.** For \( \lambda = 2 \) and for every \( v \equiv 0, 1 \mod 5 \) there exists an \( OQS(v) \) of index 2.

**Proof.** Let \( v = 10k \), for some \( k \geq 1 \). Let us consider \( \Sigma = (\mathbb{Z}_{10k-1} \cup \{\infty\}, B) \), with \( \infty \notin \mathbb{Z}_{10k} \), whose blocks are:

\[
(1) \quad [(0), 5i+1, 10i+6, (5i+2), (10i+5), 5i+4, 10i+8, (5i+3)] \quad \text{for any } i \in \{0, \ldots, k-2\} \text{ and all their translated forms (in the case } k \geq 2),
\]

\[
(2) \quad [(i), i+5k-4, \infty, (i+5k-3), (i+10k-5), i+5k-2, i+10k-3, (i+5k-1)] \quad \text{for any } i \in \mathbb{Z}_{10k-1}.
\]

Then \( \Sigma \) is an \( OQS(v) \) of index 2.

Let \( v = 10k + 1 \), for some \( k \geq 1 \). Let us consider \( \Sigma = (\mathbb{Z}_{10k+1}, B) \) whose blocks are:

\[
[(0), 5i+1, 10i+6, (5i+2), (10i+5), 5i+4, 10i+8, (5i+3)] \quad \text{for } i = 0, \ldots, k-1
\]

and all their translated forms. Then \( \Sigma \) is an \( OQS(v) \) of index 2.

Let \( v = 10k + 5 \), for some \( k \geq 1 \). Let us consider \( \Sigma = (\mathbb{Z}_{10k+4} \cup \{\infty\}, B) \), with \( \infty \notin \mathbb{Z}_{10k+4} \), whose blocks are:

\[
(1) \quad A_i = [(0), 5i+1, 10i+6, (5i+2), (10i+5), 5i+4, 10i+8, (5i+3)] \quad \text{for any } i \in \{0, \ldots, k-2\} \text{ and all their translated forms (in the case } k \geq 2),
\]

\[
(2) \quad B_i = [(i+10k-2), i+5k-2, i+10k-5, (i+5k-1), (i+10k), \infty, i+10k-1, (i+5k+2)] \quad \text{for any } i \in \mathbb{Z}_{10k+4},
\]

\[
(3) \quad C_j = [(2j+3), 2j+5k+3, 2j+1, (2j+5k+2), (2j), 2j+5k, 2j+2, (2j+5k+1)] \quad \text{for any } j \in \{0, \ldots, 5k+1\}.
\]

Then \( \Sigma \) is an \( OQS(v) \) of index 2. In fact, in this case we use again the difference method and we get:

- the differences 1, 2, \ldots, 5k-5, each repeated twice, in the blocks \( A_i \) and their translated forms,
- the differences 5k-4 and 5k-3 twice in the blocks \( B_i \),
- the differences 5k-2, 5k-1, 5k and 5k+1, each once in the blocks \( B_i \) and once in the blocks \( C_j \),
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- the difference $5k + 2$ in the blocks $C_j$, given by the edges $\{2j, 2j + 5k + 2\}$ and $\{2j + 1, 2j + 5k + 3\}$ for $j \in \{0, \ldots, 5k + 1\}$, so that each edge of difference $5k + 2$ appears twice.

Let $v = 10k + 6$, for some $k \geq 1$. Let us consider $\Sigma = (\mathbb{Z}_{10k+6}, \mathcal{B})$, whose blocks are:

1. $A_{ij} = [(2j), 2j + 5i + 3, 2j - 1, (2j + 5i + 4), (2j + 3), 2j + 5i + 6, 2j + 4, (2j + 5i + 5)](2)$ for any $i \in \{1, \ldots, k - 1\}$ and for any $j \in \{0, \ldots, 5k + 2\}$ (in the case $k \geq 2$),
2. $B_j = [(2j), 2j + 1, 2j + 6, (2j + 2), (2j + 7), 2j + 1, (2j + 5), (2j + 3)]$ for any $j \in \{0, \ldots, 5k + 2\}$,
3. $C_j = [(2j - 1), 2j + 5k, 2j - 2, (2j + 5k + 1), (2j), 2j + 1, 2j - 3, (2j + 2)]$ for any $j \in \{0, \ldots, 5k + 2\}$,
4. $D_j = [(2j), 2j + 5k + 1, 2j - 1, (2j + 5k + 2), (2j + 1), 2j + 2, 2j - 2, (2j + 3)]$ for any $j \in \{0, \ldots, 5k + 2\}$.

Then $\Sigma$ is an $OQS(v)$ of index 2. Indeed, also in this case we use the difference method and get:

- the differences 1, 2, 3, 4 and 5 once in the blocks $B_j$ and once among the blocks $C_j$ and $D_j$,
- the differences 6, 7, $\ldots$, 5k in the blocks $A_{ij}$, each of them repeated twice, because the blocks are repeated twice,
- the differences 5k + 1 and 5k + 2 once in the blocks $C_j$ and once in the blocks $D_j$,
- the difference 5k + 3, in the blocks $C_j$ given by the edges $\{2j - 2, 2j + 5k + 1\}$ and in the blocks $D_j$ given by the edges $\{2j - 1, 2j + 5k + 2\}$, so that each edge of difference 5k + 3 appears twice.

$\square$

4. INDEX $\lambda = 5$

**Theorem 4.1.** For $\lambda = 5$ and for every $v \equiv 0, 1 \pmod{4}$, there exists an $OQS(v)$ of index 5.

**Proof.** Let $v = 9$. Let us consider $\Sigma = (\mathbb{Z}_9, \mathcal{B})$ whose blocks are:

$$[(6), 0, 1, (2), (3), 4, 5, (8)] \quad \text{and} \quad [(6), 0, 2, (4), (7), 3, 5, (1)]$$

and all their translated forms. Then $\Sigma$ is an $OQS(9)$ of index 5.

Let $v = 4k + 1$, for some $k \geq 3$. Let us consider $\Sigma = (\mathbb{Z}_{4k+1}, \mathcal{B})$ whose blocks are:

1. $[(2i - 1), 0, 2i, (4i + 1), (2i + 1), 4i + 3, 6i + 2, (4i)]$ for $i = 1, \ldots, k - 1$,
2. $[(2k - 1), 4k - 2, 2k - 2, (4k), (1), 3, 2, (0)]$

and all their translated forms. Then $\Sigma$ is an $OQS(v)$ of index 5.

Let $v = 8$. Let us consider $\Sigma = (\mathbb{Z}_7 \cup \{\infty\}, \mathcal{B})$ whose blocks are:

1. $[(j + 6), \infty, j + 5, (j + 4), (j + 1), j, j + 2, (j + 3)]$ for $j \in \mathbb{Z}_7$,
2. $[(\infty), j + 3, j + 6, (j + 5), (j + 2), j, j + 1, (j + 4)]$ for $j \in \mathbb{Z}_7$. 


Then $\Sigma$ is an $OQS(8)$ of index 5.

Let $v = 4k$, for some $k \geq 3$. Let us consider $\Sigma = (Z_{4k-1} \cup \{\infty\}, \mathcal{B})$ whose blocks are:

$$
\begin{align*}
(1) \quad & [(2i-1), 0, 2i, (4i+1), (2i+1), 4i+3, 6i+2, (4i)] \text{ for } i = 1, \ldots, k-2 \\
(2) \quad & [(\infty), j, j+2k-1, (j+1), (j+2k-2), j+4k-3, j+2k, (j+4k-2)], \\
& \quad \text{for } j \in Z_{4k-1}, \\
(3) \quad & [(j+2), j, j+1, (j+3), (j+2k+2), \infty, j+5, (j+2k+4)] \text{ for } j \in Z_{4k-1}.
\end{align*}
$$

Then $\Sigma$ is an $OQS(v)$ of index 5. \hfill \Box

5. INDEX $\lambda = 10$

**Theorem 5.1.** For $\lambda = 10$ and for every $v \in \mathbb{N}$, $v \geq 8$, there exists an $OQS(v)$ of index 10.

**Proof.** Let $v \equiv 0, 1 \mod 4$. Then, in this case, the proof follows by Theorem 4.1, because, given $\Sigma = (X, \mathcal{B})$ an $OQS(v)$ of index 5, $\Sigma' = (X, \mathcal{B}')$, whose blocks are those of $\mathcal{B}$, each repeated twice, is an $OQS(v)$ of index 10.

Let $v = 10$. Let $\Sigma = (X, \mathcal{B})$ an $OQS(10)$ of index 2, as given in Theorem 3.1. Then $\Sigma' = (X, \mathcal{B}')$, whose blocks are those of $\mathcal{B}$, each repeated 5 times, is an $OQS(10)$ of index 10.

Let $v = 14$. Let us consider $\Sigma = (Z_{13} \cup \{\infty\}, \mathcal{B})$, with $\infty \notin Z_{13}$, whose blocks are:

$$
\begin{align*}
(1) \quad & [(1), 0, 5, (6), (7), 8, 3, (2)] \text{ and all its translated forms,} \\
(2) \quad & [(5), 0, 1, (6), (11), 3, 2, (10)] \text{ and all its translated forms,} \\
(3) \quad & [(j+11), \infty, j+1, (j+7), (j+3), j, j+2, (j+5)](5) \text{ for } j \in Z_{13}.
\end{align*}
$$

Then $\Sigma$ is an $OQS(14)$ of index 10.

Let $v = 18$. Let us consider $\Sigma = (Z_{17} \cup \{\infty\}, \mathcal{B})$, with $\infty \notin Z_{17}$, whose blocks are:

$$
\begin{align*}
(1) \quad & [(1), 0, 4, (5), (6), 7, 3, (2)] \text{ and all its translated forms,} \\
(2) \quad & [(4), 0, 1, (5), (9), 13, 12, (8)] \text{ and all its translated forms,} \\
(3) \quad & [(2), 0, 3, (5), (7), 9, 6, (4)] \text{ and all its translated forms,} \\
(4) \quad & [(3), 0, 2, (5), (8), 11, 9, (6)] \text{ and all its translated forms,} \\
(5) \quad & [(j+10), \infty, j+9, (j+3), (j+8), j, j+7, (j+2)](5) \text{ for } j \in Z_{17}.
\end{align*}
$$

Then $\Sigma$ is an $OQS(18)$ of index 10.

Let $v = 4k + 2$, for some $k \geq 5$. Let us consider $\Sigma = (Z_{4k+1} \cup \{\infty\}, \mathcal{B})$, with $\infty \notin Z_{4k+1}$, whose blocks are:

$$
\begin{align*}
(1) \quad & [(2i-1), 0, 2i, (4i+1), (2i+1), 4i+3, 6i+2, (4i)](2) \text{ for } i = 1, \ldots, k-3 \\
& \quad \text{and all their translated forms,} \\
(2) \quad & [(2k-5), 4k-10, 2k-6, (4k), (1), 3, 2, (0)](2) \text{ and all its translated forms,} \\
(3) \quad & [(j+2k+2), \infty, j+2k+1, (j+3), (j+2k), j, j+2k-1, (j+2)](5) \\
& \quad \text{for } j \in Z_{4k+1}.
\end{align*}
$$

Then $\Sigma$ is an $OQS(v)$ of index 10.
Let \( v = 11 \). Let us consider \( \Sigma = (\mathbb{Z}_{11}, \mathcal{B}) \) having \([(0), 1, 8, (2), (4), 10, 6, (3)]\) and all its translated forms as blocks, each repeated 5 times. Then \( \Sigma \) is an \( OQS(11) \) of index 10.

Let \( v = 15 \). Consider \( \Sigma = (\mathbb{Z}_{15}, \mathcal{B}) \) with blocks \([(0), 1, 6, (2), (7), 4, 5, (3)]\) and all its translates, each repeated 5 times, and \([(8), 0, 7, (1), (10), 4, 11, (2)]\) and all its translates, each repeated twice. Then \( \Sigma \) is an \( OQS(15) \) of index 10.

Let \( v = 4k + 3 \), for some \( k \geq 4 \). Let us consider \( \Sigma = (\mathbb{Z}_{4k+3}, \mathcal{B}) \) whose blocks are:

1. \([(2i-1), 0, 2i, (4i+1), (2i+1), 4i+3, 6i+2, (4i)](2)\) for \( i = 1, \ldots, k-1 \),
2. \([(2k+4), 0, 1, (2k+5), (6), 2k + 10, 2k + 9, (5)]\),
3. \([(2), 0, 2k + 1, (2k + 3), (2k + 5), 2k + 7, 6, (4)]\),
4. \([(2k), 0, 2k + 1, (4k + 2), (2k - 1), 4k - 1, 2k - 2, (4k + 1)]\)

and all their translated forms. Then \( \Sigma \) is an \( OQS(v) \) of index 10. \( \square \)

6. Any index \( \lambda \)

**Theorem 6.1.** For any \( \lambda \in \mathbb{N} \), with \( \lambda \geq 2 \), there exists an \( OQS(20) \) of index \( \lambda \).

**Proof.** Let us consider \( \Sigma = (\mathbb{Z}_{19} \cup \{\infty\}, \mathcal{B}) \), with \( \infty \notin \mathbb{Z}_{19} \), whose blocks are:

1. \([(i + 1), i + 3, i, (\infty), (i + 2), i + 13, i + 7, (i + 6)]\) for any \( i \in \mathbb{Z}_{19} \),
2. \([(2), 0, 1, (5), (14), 7, 15, (9)]\) and all its translated forms,
3. \([(2), 0, 1, (5), (13), 6, 16, (7)]\) and all its translated forms.

Then \( \Sigma \) is an \( OQS(20) \) of index 3.

By this construction and by Theorem 3.1 we know that the statement holds for \( \lambda = 2, 3 \). Taking any \( \lambda \in \mathbb{N} \), with \( \lambda \geq 2 \), we know that \( \lambda = 2a + 3b \), for some \( a, b \in \mathbb{N} \). Let us now consider two \( OQS(20) \), \( \Sigma_1 = (X, \mathcal{B}_1) \) and \( \Sigma_2 = (X, \mathcal{B}_2) \) on the same vertex set \( X \), of indices 2 and 3, respectively. Then \( \Sigma = (X, \mathcal{B}) \), whose blocks are those of \( \mathcal{B}_1 \), each repeated \( a \) times, and those of \( \mathcal{B}_2 \), each repeated \( b \) times, is an \( OQS(20) \) of index \( \lambda \). \( \square \)

As a consequence of all the previous results, the following statement follows easily:

**Theorem 6.2.** Let us consider \( \lambda, v \in \mathbb{N} \), with \( v \geq 8 \), such that:

1. if \( \lambda = 1 \), then \( v \equiv 0, 1, 5, 16 \) mod 20, with \( v \neq 20 \),
2. if \( \lambda \equiv 1, 3, 7, 9 \) mod 10, \( \lambda \neq 1 \), then \( v \equiv 0, 1, 5, 16 \) mod 20,
3. if \( \lambda \equiv 2, 4, 6, 8 \) mod 10, then \( v \equiv 0, 1 \) mod 5,
4. if \( \lambda \equiv 5 \) mod 10, then \( v \equiv 0, 1 \) mod 4.

Then there exists an \( OQS(v) \) of order \( \lambda \).

**Proof.** The statement has been proved in the case that \( \lambda = 1, 2, 5, 10 \).

Let \( \lambda \equiv 1, 3, 7, 9 \) mod 20, with \( \lambda \neq 1 \). If \( v = 20 \), the proof follows by Theorem 6.1. Let \( v \neq 20 \). Given \( \Sigma = (X, \mathcal{B}) \) an \( OQS(v) \) of index 1, \( \Sigma' = (X, \mathcal{B}') \), where the blocks of \( \mathcal{B}' \) are those of \( \mathcal{B} \), each repeated \( \lambda \) times, is an \( OQS(v) \) of index \( \lambda \).
Let $\lambda \equiv 2,4,6,8 \mod 10$. Given $\Sigma = (X, B)$ an $OQS(v)$ of index 2, $\Sigma' = (X, B')$, where the blocks of $B'$ are those of $B$, each repeated $\lambda/2$ times, is an $OQS(v)$ of index $\lambda$.

Let $\lambda \equiv 5 \mod 10$. Given $\Sigma = (X, B)$ an $OQS(v)$ of index 5, $\Sigma' = (X, B')$, where the blocks of $B'$ are those of $B$, each repeated $\lambda/5$ times, is an $OQS(v)$ of index $\lambda$.

Let $\lambda \equiv 0 \mod 10$. Given $\Sigma = (X, B)$ an $OQS(v)$ of index 10, $\Sigma' = (X, B')$, where the blocks of $B'$ are those of $B$, each repeated $\lambda/10$ times, is an $OQS(v)$ of index $\lambda$. □

REFERENCES


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