Abstract. A graph $G$ is called edge-magic if there is a bijective function $f : V(G) \cup E(G) \to \{1, 2, \ldots, |V(G)| + |E(G)|\}$ such that for every edge $xy \in E(G)$, $f(x) + f(xy) + f(y) = c$ is a constant, called the valence of $f$. A graph $G$ is said to be super edge-magic if $f(V(G)) = \{1, 2, \ldots, |V(G)|\}$. Let $G$ be a graph with $p$ vertices with $V(G) = \{v_1, v_2, \ldots, v_p\}$ and let $S_m$ be the star with $m$ leaves. If in $G$, every vertex $v_i$ is identified to the center vertex of $S_{m_i}$, for some $m_i \geq 0$, $1 \leq i \leq n$, where $S_0 = K_1$, then the graph obtained is denoted by $G(m_1, m_2, \ldots, m_p)$. Let $M(G) = \{(m_1, m_2, \ldots, m_p) | G(m_1, m_2, \ldots, m_p) \text{ is a super edge-magic graph}\}$. The star super edge-magic deficiency $S\mu^*(G)$ is defined as

$$S\mu^*(G) = \begin{cases} \min_{(m_1, m_2, \ldots, m_p)} (m_1 + m_2 + \cdots + m_p), & \text{if } M(G) \neq \emptyset, \\ +\infty, & \text{if } M(G) = \emptyset. \end{cases}$$

In this paper we determine the star super edge-magic deficiency of certain classes of graphs.

1. Introduction

In 1970, Kotzig and Rosa [12] introduced the concept of edge-magic labeling using a different name: magic valuations. Meanwhile, the super edge-magic labeling was introduced by Enomoto et al. [6]. In [12], Kotzig and Rosa proved that for every graph $G$ there exists an edge-magic graph $H$ such that $H \cong G \cup nK_1$ for some non-negative integer $n$. This fact motivates the emergence of the concept of the edge-magic deficiency of a graph.

The edge-magic deficiency $\mu(G)$ of a graph $G$ is the minimum non-negative integer $n$ such that $G \cup nK_1$ has an edge-magic labeling. Motivated by Kotzig and Rosa’s concept of edge-magic deficiency, Figueroa-Centeno et al. [8] defined a similar concept for the super edge-magic labeling.

The super edge-magic deficiency $\mu_s(G)$ of a graph $G$ is the minimum non-negative integer $n$ such that $G \cup nK_1$ has a super edge-magic labeling or $+\infty$ if there exists no such $n$. Figueroa-Centeno et al. [8] provided the exact values for the super edge-magic deficiencies of several classes of graphs, such as, cycles, some classes of forests and complete bipartite graphs $K_{m,n}$. Ahmad et al. [3] provided the exact values for super edge-magic deficiencies of graphs,
(n, 1)-kite graphs, (n, 3)-kite graphs, \( K_2 \cup C_n \) when \( n \equiv 1 \pmod{4} \). They also provided the upper bound of the super edge-magic deficiency of \( K_2 \cup C_n \) when \( n \equiv 3 \pmod{4} \). Nadeem et al. [13] provided the upper bound for the super edge-magic deficiencies of kite graphs. Ahmad et al. [3] provided the upper bound for the super edge-magic deficiencies of ladder graphs. Acharya and Hegde introduced the concept of strongly indexable graph that is equivalent to the concept of super edge-magic graph [1]. For further details, see [10].

We observe some drawbacks of the super edge-magic deficiency of a graph.

- For several graphs, \( \mu_s(G) = \infty \).
- To find \( \mu_s(G) \), we construct a disconnected graph with large number of components (consisting of isolated vertices) having a super edge-magic labeling.
- The distribution of non-utilized numbers to the isolated vertices is very trivial.

Motivated by the concept of super edge-magic deficiency, we introduce a new deficiency for a graph without some of the above drawbacks, namely the star super edge-magic deficiency, \( S\mu^*(G) \). We prove that \( S\mu^*(G) \) is finite for several classes of graphs for which \( \mu_s(G) = \infty \).

In this paper, we provide the exact values for the star super edge-magic deficiencies of several classes of graphs such as, cycles, \( nK_2 \) forests, \( nP_2 \) graphs, \((n,3)\)-kite graphs, and \((n,2)\)-kite graphs. We give an upper bound for the star super edge-magic deficiencies of kite graphs, ladder graphs, Mongolian tent graphs \( M_{t_n} \) when \( n \) is odd and triangular chain graphs \( TC_n \) when \( n \) is odd.

Figueroa-Centeno et al. [7] showed the following connection between the super edge-magic labeling and a special vertex labeling. This result characterizes super edge-magic graphs.

**Lemma 1.1** ([7]). A \((p,q)\) graph \( G \) is super edge-magic if and only if there exists a bijective function \( f: V(G) \rightarrow \{1,2,\ldots,p\} \) such that the set \( S = \{f(u) + f(v) : uv \in E(G)\} \) consists of \( q \) consecutive integers. In such a case, \( f \) extends to a super edge-magic labeling of \( G \) with valence \( k = p + q + s \), where \( s = \min S \) and \( S = \{k - (p + 1), k - (p + 2), \ldots, k - (p + q)\} \).

**2. Main Results**

**Definition 2.1.** Let \( G \) be a graph with \( p \) vertices with vertex set \( V(G) = \{v_1, v_2, \ldots, v_p\} \). In \( G \), every vertex \( v_i \) is identified to the center vertex of \( S_{m_i} \), for some \( m_i \geq 0, 1 \leq i \leq n \), where \( S_0 = K_1 \); this graph is denoted by \( G_{(m_1,m_2,\ldots,m_p)} \). Let \( M(G) = \{(m_1,m_2,\ldots,m_p)|G_{(m_1,m_2,\ldots,m_p)} \text{ is a super edge-magic graph}\} \). The star super edge-magic deficiency \( S\mu^*(G) \) is defined as
\[ S^\mu(G) = \begin{cases} \min_{m_1, m_2, \ldots, m_p} (m_1 + m_2 + \cdots + m_p), & \text{if } M(G) \neq \emptyset, \\ +\infty, & \text{if } M(G) = \emptyset. \end{cases} \]

**Remark.** If \( G \) is super edge-magic, then \( S^\mu(G) = 0 \).

In the next theorem, we show the exact value for the star super edge-magic deficiency for the forest \( nK_2 \).

**Theorem 2.2.** The star super edge-magic deficiency of the forest \( nK_2 \) is given by

\[ S^\mu(nK_2) = \begin{cases} 0, & \text{if } n \text{ is odd}, \\ 1, & \text{if } n \text{ is even}. \end{cases} \]

**Proof.** The vertex set and edge set of the forest \( nK_2 \) are \( V(nK_2) = \{x_i : 1 \leq i \leq n\} \cup \{y_i : 1 \leq i \leq n\} \) and \( E(nK_2) = \{x_iy_i : 1 \leq i \leq n\} \), respectively. Kotzig and Rosa [12] showed that the forest \( nK_2 \) is super edge-magic if and only if \( n \) is odd. Therefore, \( S^\mu(nK_2) = 0 \) when \( n \) is odd and \( S^\mu(nK_2) \geq 1 \) when \( n \) is even. When \( n \) is even, we define the graph \( G = (nK_2)(m_{x_1}, m_{x_2}, \ldots, m_{x_n}, m_{y_1}, m_{y_2}, \ldots, m_{y_n}) \), where

\[ m_i = \begin{cases} 1, & \text{if } i = \frac{n}{2} + 1, \\ 0, & \text{otherwise}. \end{cases} \]

The vertex set and edge set of \( G \) are \( V(G) = \{x_i : 1 \leq i \leq n\} \cup \{y_i : 1 \leq i \leq n\} \cup \{s\} \) and \( E(G) = \{x_iy_i : 1 \leq i \leq n\} \cup \{y_{(n/2)+1}s\} \), respectively. Consider the vertex labeling \( f : V(G) \to \{1, 2, \ldots, 2n + 1\} \) such that

- \( f(x_i) = i, \quad 1 \leq i \leq n \)
- \( f(y_i) = \begin{cases} \frac{3n}{2} + i + 1, & \text{if } 1 \leq i \leq \frac{n}{2}, \\ \frac{n}{2} + i, & \text{if } \frac{n}{2} + 1 \leq i \leq n, \end{cases} \)
- \( f(s) = \frac{3n+2}{2} \).

The set of all edge sums generated by the above formula forms a consecutive integer sequence \( (\frac{3n}{2}+4)/2, (3n+6)/2, \ldots, (5n+4)/2 \). Therefore, by Lemma 1.1, \( f \) can be extended to a super edge-magic labeling with valence \((9n/2)+4\) and consequently, \( S^\mu(nK_2) \leq 1 \). Therefore, we conclude that \( S^\mu(nK_2) = 1 \), when \( n \) is even. \( \square \)

In the next theorem, we show the exact value for the star super edge-magic deficiency of the forest \( nP_2 \) where \( P_2 \) is a path of length 2.

**Theorem 2.3.** The star super edge-magic deficiency of the forest \( nP_2 \) is given by

\[ S^\mu(nP_2) = \begin{cases} 0, & \text{if } n \text{ is odd}, \\ 1, & \text{if } n \text{ is even}. \end{cases} \]
The vertex set and edge set of \( nP_2 \) are \( V(nP_2) = \{x_i: 1 \leq i \leq n\} \cup \{y_i: 1 \leq i \leq n\} \cup \{z_i: 1 \leq i \leq n\} \cup \{y_i z_i: 1 \leq i \leq n\} \), respectively. Chen [5] proved that the graph \( nP_2 \) is super edge-magic if and only if \( n \) is odd. Therefore, \( S\mu^*(nP_2) = 0 \) when \( n \) is odd and \( S\mu^*(nP_2) \geq 1 \) when \( n \) is even. When \( n \) is even, we define the graph \( G = (nP_2)(m_{x_1}, m_{x_2}, \ldots, m_{x_n}, m_{y_1}, m_{y_2}, \ldots, m_{y_n}, m_{z_1}, m_{z_2}, \ldots, m_{z_n}) \), where

\[
m_i = \begin{cases} 
1, & \text{if } i = y_{\frac{n}{2}+1}, \\
0, & \text{otherwise}.
\end{cases}
\]

The vertex set and edge set of \( G \) are \( V(G) = \{x_i: 1 \leq i \leq n\} \cup \{y_i: 1 \leq i \leq n\} \cup \{z_i: 1 \leq i \leq n\} \cup \{s\} \), and \( E(G) = \{x_i y_i: 1 \leq i \leq n\} \cup \{y_i z_i: 1 \leq i \leq n\} \cup \{y_{\frac{n}{2}+1} s\} \).

Consider the vertex labeling \( f : V(G) \to \{1, 2, \ldots, 2n + 1\} \) such that

\[
\begin{align*}
f(x_i) &= i, 1 \leq i \leq n, \\
f(y_i) &= \frac{5n}{2} + 1 + i, 1 \leq i \leq \frac{n}{2}, \\
f(y_i) &= \frac{3n}{2} + i, \frac{n}{2} + 1 \leq i \leq n, \\
f(z_i) &= n + i, 1 \leq i \leq n, \\
f(s) &= \frac{5n+2}{2}.
\end{align*}
\]

The set of all edge sums generated by the above formula forms a consecutive integer sequence \( (5n+4)/2, (5n+6)/2, \ldots, (9n+4)/2 \). Therefore, by Lemma 1.1, \( f \) can be extended to a super edge-magic labeling with valence \( (15n + 8)/2 \). This shows that \( S\mu^*(nP_2) \leq 1 \). Therefore, \( S\mu^*(nP_2) = 1 \).

In the next theorem, we find the exact value for the star super edge-magic deficiency of the disconnected graph, \( K_2 \cup C_n \).

**Theorem 2.4.** The star super edge-magic deficiency of the disconnected graph \( K_2 \cup C_n \) is given by

\[
S\mu^*(K_2 \cup C_n) = \begin{cases} 
0, & \text{if } n \text{ is even,} \\
1, & \text{if } n \text{ is odd.}
\end{cases}
\]

**Proof.** Let \( G = K_2 \cup C_n \). The vertex set and edge set of \( G \) are \( V(G) = \{v_i: 1 \leq i \leq n\} \cup \{u, w\} \) and \( E(G) = \{v_i v_{i+1}: 1 \leq i \leq n - 1\} \cup \{v_n v_1, uw\} \). Kim and Park [11] proved that \( K_2 \cup C_n \) is super edge-magic if and only if \( n \) is even. Hence \( S\mu^*(G) = 0 \) for \( n \) even and \( S\mu^*(G) \geq 1 \) for \( n \) odd. When \( n \) is odd, define \( G^* = (G)(m_{v_1}, m_{v_2}, \ldots, m_{v_n}, m_{u}, m_{w}) \), where

\[
m_i = \begin{cases} 
1, & \text{if } i = v_{n-2}, \\
0, & \text{otherwise}.
\end{cases}
\]

The vertex set and edge set of \( G^* \) are \( V(G^*) = \{v_i: 1 \leq i \leq n\} \cup \{u, w\} \cup \{s\} \) and \( E(G^*) = \{v_i v_{i+1}: 1 \leq i \leq n - 1\} \cup \{v_n v_1, uw\} \cup \{v_{n-2} s\} \). We label the vertices of \( G^* \) in the following manner,

\[
\begin{align*}
f(u) &= \frac{n+2}{2}, \\
f(w) &= n + 3.
\end{align*}
\]
\[ f(x_i) = \begin{cases} 
\frac{i+1}{2}, & \text{if } i \text{ is odd and } i \leq n - 2, \\
\frac{n+3}{2}, & \text{if } i = n, \\
\frac{n+3+i}{2}, & \text{if } i \text{ is even.}
\end{cases} \]

The set of all edge sums generated by the above formula forms a set of \( n+1 \) consecutive integers \((n+5)/2, (n+7)/2, \ldots, (3n+7)/2\). Therefore, by Lemma 1.1, \( f \) can be extended to a super edge-magic labeling with valence \((5n+15)/2\). This shows that \( S_{\mu^*}(G) \leq 1 \). Therefore, \( S_{\mu^*}(G) = 1 \). \( \square \)

In the next theorem, we determine the exact value for the star super edge-magic deficiency of \( C_n \).

**Theorem 2.5.** The star super edge-magic deficiency of the cycle \( C_n \) is given by

\[ S_{\mu^*}(C_n) = \begin{cases} 
0, & \text{if } n \text{ is odd,} \\
2, & \text{if } n \text{ is even.}
\end{cases} \]

**Proof.** The vertex set and edge set of \( C_n \) are \( V(C_n) = \{x_i: 1 \leq i \leq n\} \) and \( E(C_n) = \{x_ix_{i+1}: 1 \leq i \leq n-1\} \cup \{x_nx_1\} \), respectively. Enomoto et al. [6] proved that \( C_n \) is super edge-magic if and only if \( n \) is odd. Hence \( S_{\mu^*}(C_n) = 0 \) if \( n \) is odd and \( S_{\mu^*}(C_n) \geq 1 \) if \( n \) is even. Thus, assume that \( n \) is even.

**Case 1:** \( n \equiv 0 \pmod{4} \).

Now we define the graph \( G \equiv (C_n)(m_1, m_2) \), where

\[ m_i = \begin{cases} 
1, & \text{if } i = x_1, \\
1, & \text{if } i = x_{\frac{n}{2}+2}, \\
0, & \text{otherwise.}
\end{cases} \]

The vertex set and edge set of \( G \) are \( V(G) = \{x_i: 1 \leq i \leq n\} \cup \{s_1, s_2\} \) and \( E(G) = \{x_ix_{i+1}: 1 \leq i \leq n-1\} \cup \{x_nx_1, x_1s_1\} \cup \{x_{(n/2)+2}s_2\} \), respectively. We label the vertices of \( G \) in the following manner,

\[ f(x_i) = \begin{cases} 
\frac{i-1}{2}, & \text{if } i = 1, \\
\frac{i-1}{2}, & \text{if } i \text{ is odd and } i > 1, \\
\frac{n+i}{2}, & \text{if } i \text{ is even and } 1 \leq i \leq \frac{n}{2}, \\
\frac{n+i+2}{2}, & \text{if } i \text{ is even and } \frac{n}{2} + 1 \leq i \leq n,
\end{cases} \]

\[ f(s_1) = n + 2, \]

\[ f(s_2) = 3\left(\frac{n}{2} - 1\right) + 4. \]

The set of all edge sums generated by the above formula forms a consecutive integer sequence \((n+4)/2, (n+6)/2, \ldots, (3n+6)/2\). Therefore, by Lemma 1.1, \( f \) can be extended to a super edge-magic labeling with valence \((5n+12)/2\) and consequently, \( S_{\mu^*}(C_n) \leq 2 \) when \( n \equiv 0 \pmod{4} \).

**Case 2:** \( n \equiv 2 \pmod{4} \).

See Figure 1 for the labeling of \((C_n)(1, 1, 0, 1, 0, 0, 0)\).
Now consider \( n > 6 \). Define the graph \( G \equiv (C_n)((m_{x_1}, m_{x_2}, \ldots, m_{x_n})) \), where

\[
m_i = \begin{cases} 
1, & \text{if } i = \frac{x_{n-2} + x_{n-2}}{2} \\
0, & \text{otherwise}.
\end{cases}
\]

The vertex set and edge set of \( G \) are \( V(G) = \{x_i: 1 \leq i \leq n\} \cup \{s_1, s_2\} \) and \( E(G) = \{x_ix_{i+1}: 1 \leq i \leq n\} \cup \{x_{n}x_{1}, x_{(n-8)/2}s_1\} \cup \{x_{(n-2)/2}s_2\} \), respectively. Consider the following labeling \( f \) of \( G \).

- \( f(x_i) = \begin{cases} 
\frac{i+1}{2}, & \text{if } i \text{ is odd and } 1 \leq i \leq n, \\
\frac{n+2+i}{2}, & \text{if } i \text{ is even and } 1 \leq i \leq \frac{n-4}{2}, \\
\frac{n+6+i}{2}, & \text{if } i \text{ is even and } \frac{n-4}{2} + 1 \leq i \leq n-1, \\
\frac{n+2}{2}, & \text{if } i = n.
\end{cases} \)
- \( f(s_1) = 3\left(\frac{n-2}{4} - 1\right) + 6 \)
- \( f(s_2) = 3\left(\frac{n-2}{4} - 1\right) + 5 \)

The set of all edge sums generated by the above formula forms a consecutive integer sequence \((n+4)/2, (n+6)/2, \ldots, (3n+6)/2\). Therefore, by Lemma 1.1, \( f \) can be extended to a super edge-magic labeling with valence \((5n+12)/2\) and consequently, \( S\mu^*(C_n) \leq 2 \) when \( n \equiv 2 \) (mod 4).

In both the cases \( S\mu^*(C_n) \leq 2 \), Kim and Park [11], proved that \((n, 1)\)-kite is super edge-magic if and only if \( n \) is odd. That is \((C_n)_{(1,0,0,0,0)}\) is not super edge-magic if \( n \) is even. Therefore, \( S\mu^*(C_n) = 2 \), for \( n \) even.

\[\Box\]

In the next theorem, we prove an upper bound for the star super edge-magic deficiency of Fan graphs.

**Theorem 2.6.** The star super edge-magic deficiency of the fan graph \( F_n \) is given by

\[ S\mu^*(F_n) \leq \begin{cases} 
\frac{n-1}{2}, & \text{if } n \text{ is odd,} \\
\frac{n-2}{2}, & \text{if } n \text{ is even.}
\end{cases} \]
Proof. The vertex set and edge set of \( F_n \) are \( V(F_n) = \{ x_i : 1 \leq i \leq n \} \cup \{ c \} \), and \( E(F_n) = \{ x_i x_{i+1} : 1 \leq i \leq n-1 \} \cup \{ cx_i : 1 \leq i \leq n \} \), respectively.

Case 1: \( n \) is odd.

We define the graph \( G = (F_n)_{(m_1, m_2, \ldots, m_n, m_c)} \), where

\[
m_i = \begin{cases} \frac{n-1}{2}, & \text{if } i = c, \\ 0, & \text{otherwise.} \end{cases}
\]

The vertex set and edge set of \( G \) are \( V(G) = \{ x_i : 1 \leq i \leq n \} \cup \{ c \} \cup \{ s_i : 1 \leq i \leq (n-1)/2 \} \), \( E(G) = \{ x_i x_{i+1} : 1 \leq i \leq n-1 \} \cup \{ cx_i : 1 \leq i \leq n \} \cup \{ cs_i : 1 \leq i \leq \frac{n-1}{2} \} \), respectively. We label the vertices of \( G \) in the following manner,

\[
\begin{align*}
f(x_i) &= \begin{cases} i+1, & \text{if } i \text{ is odd,} \\ \frac{2i+1}{2}, & \text{if } i \text{ is even,} \end{cases} \\
f(c) &= \frac{3n+1}{2}, \\
f(s_i) &= n + i, 1 \leq i \leq \frac{n-1}{2}.
\end{align*}
\]

The set of all edge sums generated by the above formula forms a set of consecutive integers \( \{ (n+5)/2, (n+7)/2, \ldots, (6n)/2 \} \). Therefore, by Lemma 1.1, \( f \) can be extended to a super edge-magic labeling with valence \( (9n+3)/2 \). This shows that \( S\mu^*(F_n) \leq (n-1)/2 \).

Case 2: \( n \) is even.

We define the graph \( G \cong (F_n)_{(m_1, m_2, \ldots, m_n, m_c)} \), where

\[
m_i = \begin{cases} \frac{n-2}{2}, & \text{if } i = c, \\ 0, & \text{otherwise.} \end{cases}
\]

The vertex set and edge set of \( G \) are \( V(G) = \{ x_i : 1 \leq i \leq n \} \cup \{ c \} \cup \{ s_i : 1 \leq i \leq \frac{n-2}{2} \} \), \( E(G) = \{ x_i x_{i+1} : 1 \leq i \leq n-1 \} \cup \{ cx_i : 1 \leq i \leq n \} \cup \{ cs_i : 1 \leq i \leq \frac{n-2}{2} \} \), respectively. We label the vertices of \( G \) in the following manner,

\[
\begin{align*}
f(x_i) &= \begin{cases} i+1, & \text{if } i \text{ is odd,} \\ \frac{2i+1}{2}, & \text{if } i \text{ is even,} \end{cases} \\
f(c) &= \frac{n}{2}, \\
f(s_i) &= n + i, 1 \leq i \leq \frac{n-2}{2}.
\end{align*}
\]

The set of all edge sums generated by the above formula forms a consecutive integer sequence \( \{ (n+4)/2, (n+6)/2, \ldots, (6n-2)/2 \} \). Therefore, by Lemma 1.1, \( f \) can be extended to a super edge-magic labeling with valence \( 9n/2 \). This shows that \( S\mu^*(F_n) \leq (n-2)/2 \).

\( \square \)

Open Problem. Verify whether equality holds in the above inequality.

The following theorem gives an upper bound for the star super edge-magic deficiency of Wheel graph.
**Theorem 2.7.** For \( n \) odd, the star super edge-magic deficiency of the Wheel graph \( W_n \) is given by \( S\mu^*(W_n) \leq (n - 1)/2 \).

**Proof.** The vertex set and edge set of \( W_n \) are \( V(W_n) = \{x_i: 1 \leq i \leq n\} \cup \{c\}, E(W_n) = \{x_ix_{i+1}: 1 \leq i \leq n - 1\} \cup \{cx_i: 1 \leq i \leq n\} \cup \{x_nx_1\} \), respectively. Define the graph \( G \cong (W_n)(m_{x_1}, m_{x_2}, ..., m_{x_n}, m_c) \), where

\[
m_i = \begin{cases} 
\frac{n-1}{2}, & \text{if } i = c, \\
0, & \text{otherwise}.
\end{cases}
\]

The vertex set and edge set of \( G \) are \( V(G) = \{x_i: 1 \leq i \leq n\} \cup \{c\} \cup \{s_i: 1 \leq i \leq (n - 1)/2\}, E(G) = \{x_ix_{i+1}: 1 \leq i \leq n - 1\} \cup \{cx_i: 1 \leq i \leq n\} \cup \{cs_i: 1 \leq i \leq (n - 1)/2\} \cup \{x_nx_1\} \) respectively. We label the vertices of \( G \) in the following manner,

\[
\begin{align*}
&f(x_i) = \begin{cases} 
\frac{i+1}{2}, & \text{if } i \text{ is odd,} \\
\frac{n+i+1}{2}, & \text{if } i \text{ is even,}
\end{cases} \\
&f(c) = \frac{3n+1}{2}, \\
&f(s_i) = n + i.
\end{align*}
\]

The set of all edge sums generated by the above formula forms a consecutive integer sequence \( (n + 3)/2, (n + 5)/2, \ldots, 6n/2 \). Therefore, by Lemma 1.1, \( f \) can be extended to a super edge-magic labeling with valence \( (9n + 3)/2 \). This shows that \( S\mu^*(W_n) \leq (n - 1)/2 \) if \( n \) is odd. \( \square \)

In the next theorem, we show an upper bound for the super edge-magic deficiency of \( (n, t) \)-kite graph for odd \( n \) and for even \( t \).

**Theorem 2.8.** Let \( G \) be the \( (n, t) \)-kite graph. If \( n \) is odd and \( t \) is even, then \( S\mu^*(G) \leq t/2 \).

**Proof.** Case 1: \( G = (3, t) \)-kite, \( t \) is even.

The vertex set and edge set of \( G \) is \( V(G) = \{v_i: 1 \leq i \leq 3\} \cup \{u_i: 1 \leq i \leq t\} \) and \( E(G) = \{v_iv_{i+1}: 1 \leq i \leq 2\} \cup \{v_3v_1, v_1u_1\} \cup \{u_iu_{i+1}: 1 \leq i \leq t - 1\} \) respectively. Define \( G^* = (G)(m_{u_1}, m_{v_2}, m_{v_3}, m_{u_1}, m_{u_2}, \ldots, m_{u_t}) \), where

\[
m_i = \begin{cases} 
\frac{t}{2}, & \text{if } i = v_1, \\
0, & \text{otherwise}.
\end{cases}
\]

Let \( v_1s_1, v_1s_2, v_1s_{t/2} \) be the edges of the star attached at \( v_1 \). The labeling of \( G^* \) is

\[
\begin{align*}
&f(u_i) = \begin{cases} 
\frac{i+1}{2}, & \text{if } i \text{ is odd,} \\
\frac{t+i+2}{2}, & \text{if } i \text{ is even,} \\
\frac{t+i+1}{2}, & \text{if } i = 1,
\end{cases} \\
&f(v_i) = \begin{cases} 
t+2, & \text{if } i = 2, \\
t+3, & \text{if } i = 3,
\end{cases} \\
&f(s_i) = t + 3 + i, \quad 1 \leq i \leq \frac{t}{2}.
\end{align*}
\]
The set of all edge sums generated by the above formula forms a consecutive integer sequence \((t + 6)/2, (t + 8)/2, \ldots, (4t + 10)/2\). Therefore, by Lemma 1.1, \(f\) can be extended to a super edge-magic labeling with valence \((7t + 18)/2\). This shows that \(S\mu^*((3, t)-\text{kite}) \leq t/2\).

Case 2: \(G = (n, t)-\text{kite}, n > 3\) and \(t\) is even.

The vertex set and edge set of \((n, t)-\text{kite}\) are \(V(G) = \{v_i: 1 \leq i \leq n\} \cup \{u_i: 1 \leq i \leq t\}, E(G) = \{v_i v_{i+1}: 1 \leq i \leq n-1\} \cup \{v_n v_1, v_1 u_t\} \cup \{u_i u_{i+1}: 1 \leq i \leq t-1\}\), respectively. Now, we define

\[
G^* = ((n, t)-\text{kite})_{(m_1, m_2, \ldots, m_{n-1}, m_{n-1}}} \]

where

\[
m_i = \begin{cases} 
\frac{t}{2}, & \text{if } i = v_{n-2}, \\
0, & \text{otherwise}.
\end{cases}
\]

The vertex set and edge set of \(G^*\) are \(V(G^*) = \{v_i: 1 \leq i \leq n\} \cup \{u_i: 1 \leq i \leq t\} \cup \{s_i: 1 \leq i \leq \frac{t}{2}\}, E(G^*) = \{v_i v_{i+1}: 1 \leq i \leq n-1\} \cup \{v_n v_1, v_1 u_t\} \cup \{u_i u_{i+1}: 1 \leq i \leq t-1\}\), respectively. We label the vertices of \(G^*\) in the following manner,

- \(f(v_i) = \begin{cases} 
\frac{i+t+1}{2}, & \text{if } i \text{ is odd and } 1 \leq i \leq n-1, \\
\frac{n+2t+i+1}{2}, & \text{if } i \text{ is even}, \\
\frac{n+2t+1}{2}, & \text{if } i = n,
\end{cases}\)
- \(f(u_i) = \begin{cases} 
\frac{i+1}{2}, & \text{if } i \text{ is odd}, \\
\frac{n+t+i-1}{2}, & \text{if } i \text{ is even},
\end{cases}\)
- \(f(s_i) = n + t + i, \quad 1 \leq i \leq \frac{t}{2}\).

The set of all edge sums generated by the above formula forms a consecutive integer sequence \((n + t + 3)/2, (n + t + 5)/2, \ldots, (3n + 4t + 1)/2\). Therefore, by Lemma 1.1, \(f\) can be extended to a super edge-magic labeling with valence \((5n + 7t + 3)/2\). This shows that \(S\mu^*((n, t)-\text{kite}) \leq t/2\). \(\square\)

**Corollary 2.9.** The star super edge-magic deficiency of the \((n, 2)-\text{kite}\) is

\[
S\mu^*((n, 2)-\text{kite}) = \begin{cases} 
0, & \text{if } n \text{ is even,} \\
1, & \text{if } n \text{ is odd.}
\end{cases}
\]

**Proof.** The \((n, 2)-\text{kite}\) is not super edge-magic if and only if \(n\) is even \([14]\). Hence \(S\mu^*((n, 2)-\text{kite}) = 0\) if \(n\) is even and \(S\mu^*((n, 2)-\text{kite}) \geq 1\) if \(n\) is odd. Thus assume that \(n\) is even. By Theorem 2.9, \(S\mu^*((n, 2)-\text{kite}) \leq 1\). Therefore, \(S\mu^*((n, 2)-\text{kite}) = 1\). \(\square\)

In the next theorem we find the exact value for the star super edge-magic deficiency of the \((n, 3)-\text{kite}\).
Theorem 2.10. The star super edge-magic deficiency of the \((n, 3)\)-kite is given by

\[
S^{\mu}_\ast((n, 3)\text{-kite}) = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ 1, & \text{if } n \text{ is even.} \end{cases}
\]

Proof. The vertex set and edge set of the \((n, 3)\)-kite are

\[
V((n, 3)\text{-kite}) = \{v_i: 1 \leq i \leq n\} \cup \{u_i: 1 \leq i \leq 3\},
\]

\[
E((n, 3)\text{-kite}) = \{v_iv_{i+1}: 1 \leq i \leq n-1\} \cup \{v_n, v_1\} \cup \{u_iu_{i+1}: 1 \leq i \leq 2\},
\]

respectively. Kim and Park [11] proved that the \((n, 3)\)-kite is super edge-magic if and only if \(n\) is odd. Hence \(S^{\mu}_\ast((n, 3)\text{-kite}) = 0\) if \(n\) is odd and \(S^{\mu}_\ast((n, 3)\text{-kite}) \geq 1\) if \(n\) is even. Thus assume that \(n\) is even.

Case 1: \(n \equiv 0 \pmod{4}\).

We define \(G = ((n, 3)\text{-kite})(m_{v_1}, m_{v_2}, \ldots, m_{v_n}, m_{u_1}, m_{u_2}, m_{u_3})\) where

\[
m_i = \begin{cases} 1, & \text{if } i = v_{n/2}, \\ 0, & \text{otherwise.} \end{cases}
\]

The vertex set and edge set of \(G\) are

\[
V(G) = \{v_i: 1 \leq i \leq n\} \cup \{u_i: 1 \leq i \leq 3\} \cup \{s_1\},
\]

\[
E(G) = \{v_iv_{i+1}: 1 \leq i \leq n-1\} \cup \{v_n, v_1\} \cup \{u_iu_{i+1}: 1 \leq i \leq 2\},
\]

respectively. We label the vertices of \(G\) in the following manner,

- \(f(v_i) = \begin{cases} \frac{i}{2} + 3, & \text{if } 1 \leq i \leq n-1 \text{ and } i \text{ is even,} \\ \frac{i+7}{2}, & \text{if } 1 \leq i \leq \frac{n}{2} \text{ and } i \text{ is odd,} \\ \frac{n+9+i}{2}, & \text{if } \frac{n}{2} + 1 \leq i \leq n-1 \text{ and } i \text{ is odd,} \\ 3, & \text{if } i = n, \end{cases}\)

- \(f(u_i) = \begin{cases} \frac{i+1}{2}, & \text{if } i = 1, 3, \\ \frac{n}{2} + 3, & \text{if } i = 2, \end{cases}\)

- \(f(s_1) = \frac{3n+16}{4}\).

The set of all edge sums generated by the above formula forms a consecutive integer sequence \((n+8)/2, (n+10)/2, \ldots, (3n+14)/2\). Therefore, \(f\) can be extended to a super edge-magic labeling with valence \((5n+24)/2\). This shows that \(S^{\mu}_\ast((n, 3)\text{-kite}) = 1\).

Case 2: \(n \equiv 2 \pmod{4}\).

The labeling of \(((6, 3)\text{-kite})(0, 0, 0, 0, 0, 1)\) is given in Figure 2. Now consider \((n, 3)\)-kite, \(n > 6\). We define

\[G = ((n, 3)\text{-kite})(m_{v_1}, m_{v_2}, \ldots, m_{v_n}, m_{u_1}, m_{u_2}, m_{u_3})\]

where

\[
m_i = \begin{cases} 1, & \text{if } i = v_{n/2}, \\ 0, & \text{otherwise.} \end{cases}
\]

The vertex set and edge set of \(G\) are

\[
V(G) = \{v_i: 1 \leq i \leq n\} \cup \{u_i: 1 \leq i \leq 3\} \cup \{s_1\},
\]

\[
E(G) = \{v_iv_{i+1}: 1 \leq i \leq n-1\} \cup \{v_n, v_1\} \cup \{u_iu_{i+1}: 1 \leq i \leq 2\},
\]

respectively.
Figure 2. Labeling of \(((6,3)-\text{kite})_{(0,0,0,0,0,1)}\)

We label the vertices of $G$ in the following manner,

\[ f(v_i) = \begin{cases} 
\frac{n+i+9}{2}, & \text{if } 1 \leq i \leq \frac{n}{2} - 1 \text{ and } i \text{ is odd}, \\
\frac{n+i+11}{2}, & \text{if } \frac{n}{2} \leq i \leq \frac{n}{2} + 1 \text{ and } i \text{ is odd}, \\
\frac{i+5}{2}, & \text{if } \frac{n}{2} + 2 \leq i \leq n - 1 \text{ and } i \text{ is odd}, \\
\frac{i+4}{2}, & \text{if } 1 \leq i \leq \frac{n}{2} + 1 \text{ and } i \text{ is even}, \\
\frac{n+10+i}{2}, & \text{if } \frac{n}{2} + 2 \leq i \leq n - 1 \text{ and } i \text{ is even}, \\
\frac{n+8}{2}, & \text{if } i = n,
\end{cases} \]

\[ f(u_i) = \begin{cases} 
\frac{i+1}{2}, & \text{if } i = 1, 3, \\
\frac{n+3}{2}, & \text{if } i = 2,
\end{cases} \]

\[ f(s_1) = \frac{3n+18}{4}. \]

The set of all edge sums generated by the above formula forms a consecutive integers \((n+8)/2, (n+10)/2, \ldots, (3n+14)/2\). Therefore, by Lemma 1.1, $f$ can be extended to a super edge-magic labeling with valence $(5n+24)/2$. This shows that $S\mu^*((n,3)-\text{kite}) = 1$.

\[\square\]

In the next theorem we show that $S\mu^*(L_n) \leq 1$.

**Theorem 2.11.** For $n$ even, the star super edge-magic deficiency of the ladder graph $L_n$ is $S\mu^*(L_n) \leq 1$.

**Proof.** The vertex set and edge set of $L_n$ are $V(L_n) = \{v_i, u_i : 1 \leq i \leq n\}$ and $E(L_n) = \{u_iu_{i+1}, v_iv_{i+1} : 1 \leq i \leq n - 1\} \cup \{v_iu_i : 1 \leq i \leq n\}$, respectively. Now we define $G = (L_n)(m_{v_1},m_{v_2},\ldots,m_{v_n},m_{u_1},m_{u_2},\ldots,m_{u_n})$ where

\[ m_i = \begin{cases} 
1, & \text{if } i = u_2, \\
0, & \text{otherwise}.
\end{cases} \]
The vertex set and edge set of $G$ are $V(G) = \{v_i, u_i: 1 \le i \le n\} \cup \{s_1\}$ and $E(G) = \{u_iu_{i+1}, v_iu_{i+1}: 1 \le i \le n - 1\} \cup \{v_iu_i: 1 \le i \le n\} \cup \{u_2s_1\}$, respectively. We label the vertices of $G$ in the following manner,

- $f(v_i) = \begin{cases} 
  i, & \text{if } 1 \le i \le \frac{n}{2} \text{ and } i \text{ is odd}, \\
  \frac{4i-n}{2}, & \text{if } \frac{n}{2} + 1 \le i \le n \text{ and } i \text{ is odd}, \\
  \frac{4i+n+2}{2}, & \text{if } 1 \le i \le \frac{n}{2} \text{ and } i \text{ is even}, \\
  n + 1 + i, & \text{if } \frac{n}{2} + 1 \le i \le n \text{ and } i \text{ is even}, 
\end{cases}$

- $f(u_i) = \begin{cases} 
  i, & \text{if } 1 \le i \le \frac{n}{2} \text{ and } i \text{ is even}, \\
  \frac{4i-n}{2}, & \text{if } \frac{n}{2} + 1 \le i \le n \text{ and } i \text{ is even}, \\
  \frac{4i+n+2}{2}, & \text{if } 1 \le i \le \frac{n}{2} \text{ and } i \text{ is odd}, \\
  n + 1 + i, & \text{if } \frac{n}{2} + 1 \le i \le n \text{ and } i \text{ is odd}, 
\end{cases}$

- $f(s_1) = \frac{n+2}{2}$.

The set of all edge sums generated by the above formula forms a set of consecutive integers $\{(n+6)/2, (n+8)/2, \ldots, (7n+2)/2\}$. Therefore, by Lemma 1.1, $f$ can be extended to a super edge-magic labeling with valence $(11n+6)/2$. This shows that $S\mu^*(L_n) \le 1$, when $n$ is even. \qed

**Open Problem.** Prove that the same bound holds when $n$ is odd.

The following theorem gives an upper bound for the star super edge-magic deficiency of the Mongolian tent graph.

**Theorem 2.12.** The star super edge-magic deficiency of the Mongolian tent graph $Mt_n$, for $n$ odd, is bounded by $S\mu^*(Mt_n) \le (n-3)/2$.

**Proof.** The vertex set and edge set of $Mt_n$ are $V(Mt_n) = \{v_i, u_i: 1 \le i \le n\} \cup \{u\}$ and $E(Mt_n) = \{v_iv_{i+1}, u_iv_{i+1}: 1 \le i \le n - 1\} \cup \{u_2u_i, u_i: 1 \le i \le n\}$, respectively. Let $n$ be any odd non-negative integer. According to Lemma 1.1, it is sufficient to prove that there is a vertex labeling with the property that the edge sums under this labeling are consecutive integers. Define $G = (Mt_n)(m_u_1, m_u_2, \ldots, m_u_n, m_v_1, m_v_2, \ldots, m_v_n, m_u)$, where

$$m_i = \begin{cases} 
  \frac{n-3}{2}, & \text{if } i = u, \\
  0, & \text{otherwise}. 
\end{cases}$$

The vertex set and edge set of $G$ is $V(G) = \{v_i, u_i: 1 \le i \le n\} \cup \{u\} \cup \{s_i: 1 \le i \le (n-3)/2\}$, and $E(G) = \{u_1u_{i+1}, v_iv_{i+1}: 1 \le i \le n - 1\} \cup \{v_iu_i, u_i: 1 \le i \le n\} \cup \{u_1s_i: 1 \le i \le (n-3)/2\}$, respectively. We label the vertices of $G$ in the following manner,

- $f(v_i) = \begin{cases} 
  \frac{i+1}{2}, & \text{if } 1 \le i \le n \text{ and } i \text{ is odd}, \\
  \frac{n+i+1}{2}, & \text{if } 1 \le i \le n \text{ and } i \text{ is even}, 
\end{cases}$

- $f(u_i) = \begin{cases} 
  \frac{3n+i}{2}, & \text{if } 1 \le i \le n \text{ and } i \text{ is odd}, \\
  \frac{2n+i}{2}, & \text{if } 1 \le i \le n \text{ and } i \text{ is even}, 
\end{cases}$

- $f(u) = \frac{5n-1}{2}$.
The vertices $s_i$ under the labeling $f$ are labeled by $f(s_i) = 2n + i$, for $1 \leq i \leq (n - 3)/2$. The edge sums form a consecutive integer sequence $(n + 5)/2, (n + 7)/2, \ldots, (10n - 4)/2$. Therefore, by Lemma 1.1, $f$ can be extended to a super edge-magic labeling with valence $(15n - 3)/2$. This shows that $S\mu^*(Mt_n) \leq (n - 3)/2$. □

**Open Problem.** Prove that the above upper bound is $(n - 2)/2$ when $n$ is even.

The following theorem gives an upper bound for the star super edge-magic deficiency of the triangular chain graph.

**Theorem 2.13.** For $n$ odd, the star super edge-magic deficiency of the triangular chain graph, $TC_n$, is $S\mu^*(TC_n) \leq \lfloor n/2 \rfloor$.

**Proof.** The vertex set and edge set of $TC_n$ are $V(TC_n) = \{v_i : 1 \leq i \leq 2n\} \cup \{u_i : 1 \leq i \leq n\}$, $E(TC_n) = \{v_i v_{i+1} : 1 \leq i \leq 2n - 1\} \cup \{u_i v_{2i-1}, u_i v_{2i} : 1 \leq i \leq n\}$, respectively. We define $G = (TC_n)(m_{u_1}, m_{u_2}, \ldots, m_{u_n}, m_{v_1}, m_{v_2}, \ldots, m_{v_{2n}})$, where,

$$m_i = \begin{cases} \lfloor \frac{n}{2} \rfloor, & \text{if } i = u_n; \\ 0, & \text{otherwise.} \end{cases}$$

The vertex set and edge set of $G$ are $V(G) = \{v_i : 1 \leq i \leq 2n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq \lfloor n/2 \rfloor\}$ and $E(G) = \{v_i v_{i+1} : 1 \leq i \leq 2n - 1\} \cup \{u_i v_{2i-1}, u_i v_{2i} : 1 \leq i \leq n\} \cup \{u_n s_i : 1 \leq i \leq \lfloor n/2 \rfloor\}$ respectively. We label the vertices of $G$ in the following manner,

- $f(v_{2i-1}) = i$ and $f(v_{2i}) = n + i$, for $1 \leq i \leq n$,
- $f(u_i) = \begin{cases} 3n + i, & \text{if } 1 \leq i \leq \lfloor n/2 \rfloor, \\ 2n + i, & \text{if } \lfloor n/2 \rfloor + 1 \leq i \leq n. \end{cases}$

The vertices $s_i$ are labeled by $f(s_i) = 2n + i$, for $1 \leq i \leq \lfloor n/2 \rfloor$. The set of all edge sums generated by the above formula forms a consecutive integer sequence $(2n + 4)/2, (2n + 6)/2, \ldots, (11n - 1)/2$. Therefore, by Lemma 1.1, $f$ can be extended to a super edge-magic labeling with valence $8n + 1 + 2\lfloor n/2 \rfloor$. Hence $G$ admits a super edge-magic labeling. This shows that $S\mu^*(TC_n) \leq \lfloor n/2 \rfloor$. □

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