ON UNIFORMLY RESOLVABLE \( \{K_2, P_k\} \)-DESIGNS WITH \( k = 3, 4 \)

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Abstract. Given a collection of graphs \( \mathcal{H} \), a uniformly resolvable \( \mathcal{H} \)-design of order \( v \) is a decomposition of the edges of \( K_v \) into isomorphic copies of graphs from \( \mathcal{H} \) (also called blocks) in such a way that all blocks in a given parallel class are isomorphic to the same graph from \( \mathcal{H} \). We consider the case \( \mathcal{H} = \{K_2, P_k\} \) with \( k = 3, 4 \), and prove that the necessary conditions on the existence of such designs are also sufficient.

1. Introduction

Given a collection of graphs \( \mathcal{H} \), an \( \mathcal{H} \)-design of order \( v \) is a decomposition of the edges of \( K_v \) into isomorphic copies of graphs from \( \mathcal{H} \), the copies of \( H \in \mathcal{H} \) in the decomposition are called blocks. An \( \mathcal{H} \)-design is called resolvable if it is possible to partition the blocks into classes \( P_i \) such that every point of \( K_v \) appears exactly once in some block of each \( P_i \).

A resolvable \( \mathcal{H} \)-decomposition of \( K_v \) is sometimes also referred to as an \( \mathcal{H} \)-factorization of \( K_v \). The case where \( \mathcal{H} \) is a single edge \( (K_2) \) is known as a 1-factorization of \( K_v \) and it is well known to exist if and only if \( v \) is even. A single class of a 1-factorization, a pairing of all points, is also known as a 1-factor or a perfect matching. A resolvable \( \mathcal{H} \)-design is called uniform if every block of the class is isomorphic to the same graph from \( \mathcal{H} \). Of particular note is the result of Rees [10] which finds necessary and sufficient conditions for the existence of uniformly resolvable \( \{K_2, K_3\} \)-designs of order \( v \). Uniformly resolvable decompositions of \( K_v \) have also been studied in [2, 3, 4, 5, 6, 7, 8, 9, 12, 11, 14, 13].

In what follows, we will denote by \([a_1, \ldots, a_k]\), \( k \geq 2 \), the path \( P_k \) having vertex set \( \{a_1, \ldots, a_k\} \) and edge set \( \{(a_1, a_2), (a_2, a_3), \ldots, (a_{k-1}, a_k)\} \). If \( v \) is even and \( k \in \{3, 4\} \), let \((K_2, P_k)\)-URD\((v; r, s)\) denote a uniformly resolvable decomposition of \( K_v \) into \( r \) classes containing only copies of 1-factors and \( s \) classes containing only copies of paths \( P_k \). Let \( \text{URD}(v; K_2, P_k) \) denote the set of all pairs \((r, s)\) such that there exists a \((K_2, P_k)\)-URD\((v; r, s)\).

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Given \( v \equiv 0 \pmod{6} \), define \( J_1(v) \) according to the following table:

<table>
<thead>
<tr>
<th>( v )</th>
<th>( J_1(v) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 (mod 12)</td>
<td>{(v - 1 - 4x, 3x), x = 0, 1, \ldots, (v - 4)/4}</td>
</tr>
<tr>
<td>6 (mod 12)</td>
<td>{(v - 1 - 4x, 3x), x = 0, 1, \ldots, (v - 2)/4}</td>
</tr>
</tbody>
</table>

Table 1. The set \( J_1(v) \).

Given \( v \equiv 0 \pmod{4} \), define \( J_2(v) \) according to the following table:

<table>
<thead>
<tr>
<th>( v )</th>
<th>( J_2(v) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 (mod 12)</td>
<td>{(v - 1 - 3x, 2x), x = 0, 1, \ldots, (v - 3)/3}</td>
</tr>
<tr>
<td>4 (mod 12)</td>
<td>{(v - 1 - 3x, 2x), x = 0, 1, \ldots, (v - 1)/3}</td>
</tr>
<tr>
<td>8 (mod 12)</td>
<td>{(v - 1 - 3x, 2x), x = 0, 1, \ldots, (v - 2)/3}</td>
</tr>
</tbody>
</table>

Table 2. The set \( J_2(v) \).

In this paper, the main purpose is to investigate the existence problem of a \((K_2, P_k)\)-URD\((v; r, s)\) of \( K_v \) for \( k = 3, 4 \). We completely solve the spectrum problem for such design; i.e., characterize the existence of uniformly resolvable \( \{K_2, P_k\}\)-designs of order \( v \), by proving the following result:

**Main Theorem.**

(i) A \((K_2, P_3)\)-URD\((v; r, s)\) exists if and only if \( v \equiv 0 \pmod{6} \) and \( \text{URD}(v; K_2, P_3) = J_1(v) \).

(ii) A \((K_2, P_4)\)-URD\((v; r, s)\) exists if and only if \( v \equiv 0 \pmod{4} \) and \( \text{URD}(v; K_2, P_3) = J_2(v) \).

2. Preliminaries and Necessary Conditions

In this section we will introduce some useful definitions, results, and give necessary conditions for the existence of a uniformly resolvable decomposition of \( K_v \) into \( r \) classes of 1-factors and \( s \) classes of paths \( P_k \), \( k = 3, 4 \). For missing terms or results that are not explicitly explained in the paper, the reader is referred to [1] and its online updates. For some results below, we also cite this handbook instead of the original papers. A (resolvable) \( H \)-decomposition of the complete multipartite graph with \( u \) parts each of size \( g \) is known as a resolvable group divisible design \( H \)-RGDD of type \( g^u \), the parts of size \( g \) are called the groups of the design. When \( H = K_n \) we will call it an \( n-(R)GDD \). A \((K_2, P_k)\)-URGDD \((r, s)\) of type \( g^u \) is a uniformly resolvable decomposition of the complete multipartite graph with \( u \) parts each of size \( g \) into \( r \) classes containing only copies of 1-factors and \( s \) classes containing only copies of paths \( P_k \).

If the blocks of an \( H \)-GDD of type \( g^u \) can be partitioned into partial parallel classes, each of which contain all points except those of one group, we refer to the decomposition as a *frame*. 
A incomplete resolvable \((K_2, P_4)\)-decomposition of \(K_v\) with a hole of size \(h\) is an \((K_2, P_4)\)-decomposition of \(K_{v+h} - K_h\) in which there are two types of classes, full classes and partial classes which cover every point except those in the hole (the points of \(K_h\) are referred to as the hole). Specifically a \((K_2, P_4)\)-IURD\((v + h, h; [r_1, s_1], [\bar{r}_1, \bar{s}_1])\) is a uniformly resolvable \((K_2, P_4)\)-decomposition of \(K_{v+h} - K_h\) with \(r_1\) 1-factors which cover only the points not in the hole, \(\bar{r}_1\) partial classes of paths \(P_4\) which cover only the points not in the hole, \(\bar{s}_1\) full classes of paths \(P_4\) which cover every point of \(K_{v+h}\).

**Lemma 2.1.** If there exists a \((K_2, P_3)\)-URD\((v; r, s)\) of \(K_v\), then \(v \equiv 0 \pmod{6}\) and \((r, s) \in J_1(v)\).

**Proof.** The condition \(v \equiv 0 \pmod{6}\) is trivial. Let \(D\) be a \((K_2, P_3)\)-URD\((v; r, s)\) of \(K_v\). Counting the edges of \(K_v\) that appear in \(D\) we obtain

\[
\frac{rv}{2} + \frac{2sv}{3} = \frac{v(v-1)}{2},
\]

and hence

\[
3r + 4s = 3(v-1).
\]

This equation implies that \(3r \equiv 3(v-1) \pmod{4}\) and \(4s \equiv 3(v-1) \pmod{3}\). Then we obtain

- \(r \equiv 3 \pmod{4}\) and \(s \equiv 0 \pmod{3}\) for \(v \equiv 0 \pmod{12}\),
- \(r \equiv 1 \pmod{4}\) and \(s \equiv 0 \pmod{3}\) for \(v \equiv 6 \pmod{12}\).

Letting now \(s = 3x\), the equation (2) yields \(r = (v-1) - 4x\). Since \(r\) and \(s\) cannot be negative, and \(x\) is an integer, the value of \(x\) has to be in the range as given in the definition of \(J_1(v)\). This completes the proof. \(\Box\)

**Lemma 2.2.** If there exists a \((K_2, P_4)\)-URD\((v; r, s)\) of \(K_v\) then \(v \equiv 0 \pmod{4}\) and \((r, s) \in J_2(v)\).

**Proof.** The condition \(v \equiv 0 \pmod{4}\) is trivial. Let \(D\) be a \((K_2, P_4)\)-URD\((v; r, s)\) of \(K_v\). Counting the edges of \(K_v\) that appear in \(D\) we obtain

\[
\frac{rv}{2} + \frac{3sv}{4} = \frac{v(v-1)}{2},
\]

and hence

\[
2r + 3s = 2(v-1).
\]

This equation implies that

\[
2r \equiv 2(v-1) \pmod{3}\) and \(3s \equiv 2(v-1) \pmod{2}.
\]

Then we obtain

- \(r \equiv 2 \pmod{3}\) and \(s \equiv 0 \pmod{2}\) for \(v \equiv 0 \pmod{12}\),
- \(r \equiv 0 \pmod{3}\) and \(s \equiv 0 \pmod{2}\) for \(v \equiv 4 \pmod{12}\),
- \(r \equiv 1 \pmod{3}\) and \(s \equiv 0 \pmod{2}\) for \(v \equiv 8 \pmod{12}\).
Theorem 2.4. 

Proof. Lemma 3.1. Let $v \equiv 0 \pmod{6}$, $(r, s) \in \{(v - 1 - 2x, x), x = 1, 2, \ldots, \frac{v-2}{2}\}$, with the two exceptions $(v, s) = (6, 2), (12, 5)$.

Theorem 2.4. Let $v \equiv 0 \pmod{3}, v \geq 9$. The union of any two edge-disjoint parallel classes of 3-cycles of $K_v$ can be decomposed into three parallel classes of $P_3$.

We also need the following definitions. Let $(s_1, t_1)$ and $(s_2, t_2)$ be two pairs of non-negative integers. Define $(s_1, t_1) + (s_2, t_2) = (s_1 + s_2, t_1 + t_2)$. If $X$ and $Y$ are two sets of pairs of non-negative integers, then $X + Y$ denotes the set $\{(s_1, t_1) + (s_2, t_2) : (s_1, t_1) \in X, (s_2, t_2) \in Y\}$. If $X$ is a set of pairs of non-negative integers and $h$ is a positive integer, then $h \cdot X$ denotes the set of all pairs of non-negative integers which can be obtained by adding any $h$ elements of $X$ together (repetitions of elements of $X$ are allowed).

3. Small cases

Lemma 3.1. $URD(6; K_2, P_3) = \{(5, 0), (1, 3)\}$.

Proof. The case $(5, 0)$ corresponds to a 1-factorization of the complete bipartite graph $K_6$ which is known to exist [1]. For the case $(1, 3)$, let $V(K_{12}) = \mathbb{Z}_6$, and the classes as listed below:

\[
\{\{0, 1\}, \{2, 3\}, \{4, 5\}\}, \{\{1, 4, 5\}, \{2, 3, 6\}\}, \{\{3, 1, 5\}, \{4, 2, 6\}\}, \{\{1, 6, 4\}, \{2, 5, 3\}\}.
\]

Lemma 3.2. There exists a $(K_2, P_4)$-URGDD$(r, s)$ of type $6^2$ with $(r, s) \in \{(0, 4), (3, 2), (6, 0)\}$.

Proof. The case $(6, 0)$ corresponds to a 1-factorization of the complete bipartite graph $K_{6,6}$ which is known to exist [1]. The case $(0, 4)$ corresponds to a $(K_2, P_4)$-URGDD(0, 4) which is known to exist [15]. For the case $(3, 2)$ take the groups to be $\{1, 2, 3, 4, 5, 6, 7, 8\}, \{a, b, c, d, e, f\}$ and the classes listed below:

\[
\{\{1, c\}, \{2, d\}, \{3, e\}, \{4, f\}, \{5, a\}, \{6, b\}\},
\{\{1, d\}, \{2, c\}, \{3, f\}, \{4, e\}, \{5, b\}, \{6, a\}\},
\{\{1, b\}, \{2, e\}, \{3, c\}, \{4, a\}, \{5, f\}, \{6, d\}\},
\{\{1, a, 2, b\}, \{3, d, 4, c\}, \{5, e, 6, f\}\}, \{\{4, b, 3, a\}, \{6, c, 5, d\}, \{e, 1, f, 2\}\}.
\]
Lemma 3.3. \( URD(12; K_2, P_4) = \{(11, 0), (8, 2), (5, 4), (2, 6)\} \).

Proof. The case \((11, 0)\) corresponds to a 1-factorization of the complete graph \(K_{12}\) which is known to exist [1]. The rest of the cases are given explicitly below.

- \((8, 2), (5, 4)\).
  Take a \((K_2, P_4)\)-URGDD\((r, s)\) of type 6\(^2\) with \((r, s) \in \{(0, 4), (3, 2)\}\), which come from Lemma 3.2. Fill in each of the groups of size 6 with the same 1-factorization of \(K_6\). This gives a \((K_2, P_4)\)-URD\((12; r, s)\) for each \((r, s) \in \{(5, 0) + 4 \ast \{(0, 4), (3, 2), (6, 0)\}\}\).

- \((2, 6)\).
  Let \(V(K_{12}) = \{0, 1, \ldots, 11\}\) be the vertex set and the classes listed below:
  \[
  \{0, 1, 2, 3], [4, 5, 6, 7], [8, 9, 10, 11]\}, \{1, 3, 0, 2], [5, 7, 4, 6], [9, 11, 8, 10]\},
  \{0, 4, 1, 5], [8, 6, 9, 7], [10, 2, 11, 3]\}, \{1, 7, 0, 6], [2, 8, 3, 9], [11, 5, 10, 4]\},
  \{9, 4, 8, 5], [11, 0, 10, 1], [3, 6, 2, 7]\}, \{2, 5, 3, 4], [8, 1, 9, 0], [10, 7, 11, 6]\},
  \{0, 8], [1, 11], [2, 4], [3, 7], [6, 10], [5, 9]\},
  \{0, 5], [1, 6], [2, 9], [3, 10], [4, 11], [7, 8]\}.

\( \Box \)

Lemma 3.4. There exists a \((K_2, P_4)\)-IURD\((8, 2; [1, 0], [r, s])\) with \((r, s) \in \{(6, 0), (3, 2), (0, 4)\}\).

Proof. Let the point set be \(V = \{a, b, 0, 1, 2, 3, 4, 5\}\) and let \(\{a, b\}\) be the hole. Let \(\mathcal{F} = \{F_1, F_2, \ldots, F_7\}\) be a 1-factorization of \(K_8\) such that \(\{a, b\} \in F_1\).

- A \((K_2, P_4)\)-IURD\((8, 2; [1, 0], [6, 0])\)
  \(F_1 - \{a, b\}, \{F_2, \ldots, F_7\}\).

- A \((K_2, P_4)\)-IURD\((8, 2; [1, 0], [3, 2])\)
  \(F_1 - \{a, b\}, \{0, b\}, \{1, 5\}, \{2, a\}, \{3, 4\}, \{4, b\}, \{a, 5\}, \{2, 3\}, \{0, 1\}\}, \{0, 3\}, \{b, 5\}, \{2, 1\}, \{3, 0\}\),
  \(\{0, a, 1, b\}, \{3, 5, 2, 4\}\}]. \{2, b, 3, a\}, [5, 0, 4, 1\}.

- A \((K_2, P_4)\)-IURD\((8, 2; [1, 0], [0, 4])\)
  \(F_1 - \{a, b\}, \{0, a, 1, b\}, [3, 5, 2, 4\}]. \{2, b, 3, a\}, [5, 0, 4, 1\}.
  \{2, a, 5, b\}, [1, 0, 3, 4\}]. \{0, b, 4, a\}, [5, 1, 2, 3\}.

\( \Box \)

Lemma 3.5. \( URD(8; K_2, P_4) = \{(7, 0), (4, 2), (1, 4)\} \).

Proof. The assertion follows from Lemma 3.4. \( \Box \)

4. Main results

Lemma 4.1. For every \(v \equiv 0 \pmod{6}\) \(J_1(v) \subseteq URD(v; K_2, P_3)\).

Proof. For \(v = 6\) the conclusion follows from Lemma 3.1. For \(v \geq 12\), take a \((K_2, K_3)\)-URD\((v; v - 1 - 4t, 2t)\) with \(t \in \{0, 1, \ldots, (v - 4)/4\}\) for \(v \equiv 0 \pmod{12}\) and \(t \in \{0, 1, \ldots, (v - 2)/4\}\) for \(v \equiv 6 \pmod{12}\), which exists
by Theorem 2.3. Applying Theorem 2.4 we obtain a \((K_2, P_3)\)-URD\((v; v - 1 - 4t, 3t)\).

\[ \square \]

**Lemma 4.2.** For every \( v \equiv 4 \pmod{12} \), \( J_2(v) \subseteq URD(v; K_2, P_4) \).

**Proof.** Let \( R_1, R_2, \ldots, R_{v-1} \) be the parallel classes of a resolvable \( \{K_4\}\)-design \( R \) of order \( v \). Place on each block of a given resolution class of \( R \) the same \((K_2, P_4)\)-URD\((4; r, s)\) with \((r, s) \in \{(3, 0), (0, 2)\}\). Since \( R \) contains \((v - 1)/3\) parallel classes the result is a \((K_2, P_4)\)-URD\((v; r, s)\) of \( K_v \) for each \((r, s) \in (v - 1)/3 \ast \{(3, 0), (0, 2)\}\). This implies

\[ URD(v; K_2, P_4) \supseteq \left\{ \frac{v - 1}{3} \ast \{(3, 0), (0, 2)\} \right\}. \]

Since

\[ \frac{v - 1}{3} \ast \{(3, 0), (0, 2)\} = \left\{ (v - 1 - 3x, 2x), x = 0, \ldots, \frac{v - 1}{3} \right\} = J_2(v), \]

we obtain the proof. \[ \square \]

**Lemma 4.3.** For every \( v \equiv 0 \pmod{12} \), \( J_2(v) \subseteq URD(v; K_2, P_4) \).

**Proof.** For \( v = 12 \) the conclusion follows from Lemma 3.3. For \( v \geq 24 \) start with a 2-RGDD \( G \) of type \( 2^{12} r, s \) [1]. Give weight 6 to each point of this 2-GDD and place on each edge of a given resolution class the same \((K_2, P_4)\)-URGDD\((r, s)\) of type \( 6^2 \), with \((r, s) \in \{(6, 0), (3, 2), (0, 4)\}\), which exists by Lemma 3.2. Fill the groups of sizes 12 with the same \((K_2, P_4)\)-URD\((12; r, s)\), with \((r, s) \in \{(11, 0), (8, 2), (5, 4), (2, 6)\}\), which exists by Lemma 3.3. Since \( G \) contains \((v - 12)/6\) resolution classes the result is a \((K_2, P_4)\)-URD\((v; r, s)\) of \( K_v \) for each \((r, s) \in \{(11, 0), (8, 2), (5, 4), (2, 6)\} + (v - 12)/6 \ast \{(6, 0), (3, 2), (0, 4)\}\). This implies

\[ URD(v; K_2, P_4) \supseteq \left\{ \{(11, 0), (8, 2), (5, 4), (2, 6)\} + \frac{(v - 12)}{6} \ast \{(6, 0), (3, 2), (0, 4)\} \right\}. \]

Since

\[ \frac{v - 12}{6} \ast \{(6, 0), (3, 2), (0, 4)\} = \left\{ (v - 12 - 3x, 2x), x = 0, \ldots, \frac{v - 12}{3} \right\}, \]

it easy to see that

\[ \left\{ \{(11, 0), (8, 2), (5, 4), (2, 6)\} + \frac{(v - 12)}{6} \ast \{(6, 0), (3, 2), (0, 4)\} \right\} = J_2(v). \]

This completes the proof. \[ \square \]

**Lemma 4.4.** For every \( v \equiv 8 \pmod{12} \), \( J_2(v) \subseteq URD(v; K_2, P_4) \).
Proof. For \( v = 8 \) the conclusion follows from Lemma 3.5. For \( v > 8 \) start with a 2–frame \( F \) of type \( 1 + \frac{v^2}{2} \) [14] with groups \( G_i, \ i = 1, \ldots, (v - 2)/6 \). Let \( p_i \) be the partial parallel class which miss the group \( G_i \). Expand each point 6 times and add a set \( H \) of 2 ideal points \( a_1, a_2 \). For each \( i = 1, \ldots, (v - 2)/6 \), place on \( G_i \times \{1, \ldots, 6\} \cup H \) the same \((K_2, P_4)\)-\text{URD}(8, 2; (1, 0), [x, y]) \( D_i \) of \( K_8 - K_2 \) with \((x, y) \in \{(6, 0), (3, 2), (0, 4)\}\), which exists by Lemma 3.4, in such a way the hole covers the point of \( H \). For each \( i = 1, \ldots, (v - 2)/6 \), place on each block of the \( p_i \) partial parallel class the same \((K_2, P_4)\)-\text{URGDD}(r_2, s_2) of type \( 6^2 \) with \((r_2, s_2) \in \{(6, 0), (3, 2), (0, 4)\}\), which exists by Lemma 3.2.

Add the edge \( \{a_1, a_2\} \) of \( H \) to the partial classes of \( D_i \) and form, on \( \cup_{i = 1}^{v/6} G_i \times \{1, \ldots, 6\} \cup H \), 1 class of 1-factors. For each \( i = 1, \ldots, (v - 2)/6 \), add the full classes of \( D_i \) to the classes of \( p_i \) and form \( r_3 \) classes of 1-factors and \( s_3 \) classes of \( P_4 \)-factors with \((r_3, s_3) \in \{(6, 0), (3, 2), (0, 4)\}\). Since each group \( G_i \) is missed by 1 partial parallel class of \( F \) we obtain a \((K_2, P_4)\)-\text{URD} \((v; r, s)\) for each \((r, s) \in \{(1, 0) + (v - 2)/6 \ast \{(6, 0), (3, 2), (0, 4)\}\}\). This implies

\[
URD(v; K_2, P_4) \supseteq \left\{(1, 0) + \frac{v - 2}{6} \ast \{(0, 4), (3, 2), (6, 0)\}\right\}.
\]

Since

\[
\frac{v - 2}{6} \ast \{(0, 4), (3, 2), (6, 0)\} = \left\{(v - 1 - 3x, 2x), x = 0, \ldots, \frac{v - 2}{3}\right\},
\]

it easy to see that \( \{(1, 0) + (v - 2)/6 \ast \{(6, 0), (3, 2), (0, 4)\}\} = J_2(v) \). This completes the proof.

\(\Box\)

5. Conclusion

We are now in a position to prove the main result of the paper.

**Theorem 5.1.** For every \( v \equiv 0 \pmod{6} \), we have \( URD(v; K_2, P_3) = J_1(v) \) and, for every \( v \equiv 0 \pmod{4} \), we have \( URD(v; K_2, P_4) = J_2(v) \).

**Proof.** Necessity follows from Lemmas 2.1 and 2.2. Sufficiency follows from Lemmas 4.1, 4.2, 4.3 and 4.4. This completes the proof. \(\Box\)

**Remark:** Note that the existence of uniformly resolvable \( \{K_2, P_k\} \)-designs with \( k > 4 \) is very difficult to study and it is currently under investigation.

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