A THEOREM ON FRACTIONAL ID-$(g,f)$-FACTOR-CRITICAL GRAPHS

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Abstract. Let $a$, $b$ and $r$ be three nonnegative integers with $2 \leq a \leq b - r$, let $G$ be a graph of order $p$ satisfying the inequality $p(a + r) \geq (a + b - 3)(2a + b + r) + 1$, and let $g$ and $f$ be two integer-valued functions defined on $V(G)$ satisfying $a \leq g(x) \leq f(x) - r \leq b - r$ for every $x \in V(G)$. A graph $G$ is said to be fractional ID-$(g,f)$-factor-critical if $G - I$ contains a fractional $(g,f)$-factor for every independent set $I$ of $G$. In this paper, we prove that $G$ is fractional ID-$(g,f)$-factor-critical if $\text{bind}(G)((a + r)p - (a + b - 2)) > (2a + b + r - 1)(p - 1)$, which is a generalization of a previous result of Zhou.

1. Introduction

The graphs considered here are finite undirected graphs which have neither loops nor multiple edges. Let $G = (V(G), E(G))$ be a graph, where $V(G)$ and $E(G)$ denote its vertex set and edge set. For every $x \in V(G)$, we denote by $d_G(x)$ the degree of $x$ and by $N_G(x)$ the set of vertices adjacent to $x$ in $G$. For a subset $S$ of $V(G)$, we write $N_G(S) = \bigcup_{x \in S} N_G(x)$, $G[S]$ for the subgraph of $G$ induced by $S$, and we define $G - S = G[V(G) \setminus S]$.

The minimum degree of $G$ is denoted by $\delta(G)$, while a subset $S$ of $V(G)$ is said to be independent if $G[S]$ has no edges. The binding number of $G$ is denoted by $\text{bind}(G)$ and defined as

$$\text{bind}(G) = \min \left\{ \frac{|N_G(X)|}{|X|} : \emptyset \neq X \subseteq V(G), N_G(X) \neq V(G) \right\}.$$ 

Let $g$ and $f$ be two nonnegative integer-valued functions defined on $V(G)$ satisfying $g(x) \leq f(x)$ for any $x \in V(G)$. A spanning subgraph $F$ of $G$ is a
(g, f)-factor if \( g(x) \leq d_F(x) \leq f(x) \) for any \( x \in V(G) \). Assume there exists a function \( h : E(G) \to [0, 1] \) such that
\[
g(x) \leq \sum_{e \ni x} h(e) \leq f(x)
\]
for every vertex \( x \) of \( G \). The spanning subgraph of \( G \) induced by the set of edges \( \{ e : e \in E(G), h(e) > 0 \} \) is called a fractional \((g, f)\)-factor of \( G \) with indicator function \( h \).

**Definition 1.1.** A graph \( G \) is said to be fractional ID-\((g, f)\)-factor-critical if \( G - I \) contains a fractional \((g, f)\)-factor for every independent set \( I \) of \( G \).

A fractional ID-(f, f)-factor-critical graph is a fractional ID-f-factor-critical graph. If \( f(x) \equiv k \), then we say a fractional ID-k-factor-critical graph instead of a fractional ID-f-factor-critical graph. For any function \( f(x) \) and \( S \subseteq V(G) \), we define
\[
f(S) = \sum_{x \in S} f(x).
\]
In particular, note that
\[
d_G(S) = \sum_{x \in S} d_G(x).
\]
A huge amount of work has been done concerning factors and fractional factors in graphs (see [1, 4, 5, 6, 8]). In [3] Chang, Liu, and Zhu first investigated the fractional ID-k-factor-critical graph and obtained a minimum degree condition for a graph to be a fractional ID-k-factor-critical graph. This result is summarized below:

**Theorem 1.2** (Chang, Liu, and Zhu [3]). Let \( k \) be a positive integer and \( G \) be a graph of order \( p \) with \( p \geq 6k - 8 \). If \( \delta(G) \geq 2p/3 \), then \( G \) is fractional ID-k-factor-critical.

In [11] Zhou, Xu, and Sun proved the following result on the fractional ID-k-factor-critical graphs:

**Theorem 1.3** (Zhou, Xu, and Sun [11]). Let \( G \) be a graph, and let \( k \) be an integer with \( k \geq 1 \). If
\[
\alpha(G) \leq \frac{4k(\delta(G) - k + 1)}{k^2 + 6k + 1},
\]
then \( G \) is fractional ID-k-factor-critical.

Zhou studied the relationship between binding number and the fractional ID-k-factor-critical graph in [10] and proved the following theorem:

**Theorem 1.4** (Zhou [10]). Let \( k \) be an integer with \( k \geq 2 \), and let \( G \) be a graph of order \( p \) with \( p \geq 6k - 9 \). If
\[
\text{bind}(G) > \frac{(3k - 1)(p - 1)}{kp - 2k + 2},
\]
then \( G \) is fractional ID-k-factor-critical.
In this work, we generalize the fractional ID-$k$-factor-critical graph to the fractional ID-$(g, f)$-factor-critical graph and obtain a binding number condition for a graph to be fractional ID-$(g, f)$-factor-critical:

**Theorem 1.5.** Let $a$, $b$, and $r$ be three integers such that $2 \leq a \leq b - r$ and $r \geq 0$, let $G$ be a graph of order $p$, where

$$p \geq \frac{(a + b - 3)(2a + b + r) + 1}{a + r},$$

and let both $g$ and $f$ be nonnegative integer-valued functions defined on $V(G)$, where $a \leq g(x) \leq f(x) - r \leq b - r$ for any $x \in V(G)$. If

$$\text{bind}(G) > \frac{(2a + b + r - 1)(p - 1)}{(a + r)p - (a + b - 2)},$$

then $G$ is fractional ID-$(g, f)$-factor-critical.

We obtain the following corollary by setting $r = 0$ in Theorem 1.5:

**Corollary 1.6.** Let $a$ and $b$ be two integers with $2 \leq a \leq b$, and let $G$ be a graph of order $p$, where

$$p \geq \frac{(a + b - 3)(2a + b) + 1}{a},$$

and let $g$ and $f$ be nonnegative integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) \leq b$ for any $x \in V(G)$. If

$$\text{bind}(G) > \frac{(2a + b - 1)(p - 1)}{ap - (a + b - 2)},$$

then $G$ is fractional ID-$(g, f)$-factor-critical.

If $g(x) \equiv f(x)$ in Corollary 1.6, then we have the following result:

**Corollary 1.7.** Let $a$ and $b$ be two integers satisfying $2 \leq a \leq b$, and let $G$ be a graph of order $p$ with

$$p \geq \frac{(a + b - 3)(2a + b) + 1}{a},$$

and let $f$ be a nonnegative integer-valued function defined on $V(G)$, where $a \leq f(x) \leq b$ for any $x \in V(G)$. If

$$\text{bind}(G) > \frac{(2a + b - 1)(p - 1)}{ap - (a + b - 2)},$$

then $G$ is fractional ID-$f$-factor-critical.

2. **Proof of Theorem 1.4**

The following result was first obtained by Anstee [2], and it is very useful for proving Theorem 1.5. An alternative proof was provided by Liu and Zhang in [7].
Lemma 2.1 (Anstee [2], Liu and Zhang [7]). Let $G$ be a graph. Then $G$ has a fractional $(g,f)$-factor if and only if for every subset $S$ of $V(G)$,

$$\delta_G(S,T) = f(S) + d_{G-S}(T) - g(T) \geq 0,$$

where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq g(x)\}$.

In [9] Woodall presented the following result, which will also be used in the proof of Theorem 1.5:

Lemma 2.2 (Woodall [9]). Let $c$ be a positive real number and let $G$ be a graph of order $p$ with $\text{bind}(G) > c$. Then

$$\delta(G) \geq p - \frac{p-1}{\text{bind}(G)} > p - \frac{p-1}{c}.$$

Proof of Theorem 1.5. Let $X$ be an independent set of $G$ and $H = G - X$. In order to prove Theorem 1.5, by Definition 1.1 we only need to prove that $H$ admits a fractional $(g,f)$-factor.

Suppose that $H$ has no fractional $(g,f)$-factor. Then from Lemma 2.1, there exists some subset $S$ of $V(H)$ satisfying

$$\delta_H(S,T) = f(S) + d_{H-S}(T) - g(T) \leq -1, \tag{1}$$

where $T = \{x : x \in V(H) \setminus S, d_{H-S}(x) \leq g(x)\}$.

Henceforth we write $\text{bind}(G) = \lambda$. In terms of Lemma 2.2 and the hypotheses of Theorem 1.5, we obtain the inequality

$$\delta(G) \geq p - \frac{p-1}{\lambda} > \frac{(a+b-1)p+a+b-2}{2a+b+r-1}. \tag{2}$$

Assume, in order to derive a contradiction, that $T = \emptyset$. Then using Equation (1) we derive that

$$-1 \geq \delta_H(S,T) = f(S) \geq 0,$$

which is a contradiction. Therefore $T \neq \emptyset$.

In the following, we set $h = \min \{d_{H-S}(x) : x \in T\}$. Obviously, $0 \leq h \leq b - r$. We now must prove the following claims:

Claim 2.3. $|S| \geq \delta(G) - |X| - h$.

Proof. We choose $x_1 \in T$ with $d_{H-S}(x_1) = h$. Clearly, we have

$$\delta(G) \leq d_G(x_1) \leq d_{G-X-S}(x_1) + |X| + |S|$$

$$= d_{H-S}(x_1) + |X| + |S| = h + |X| + |S|,$$

which implies

$$|S| \geq \delta(G) - |X| - h.$$ 

This completes the proof of Claim 2.3. \hfill \Box

Claim 2.4. $|X| \leq p - \delta(G)$.
Proof. Obviously, \(d_G(x) \geq \delta(G)\) for any \(x \in V(G)\). Consequently, \(d_G(x) \geq \delta(G)\) for any \(x \in X\). Because \(X\) is an independent set of \(G\) we have
\[
p \geq d_G(x) + |X| \geq \delta(G) + |X|
\]
for all \(x \in X\), which implies
\[
|X| \leq p - \delta(G).
\]
This proves Claim 2.4.

We now consider the following two cases regarding the value of \(h\):

**Case 1**: \(h = 0\):

In this case, we first prove the following claim:

**Claim 2.5.** \(\lambda \leq a + b - 1\).

**Proof.** Suppose that \(\lambda > a + b - 1\). In view of Equation (2) and \(2 \leq a \leq b - r\), we obtain
\[
\delta(G) \geq p - \frac{p - 1}{\lambda} > \frac{(a + b - 2)p}{a + b - 1} > \frac{(a + b)p}{2a + b + r}.
\]
Combining this with Equation (1), the inequality \(p \geq |X| + |S| + |T|\), and Claims 2.3 and 2.4, we have:
\[
-1 \geq \delta_H(S, T) = f(S) + d_{H-S}(T) - g(T) \\
\geq (a + r)|S| - (b - r)|T| \\
\geq (a + r)|S| - (b - r)(p - |X| - |S|) \\
= (a + b)|S| - (b - r)p + (b - r)|X| \\
\geq (a + b)(\delta(G) - |X|) - (b - r)p + (b - r)|X| \\
= (a + b)\delta(G) - (b - r)p - (a + r)|X| \\
\geq (a + b)\delta(G) - (b - r)p - (a + r)(p - \delta(G)) \\
= (2a + b + r)\delta(G) - (a + b)p > 0,
\]
which is a contradiction. This completes the proof of Claim 2.5.

Now set \(Y = \{x : x \in T, d_{H-S}(x) = 0\}\). Note that \(Y \neq \emptyset\) and \(N_G(V(G) \setminus (X \cup S)) \cap Y = \emptyset\), which gives \(|N_G(V(G) \setminus (X \cup S))| \leq p - |Y|\). Thus,
\[
\text{bind}(G) = \lambda \leq \frac{|N_G(V(G) \setminus (X \cup S))|}{|V(G) \setminus (X \cup S)|} \leq \frac{p - |Y|}{p - |X| - |S|},
\]
that is,
\[
|S| \geq \left(1 - \frac{1}{\lambda}\right)p - |X| + \frac{1}{\lambda}|Y|.
\]
It then follows from Equation (1) and the inequality \(|X| + |S| + |T| \leq p\) that:

\[
-1 \geq \delta_H(S, T) = f(S) + d_{H-S}(T) - g(T) \\
\geq (a + r)|S| + |T| - |Y| - (b - r)|T| \\
= (a + r)|S| - (b - r - 1)|T| - |Y| \\
\geq (a + r)|S| - (b - r - 1)(p - |X| - |S|) - |Y| \\
= (a + b - 1)|S| - (b - r - 1)p + (b - r - 1)|X| - |Y|.
\]

Invoking Equation (3) then gives that:

\[
(a + b - 1)|S| - (b - r - 1)p + (b - r - 1)|X| - |Y| \\
\geq (a + b - 1) \left( \left( 1 - \frac{1}{\lambda} \right) p - |X| + \frac{|Y|}{\lambda} \right) + (b - r - 1)(|X| - p) - |Y| \\
= (a + r)p - \frac{(a + b - 1)p}{\lambda} - (a + r)|X| + \left( \frac{a + b - 1}{\lambda} - 1 \right) |Y|.
\]

Claim 2.5 and the fact that \(Y \neq \emptyset\) imply together the inequality

\[
(a + r)p - \frac{(a + b - 1)p}{\lambda} - (a + r)|X| + \left( \frac{a + b - 1}{\lambda} - 1 \right) |Y| \\
\geq (a + r)p - \frac{(a + b - 1)p}{\lambda} - (a + r)|X| + \frac{a + b - 1}{\lambda} - 1;
\]

applying Claim 2.4 then yields the following:

\[
(a + r)p - \frac{(a + b - 1)p}{\lambda} - (a + r)|X| + \frac{a + b - 1}{\lambda} - 1 \\
\geq (a + r)p - \frac{(a + b - 1)p}{\lambda} - (a + r)(p - \delta(G)) + \frac{a + b - 1}{\lambda} - 1 \\
= - \frac{(a + b - 1)p}{\lambda} + (a + r)\delta(G) + \frac{a + b - 1}{\lambda} - 1.
\]

Using Equation (2) allows us to conclude

\[
- \frac{(a + b - 1)p}{\lambda} + (a + r)\delta(G) + \frac{a + b - 1}{\lambda} - 1 \\
\geq -(a + r)p + (a + r) \left( p - \frac{p - 1}{\lambda} \right) + \frac{a + b - 1}{\lambda} - (a + b - 1) \\
= - \frac{(2a + b + r - 1)(p - 1)}{\lambda} + (a + r)p - (a + b - 1),
\]

which implies

\[
\lambda \leq \frac{(2a + b + r - 1)(p - 1)}{(a + r)p - (a + b - 2)},
\]

contradicting the hypotheses of Theorem 1.5.

**Case 2:** \(1 \leq h \leq b - r\);
According to Equation (1), Claims 2.3 and 2.4, and the inequality \( p \geq |S| + |T| + |X| \), we obtain:

\[
\begin{align*}
-1 & \geq \delta_H(S, T) = f(S) + d_{H-S}(T) - g(T) \\
& \geq (a + r)|S| - (b - r - h)|T| \\
& \geq (a + r)|S| - (b - r - h)(p - |X| - |S|) \\
& = (a + b - h)|S| + (b - r - h)|X| - (b - r - h)p \\
& \geq (a + b - h)(\delta(G) - |X| - h) + (b - r - h)|X| - (b - r - h)p \\
& = (a + b - h)\delta(G) - (a + r)|X| - h(a + b - h) - (b - r - h)p \\
& \geq (2a + b + r - h)\delta(G) - h(a + b - h) - (a + b - h)p,
\end{align*}
\]

that is,

\[
\delta(G) \leq \frac{(a + b - h)(p + h) - 1}{2a + b + r - h}.
\] (4)

If \( h = 1 \) in Equation (4), then we have

\[
\delta(G) \leq \frac{(a + b - 1)(p + 1) - 1}{2a + b + r - 1},
\]

which contradicts Equation (2). Hence we assume \( 2 \leq h \leq b - r \). Let

\[
F(h) = \frac{(a + b - h)(p + h) - 1}{2a + b + r - h}.
\]

Using

\[
p \geq \frac{(a + b - 3)(2a + b + r) + 1}{a + r},
\]

we calculate \( F'(h) < 0 \), implying that \( F(h) \) attains its maximum value at \( h = 2 \). Therefore we have

\[
\delta(G) \leq F(2) = \frac{(a + b - 2)(p + 2) - 1}{2a + b + r - 2}.
\] (5)

Since

\[
p \geq \frac{(a + b - 3)(2a + b + r) + 1}{a + r},
\]

we prove easily that

\[
\frac{(a + b - 2)(p + 2) - 1}{2a + b + r - 2} \leq \frac{(a + b - 1)p + a + b - 2}{2a + b + r - 1}.
\]

Combining this with Equation (5), we obtain

\[
\delta(G) \leq \frac{(a + b - 1)p + a + b - 2}{2a + b + r - 1},
\]

which contradicts Equation (2). This completes the proof of Theorem 1.5. \( \square \)
Finally, we present the following problem:

**Problem.** Is it possible to weaken the binding number condition

\[
\text{bind}(G) > \frac{(2a + b + r - 1)(p - 1)}{(a + r)p - (a + b - 2)}
\]

for the existence of fractional ID-\((g, f)\)-factor-critical graphs in Theorem 1.5?

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