THE ERDŐS–KO–RADO BASIS FOR A LEONARD SYSTEM

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Abstract. We introduce and discuss an Erdős–Ko–Rado basis of the vector space underlying a Leonard system $\Phi = (A; A^*; \{E_i\}_{i=0}^d; \{E^*_i\}_{i=0}^d)$ that satisfies a mild condition on the eigenvalues of $A$ and $A^*$. We describe the transition matrices to/from other known bases, as well as the matrices representing $A$ and $A^*$ with respect to the new basis. We also discuss how these results can be viewed as a generalization of the linear programming method used previously in the proofs of the “Erdős–Ko–Rado theorems” for several classical families of $Q$-polynomial distance-regular graphs, including the original 1961 theorem of Erdős, Ko, and Rado.

1. Introduction

Leonard systems [23] naturally arise in representation theory, combinatorics, and the theory of orthogonal polynomials (see e.g. [25, 28]). Hence they are receiving considerable attention. Indeed, the use of the name “Leonard system” is motivated by a connection to a theorem of Leonard [12], [2, pp. 263–274], which involves the $q$-Racah polynomials [1] and some related polynomials of the Askey scheme [10]. Leonard systems also play a role in coding theory; see [11].

Let $\Phi = (A; A^*; \{E_i\}_{i=0}^d; \{E^*_i\}_{i=0}^d)$ be a Leonard system over a field $K$, and $V$ the vector space underlying $\Phi$ (see Section 2 for formal definitions). Then $V = \bigoplus_{i=0}^d E^*_i V$ and $\dim E^*_i V = 1 \ (0 \leq i \leq d)$. We have a “canonical” (ordered) basis of $V$ associated with this direct sum decomposition, called a standard basis. There are 8 variations for the standard basis. Next, let $U_\ell = (\sum_{i=0}^\ell E_i^* V) \cap (\sum_{j=0}^{\ell} E_j V) \ (0 \leq \ell \leq d)$. Then, again it follows that $V = \bigoplus_{\ell=0}^d U_\ell$ and $\dim U_\ell = 1 \ (0 \leq \ell \leq d)$. We have a “canonical” basis of $V$ associated with this split decomposition, called a split basis. The split...
decomposition is crucial in the theory of Leonard systems, and there are 16 variations for the split basis. Altogether, Terwilliger [24] defined 24 bases of $V$ and studied in detail the transition matrices between these bases as well as the matrices representing $A$ and $A^*$ with respect to them.

In the present paper, we introduce another basis of $V$, which we call an Erdős–Ko–Rado (or EKR) basis of $V$, under a mild condition on the eigenvalues of $A$ and $A^*$ (see below). As its name suggests, this basis arises in connection with the famous Erdős–Ko–Rado theorem [6] in extremal set theory. Indeed, Delsarte’s linear programming method [4], which is closely related to Lovász’s $\vartheta$-function bound [13, 16] on the Shannon capacity of graphs, has been successfully used in the proofs of the “Erdős–Ko–Rado theorems” for certain families of $Q$-polynomial distance-regular graphs [29, 7, 17, 20] (including the original 1961 theorem of Erdős et al.), and constructing appropriate feasible solutions to the dual programs amounts to describing the EKR bases for the Leonard systems associated with these graphs; see Section 4. It seems that the previous constructions of the feasible solutions depend on the geometric/algebraic structures which are more or less specific to the family of graphs in question. Our results give a uniform description of such feasible solutions in terms of the parameter arrays of Leonard systems.

The contents of the paper are as follows. Section 2 reviews basic terminology, notation and facts concerning Leonard systems. In Section 3, we first study the subspaces $W_t = (E_0 V + \sum_{i=t}^{d} E_i V) \cap (E_0 V + \sum_{j=t}^{d} E_j V)$ ($0 \leq t \leq d$). We show that $\dim W_t = 1$ ($0 \leq t \leq d$), and that $V = \bigoplus_{t=0}^{d} W_t$ if and only if $q \neq -1$, or $q = -1$ and $d$ is even, where $q$ denotes a base of $\Phi$ (which is determined by the recurrence satisfied by the eigenvalues of $A$ and $A^*$). Assuming that this is the case, we then define an EKR basis associated with this direct sum decomposition. We describe the transition matrices to/from 3 bases out of the 24 bases mentioned above (2 standard, 1 split), as well as the matrices representing $A$ and $A^*$ with respect to the EKR basis. Our main results are Theorems 3.9, 3.12, and 3.13. Section 4 is devoted to discussions of the connections and applications of these results to the Erdős–Ko–Rado theorems.

2. Leonard systems

Let $\mathbb{K}$ be a field, $d$ a positive integer, $\mathcal{A}$ a $\mathbb{K}$-algebra isomorphic to the full matrix algebra $\text{Mat}_{d+1}(\mathbb{K})$, and $V$ an irreducible left $\mathcal{A}$-module. We remark that $V$ is unique up to isomorphism, and that $V$ has dimension $d+1$. An element $A$ of $\mathcal{A}$ is said to be multiplicity-free if it has $d + 1$ mutually distinct eigenvalues in $\mathbb{K}$. Let $A$ be a multiplicity-free element of $\mathcal{A}$ and

\footnote{In some cases, $V$ has the structure of an evaluation module of the quantum affine algebra $U_q(\hat{a}_2)$, and the split decomposition corresponds to its weight space decomposition; see e.g. [9].}

\footnote{$Q$-polynomial distance-regular graphs are thought of as finite/combinatorial analogues of compact symmetric spaces of rank one; see [2, pp. 311–312].}
an ordering of the eigenvalues of $A$. Let $E_i: V \to V(\theta_i)$ ($0 \leq i \leq d$) be the projection map onto $V(\theta_i)$ with respect to $V = \bigoplus_{i=0}^{d} V(\theta_i)$, where $V(\theta_i) = \{ u \in V : A u = \theta_i u \}$. We call $E_i$ the \textit{primitive idempotent} of $A$ associated with $\theta_i$. Notice that the $E_i$ are polynomials in $A$.

A \textit{Leonard system} in $\mathcal{A}$ ([23, Definition 1.4]) is a sequence

\[(1) \quad \Phi = \left( A; A^*; \{ E_i \}_{i=0}^{d}; \{ E_i^* \}_{i=0}^{d} \right) \]

satisfying the following axioms (LS1)–(LS5):

- (LS1) Each of $A, A^*$ is a multiplicity-free element in $\mathcal{A}$.\(^3\)
- (LS2) $\{ E_i \}_{i=0}^{d}$ is an ordering of the primitive idempotents of $A$.
- (LS3) $\{ E_i^* \}_{i=0}^{d}$ is an ordering of the primitive idempotents of $A^*$.
- (LS4) $E_i^* A E_j^* = \begin{cases} 0 & \text{if } |i-j| > 1 \\ \neq 0 & \text{if } |i-j| = 1 \end{cases}$ ($0 \leq i, j \leq d$).
- (LS5) $E_i A^* E_j = \begin{cases} 0 & \text{if } |i-j| > 1 \\ \neq 0 & \text{if } |i-j| = 1 \end{cases}$ ($0 \leq i, j \leq d$).

We say that $\Phi$ is \textit{over} $\mathbb{K}$. We refer the reader to [23, 26, 28] for background on Leonard systems.

Throughout the paper, $\Phi = \left( A; A^*; \{ E_i \}_{i=0}^{d}; \{ E_i^* \}_{i=0}^{d} \right)$ shall always denote the Leonard system (1). Notice that the following are Leonard systems:

\[
\Phi^* = \left( A^*; A; \{ E_i^* \}_{i=0}^{d}; \{ E_i \}_{i=0}^{d} \right), \\
\Phi^\dagger = \left( A; A^*; \{ E_i \}_{i=0}^{d}; \{ E_i^* \}_{i=0}^{d} \right), \\
\Phi^\ddagger = \left( A; A^*; \{ E_{d-i} \}_{i=0}^{d}; \{ E_i^* \}_{i=0}^{d} \right).
\]

Viewing $\star, \dagger, \ddagger$ as permutations on all Leonard systems,

\[
\star^2 = \dagger^2 = \ddagger^2 = 1, \quad \dagger \star = \star \dagger, \quad \dagger \star = \star \dagger, \quad \dagger \ddagger = \ddagger \dagger.
\]

The group generated by the symbols $\star, \dagger, \ddagger$ subject to the above relations is the dihedral group $D_4$ with 8 elements. We shall use the following notational convention:

\textbf{Notation 2.1.} For any $g \in D_4$ and for any object $f$ associated with $\Phi$, we let $f^g$ denote the corresponding object for $\Phi^g^{-1}$; an example is $E_i^* (\Phi) = E_i (\Phi^*)$.

It is known ([26, Theorem 6.1]) that there is a unique antiautomorphism $\dagger$ of $\mathcal{A}$ such that $A^\dagger = A$ and $A^{\dagger \dagger} = A^*$. From now on, let $\langle \cdot, \cdot \rangle: V \times V \to \mathbb{K}$ be a nondegenerate bilinear form on $V$ such that ([26, Section 15])

\[
\langle Xu_1, u_2 \rangle = \langle u_1, Xu^\dagger u_2 \rangle \quad (u_1, u_2 \in V, \ X \in \mathcal{A}).
\]

We shall write

\[
\|u\|^2 = \langle u, u \rangle \quad (u \in V).
\]

\(^3\)It is customary that $A^*$ denotes the conjugate transpose of $A$. It should be stressed that we are not using this convention.
Notation 2.2. Henceforth we fix a nonzero vector \( v^9 \) in \( E_0^2 V \) for each \( g \in D_4 \). We abbreviate \( v = v^1 \) where 1 is the identity of \( D_4 \). For convenience, we also assume \( v^{g_1} = v^{g_2} \) whenever \( E_0^g V = E_0^{g_2} V \) \((g_1, g_2 \in D_4)\). We remark that \( ||v^9||^2 \), \( \langle v^9, v^g \rangle \) are nonzero for any \( g \in D_4 \); cf. [26, Lemma 15.5].

We now recall a few direct sum decompositions of \( V \), as well as (ordered) bases of \( V \) associated with them. First, \( \dim E_i^a V = 1 \) \((0 \leq i \leq d)\) and \( V = \bigoplus_{i=0}^d E_i^a V \). By [26, Lemma 10.2], \( E_i^a v \neq 0 \) \((0 \leq i \leq d)\), so that \( \{E_i^a v\}_{i=0}^d \) is a basis of \( V \), called a \( \Phi \)-standard basis of \( V \). Next, let \( U_\ell = (\sum_{i=0}^d E_i^2 V) \cap (\sum_{j=\ell}^d E_j V) \) \((0 \leq \ell \leq d)\). Then, again \( \dim U_\ell = 1 \) \((0 \leq \ell \leq d)\) and \( V = \bigoplus_{\ell=0}^d U_\ell \), which is referred to as the \( \Phi \)-split decomposition of \( V \) [28]. We observe \( U_0 = E_0^2 V \) and \( U_d = E_d V \) \((0 \leq i \leq d)\), \( \theta_i \) be the eigenvalue of \( A \) associated with \( E_i \). Then it follows that \( (A - \theta_i I)U_\ell = U_{\ell + 1} \) and \( (A^* - \theta_i^* I)U_\ell = U_{\ell - 1} \) \((0 \leq \ell \leq d)\), where \( U_{d+1} = U_{d+2} = 0 \) [23, Lemma 3.9]. For \( 0 \leq i \leq d \), let \( \tau_i, \eta_i \) be the following polynomials in \( \mathbb{K}[z] \):

\[
\tau_i(z) = \prod_{h=0}^{i-1} (z - \theta_h), \quad \eta_i(z) = \tau_i(0) = \prod_{h=0}^{i-1} (z - \theta_{d-h}).
\]

From the above comments it follows that \( \tau_i(A)v^* \in U_\ell \) \((0 \leq \ell \leq d)\) and \( \{\tau_i(A)v^* \}_{i=0}^d \) is a basis of \( V \), called a \( \Phi \)-split basis of \( V \). Moreover, there are nonzero scalars \( \varphi_i \) \((1 \leq i \leq d)\) in \( \mathbb{K} \) such that \( A^*\tau_i(A)v^* = \theta_i^*\tau_i(A)v^* + \varphi_i \tau_{i-1}(A)v^* \) \((1 \leq i \leq d)\).

Let \( \phi_i = \varphi_i^{(1)} \) \((1 \leq i \leq d)\). The parameter array of \( \Phi \) is

\[
p(\Phi) = \left( \{\theta_i \}_{i=0}^d; \{\theta_i^* \}_{i=0}^d; \{\varphi_i \}_{i=1}^d; \{\phi_i \}_{i=1}^d \right).
\]

Terwilliger [23, Theorem 1.9] showed that the isomorphism class\(^4\) of \( \Phi \) is determined by \( p(\Phi) \) and gave a classification of the parameter arrays of Leonard systems; cf. [27, Section 5]. In particular, the sequences \( \{\theta_i \}_{i=0}^d \) and \( \{\theta_i^* \}_{i=0}^d \) are recurrent in the sense that there is a scalar \( \beta \in \mathbb{K} \) such that

\[
\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i} = \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*} = \beta + 1 \quad (2 \leq i \leq d - 1).
\]

It also follows that

\[
\phi_i = \varphi_1 \varphi_i + (\theta_i^* - \theta_0^*)(\theta_{d-i+1} - \theta_0) \quad (1 \leq i \leq d),
\]

where

\[
\varphi_i = \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} \quad (1 \leq i \leq d).
\]

\(^4\) A Leonard system \( \Psi \) in a \( \mathbb{K} \)-algebra \( \mathcal{B} \) is isomorphic to \( \Phi \) if there is a \( \mathbb{K} \)-algebra isomorphism \( \gamma : \mathcal{B} \to \mathcal{B} \) such that \( \Psi = \Phi^\gamma := (A^\gamma; A^{*\gamma}; \{E_i^\gamma \}_{i=0}^d; \{E_i^{*\gamma} \}_{i=0}^d) \).
Notice that $\vartheta_1 = \vartheta_d = 1$. Moreover,
\begin{align*}
(4) \quad \vartheta_{d-i+1} = \vartheta_i, \quad \vartheta_i^* = \vartheta_i \quad (1 \leq i \leq d).
\end{align*}

The parameter array behaves nicely with respect to the $D_4$ action:

**Lemma 2.3** ([23, Theorem 1.11]). The following hold.

(i) $p(\Phi^*) = \left(\{\theta_i^*\}_{i=0}^d; \{\theta_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_{d-i+1}\}_{i=1}^d\right)$.

(ii) $p(\Phi^1) = \left(\{\theta_i\}_{i=0}^d; \{\theta^*_{d-i}\}_{i=0}^d; \{\phi_{d-i+1}\}_{i=1}^d; \{\varphi_{d-i+1}\}_{i=1}^d\right)$.

(iii) $p(\Phi^\dagger) = \left(\{\theta_{d-i}\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\phi_i\}_{i=1}^d; \{\varphi_i\}_{i=1}^d\right)$.

The following can be easily read off [24, 26].

**Lemma 2.4** ([24, 26]). The following hold.

(i) $E_i^* v = \frac{||E_i^* v||^2}{\langle v, v^* \rangle} \sum_{\ell = 0}^i \frac{\tau^*(\theta^*_{\ell})}{\varphi_1 \cdots \varphi_\ell} \tau_\ell(A) v^* \quad (0 \leq i \leq d)$.

(ii) $\tau_\ell(A) v^* = \langle v, v^* \rangle \cdot \varphi_1 \cdots \varphi_\ell$
\begin{align*}
&\times \sum_{i=0}^\ell \frac{\eta_{d-i}(\theta^*_i)}{\tau^*_i(\theta^*_i) \eta_{d-i}(\theta^*_i)} \cdot \frac{1}{||E_i^* v||^2} E_i^* v \quad (0 \leq \ell \leq d).
\end{align*}

(iii) $E_j v^* = \sum_{\ell=j}^d \frac{\eta_{d-\ell}(\theta^*_j)}{\tau_\ell(\theta^*_j) \eta_{d-\ell}(\theta^*_j)} \tau_\ell(A) v^* \quad (0 \leq j \leq d)$.

(iv) $\tau_\ell(A) v^* = \sum_{j=\ell}^d \tau_\ell(\theta^*_j) E_j v^* \quad (0 \leq \ell \leq d)$.

(v) $E_j v^* = \frac{\langle v, v^* \rangle}{\varphi_1 \cdots \varphi_j} \frac{\phi_{d-j+1} \cdots \phi_d}{\varphi_1 \cdots \varphi_j} E_j v^* \quad (0 \leq j \leq d)$.

Finally, it follows that ([26, Lemma 9.2, Theorem 17.12])
$$E_0^* E_i E_0^* = \frac{\varphi_1 \cdots \varphi_i \phi_1 \cdots \phi_{d-i}}{\eta_{d}(\theta^*_0) \tau_i(\theta_i) \eta_{d-i}(\theta_i)} E_0^* \quad (0 \leq i \leq d),$$
from which it follows that
\begin{align*}
(5) \quad ||E_i^* v||^2 = \frac{\varphi_1 \cdots \varphi_i \phi_{i+1} \cdots \phi_d}{\eta_{d}(\theta^*_0) \tau_i(\theta_i) \eta_{d-i}(\theta_i)} ||v||^2 \quad (0 \leq i \leq d),
\end{align*}
by virtue of Lemma 2.3 (i).

3. The Erdös–Ko–Rado Basis

Let $F_\ell : V \to U_\ell \quad (0 \leq \ell \leq d)$ be the projection map onto $U_\ell$ with respect to the $\Phi$-split decomposition $V = \bigoplus_{\ell=0}^d U_\ell$.

**Lemma 3.1** (cf. [8, Lemma 5.4]). The following hold.

(i) $F_i E_i^* = 0$ if $\ell > i \quad (0 \leq i, \ell \leq d)$.

(ii) $F_\ell E_j = 0$ if $\ell < j \quad (0 \leq j, \ell \leq d)$. 
Proof. Immediate from $E_i^*V \subseteq \sum_{\ell=0}^i U_\ell$ and $E_jV \subseteq \sum_{\ell=j}^d U_\ell$. □

We shall mainly work with the $\Phi$-split decomposition $V = \bigoplus_{\ell=0}^d U_\ell^\perp$, where

$$U_\ell^\perp = \left(\sum_{i=d-\ell}^d E_i^*V\right) \cap \left(\sum_{j=t+1}^d E_jV\right) \quad (0 \leq \ell \leq d).$$

We now “modify” the $U_\ell^\perp$ and introduce the subspaces $W_t$ ($0 \leq t \leq d$) of $V$ defined by

$$W_t = \left(\sum_{i=d-t+1}^d E_i^*V\right) \cap \left(\sum_{j=t+1}^d E_jV\right) \quad (0 \leq t \leq d).$$

Observe $W_t \neq 0$ ($0 \leq t \leq d$), $W_0 = E_0^*V$, and $W_d = E_0V$. Notice also that

$$W_t^* = W_{d-t} \quad (0 \leq t \leq d).$$

Our aim is to show $\dim W_t = 1$ ($0 \leq t \leq d$), and then to determine precisely when $V = \bigoplus_{\ell=0}^d W_\ell$. Pick $w \in W_t$. Then from Lemma 3.1 (applied to $\Phi^\perp$) it follows that

$$F_\ell^\perp w = \sum_{i=0}^{d-\ell} F_\ell^\perp E_i^* w = \sum_{j=0}^\ell F_\ell^\perp E_jw \quad (0 \leq \ell \leq d).$$

Hence

$$F_\ell^\perp w = \begin{cases} F_\ell^\perp E_0w & \text{if } 0 \leq \ell \leq t, \\ F_\ell^\perp E_0^*w & \text{if } t \leq \ell \leq d, \end{cases}$$

from which it follows that

$$w = \sum_{\ell=0}^t F_\ell^\perp E_0w + \sum_{\ell=t+1}^d F_\ell^\perp E_0^*w = E_0w + \sum_{\ell=t+1}^d F_\ell^\perp (E_0^* - E_0)w.$$

By Lemma 2.4 (i) and Lemma 2.3 (ii), we have

$$F_\ell^\perp E_0^*w = F_\ell^\perp E_d^* w$$

$$= \frac{\langle w, E_d^* v^\perp \rangle}{||E_d^* v^\perp||^2} F_\ell^\perp E_d^* v^\perp$$

$$= \frac{\langle w, E_d^* v^\perp \rangle}{\langle v^*, v^*\perp \rangle} \cdot \frac{\tau_\ell^\perp(\theta^*_d)}{\phi_1 \cdots \phi_\ell} \tau_\ell(A) v^*\perp$$

$$= \frac{\langle w, E_0^* v \rangle}{\langle v^*, v^*\perp \rangle} \cdot \frac{\eta^*_d(\theta^*_0)}{\phi_d - \ell + 1 \cdots \phi_d} \tau_\ell(A) v^*\perp$$

The subscript $t$ is chosen in accordance with the concept of $t$-intersecting families in the Erdős–Ko–Rado theorem; see Section 4.
for $0 \leq \ell \leq d$. Likewise, by Lemma 2.4 (iii) and Lemma 2.3 (ii), we have

$$F^\downarrow_\ell E_0^\downarrow w = F^\downarrow_\ell E_0^\downarrow v^\downarrow$$

$$= \frac{\langle w, E_0^\downarrow v^\downarrow \rangle}{\|E_0^\downarrow v^\downarrow\|^2} F^\downarrow_\ell E_0^\downarrow v^\downarrow$$

$$= \frac{\langle w, E_0^\downarrow v^\downarrow \rangle}{\|E_0^\downarrow v^\downarrow\|^2} \eta_{d-\ell}(\theta_0) \tau_\ell(A) v^\downarrow$$

for $0 \leq \ell \leq d$. Since $F^\downarrow_\ell E_0^\downarrow w = F^\downarrow_\ell E_0^\downarrow w$ by (7), we have in particular:

$$\frac{\langle w, E_0^\downarrow v^\downarrow \rangle}{\|E_0^\downarrow v^\downarrow\|^2} \cdot \eta_{d-\ell}(\theta_0) = \frac{\langle w, E_0^\downarrow v^\downarrow \rangle}{\|E_0^\downarrow v^\downarrow\|^2} \eta_{d-\ell}(\theta_0).$$

Combining these comments, it follows from (8), Lemma 2.4 (iv) and (v) that

$$w = E_0 w + \frac{\langle w, E_0^\downarrow v^\downarrow \rangle}{\|E_0^\downarrow v^\downarrow\|^2} \cdot \eta_{d-\ell}(\theta_0) \sum_{\ell=t+1}^d \phi_{d-\ell+1} \cdots \phi_{d-t} \varphi_1 \cdots \varphi_j \tau_\ell(\theta_j) \left( \frac{\eta_{d-\ell}(\theta_0)}{\eta_{d-\ell}(\theta_0)} - \frac{\eta_{d-\ell}(\theta_0)}{\eta_{d-\ell}(\theta_0)} \right) E_j v^\downarrow.\]
where we have used (3) and (4). Hence

**Proposition 3.2.** Let \( w \in W_t \). Then the following hold.

(i) \( w = E_0 w + \frac{\langle w, E_0 v^* \rangle}{||E_0 v^*||^2} \cdot \frac{\eta_{d-t}(\theta_0)}{\eta_d(\theta_0)\eta_d^*(\theta_0)} \)
\( \times \sum_{j=t+1}^d \frac{\phi_{d-j+1} \cdots \phi_d}{\varphi_2 \cdots \varphi_j(\theta_j - \theta_0)} \left( \sum_{\ell=t+1}^j \frac{\tau_{\ell}(\theta_j)\eta_{\ell-1}(\theta_0)^\ell}{\phi_{\ell-t+1} \cdots \phi_{\ell-t}} \right) E_j v^* \).

(ii) \( w = E_0^* w + \frac{\langle w, E_0^* v \rangle}{||E_0^* v||^2} \cdot \frac{\eta_d(\theta_0)^*}{\eta_d^*(\theta_0)\eta_d(\theta_0)} \)
\( \times \sum_{i=d-t+1}^d \frac{\phi_1 \cdots \phi_i}{\varphi_2 \cdots \varphi_i(\theta_i^* - \theta_0^*)} \left( \sum_{\ell=d-t+1}^i \frac{\tau_{\ell}^*(\theta_i)\eta_{\ell-1}(\theta_0)^\ell}{\phi_{\ell-t+1} \cdots \phi_{\ell-t}} \right) E_i^* v. \)

In particular, \( E_0 W_t \neq 0, E_0^* W_t \neq 0, \) and \( \dim W_t = 1. \)

**Proof.** (i): Clear.

(ii): By virtue of (6), the result follows from (i) above, together with Lemma 2.3 (i) and (4).

The last line follows by noting that each of \( E_0 w, E_0^* w \) determines \( w. \) \( \Box \)

**Notation 3.3.** Henceforth we let \( q \) be a nonzero scalar in the algebraic closure \( \overline{\mathbb{K}} \) of \( \mathbb{K} \) such that \( q + q^{-1} = \beta, \) where the scalar \( \beta \) is from (2). We call \( q \) a base for \( \Phi. \)\(^6\) By convention, if \( d < 3 \) then \( q \) can be taken to be any nonzero scalar in \( \overline{\mathbb{K}}. \)

**Lemma 3.4** (cf. [18, (6.4)]). For \( 1 \leq i \leq d, \) we have \( \vartheta_i = 0 \) precisely when \( q = -1, d \) is odd, and \( i \) is even.

From Proposition 3.2 and Lemma 3.4, it follows that

**Lemma 3.5.** Let \( q \) be as above. Then for \( 1 \leq t \leq d - 1, \) the following hold.

(i) Suppose \( q \neq -1, \) or \( q = -1 \) and \( d \) is even. Then \( E_{d-t+1}^* W_t \neq 0 \) and \( E_{t+1} W_t \neq 0. \)

(ii) Suppose \( q = -1 \) and \( d \) is odd. Then \( E_{d-t+1}^* W_t \neq 0 \) (resp. \( E_{t+1} W_t \neq 0 \)) if and only if \( t \) is odd (resp. even).

**Corollary 3.6.** Let \( q \) be as above. Then the following hold.

(i) Suppose \( q \neq -1, \) or \( q = -1 \) and \( d \) is even. Then \( V = \bigoplus_{t=0}^d W_t. \)

Moreover,

\[
\sum_{t=0}^h W_t = E_0^* V + \sum_{i=d-h+1}^d E_i^* V
\]

\(^6\)We may remark that if \( d \geq 3 \) then \( \Phi \) has at most two bases, i.e., \( q \) and \( q^{-1}. \)
and
\[ \sum_{t=h}^{d} W_t = E_0 V + \sum_{j=h+1}^{d} E_j V \]
for \( 0 \leq h \leq d \).

(ii) Suppose \( q = -1 \) and \( d \) is odd. Then \( W_{2s-1} = W_{2s} \) for \( 1 \leq s \leq \lfloor d/2 \rfloor \).

**Proof.** (i): Immediate from Lemma 3.5 (i).

(ii): It follows from Lemma 3.5 (ii) that
\[ W_{2s-1} = \left( E_0^s V + \sum_{i=d-2s+2}^{d} E_i^s V \right) \cap \left( E_0 V + \sum_{j=2s+1}^{d} E_j V \right) = W_{2s} \]
for \( 1 \leq s \leq \lfloor d/2 \rfloor \). \( \square \)

By virtue of Corollary 3.6, we make the following assumption.

**Assumption 3.7.** With reference to Notation 3.3, for the rest of the paper we shall assume \( q \neq -1 \), or \( q = -1 \) and \( d \) is even.\(^7\)

We are now ready to introduce an Erdős–Ko–Rado basis of \( V \).

**Definition 3.8.** With reference to Assumption 3.7, for \( 0 \leq t \leq d \) let \( w_t \) be the (unique) vector in \( W_t \) such that \( E_0 w_t = E_0 v^* \). We call \( \{ w_t \}_{t=0}^{d} \) a (\( \Phi \)-)Erdős–Ko–Rado (or \( \Phi \)-EKR) basis of \( V \).

Notice that the basis \( \{ w_t \}_{t=0}^{d} \) linearly depends on the choice of \( v^* \in E_0 V \).

In particular, we have \( w_0 = v^* \) and \( w_d = E_0 v^* \). Our preference for the normalization \( E_0 w_t = E_0 v^* \) comes from the applications to the Erdős–Ko–Rado theorem; see Section 4. The following theorem gives the transition matrix from each of the \( \Phi \)-split basis \( \{ \tau_t(A) v^* \}^d_{t=0} \), the \( \Phi \)-standard basis \( \{ E_j v^* \}^d_{j=0} \), and the \( \Phi \)-standard basis \( \{ E_t v \}^d_{t=0} \), to the EKR basis \( \{ w_t \}^d_{t=0} \).

**Theorem 3.9.** The following hold for \( 0 \leq t \leq d \).

(i) \( w_t = \frac{\eta_{d-t}(\theta_0)}{\eta_d(\theta_0)} \left\{ \sum_{\ell=0}^{t} \frac{\eta_{d-\ell}(\theta_0)}{\eta_d(\theta_0)} \tau_{\ell}(A) v^* \right\} \)

(ii) \( w_t = E_0 v^* + \frac{\eta_{d-t}(\theta_0)}{\eta_d(\theta_0)} \left( \sum_{j=t+1}^{d} \frac{\phi_{d-j+1} \cdots \phi_d}{\varphi_2 \cdots \varphi_j (\theta_j - \theta_0)} \tau_{\ell}(A) v^* \right) E_j v^* \).

\(^7\)The Leonard systems with \( d \geq 3 \) that do not satisfy this assumption are precisely those of Bannai/Ito type [27, Example 5.14] with \( d \) odd, and those of Orphan type [27, Example 5.15].
(iii) \( w_t = \frac{\langle v, v^* \rangle}{||v||^2} \left\{ \frac{\eta_d(\theta_0)\eta_{d-t}(\theta_0)}{\phi_1 \cdots \phi_{d-t}\eta_d(\theta_0)} E_0^*v + \sum_{i=d-t+1}^d \frac{\phi_{d-t+1} \cdots \phi_i}{\varphi_2 \cdots \varphi_i(\theta_i^* - \theta_0)} \left( \sum_{\ell=d-t+1}^i \frac{\tau_\ell^*(\theta_\ell^*)\eta_{d-\ell}(\theta_0)^{-1}\eta_\ell^*(\theta_0^*)^{-1}}{\phi_{d-t+1} \cdots \phi_\ell} \right) E_i^*v \right\} \).

Proof. (i): By Lemma 2.4 (v) and since \( E_0^*w_t = E_0^*v \), we have

\[
\langle w_t, E_0^*v^* \rangle_{E_0^*v^*} = \frac{\langle w_t, E_0^*v^* \rangle}{||E_0^*v^*||^2} \cdot \frac{\langle v, v^* \rangle}{||v^*||^2} = \frac{\langle v, v^* \rangle}{||v^*||^2}.
\]

Combining this with (11), it follows that

\[
E_0^*w_t = \frac{\langle w_t, E_0^*v^* \rangle_{E_0^*v^*}}{||E_0^*v^*||^2} E_0^*v
\]

\[
= \frac{\langle w_t, E_0^*v^* \rangle_{E_0^*v^*}}{||E_0^*v^*||^2} \cdot \frac{\phi_{d-t+1} \cdots \phi_d\eta_{d-t}(\theta_0)}{\eta_d(\theta_0)\eta_d^*(\theta_0^*)} E_0^*v,
\]

from which it follows that

\[
\frac{\langle w_t, E_0^*v \rangle}{\langle v, v^* \rangle_{E_0^*v^*}} = \frac{\langle v, v^* \rangle}{\langle v, v^* \rangle_{E_0^*v^*}} \cdot \frac{\phi_{d-t+1} \cdots \phi_d\eta_{d-t}(\theta_0)}{\eta_d(\theta_0)\eta_d^*(\theta_0^*)}.
\]

Now the result follows from (8)–(10), (12), and (14).

(ii): Immediate from Corollary 3.6 (i) and \( E_0^*w_t = E_0^*v \).

(iii): Follows from Proposition 3.2 (ii), (5), and (13).

\[\square\]

Corollary 3.10. Let \( \{w_t^*\}_{t=0}^d \) be the \( \Phi^* \)-EKR basis of \( V \) normalized so that \( E_0^*w_t^* = E_0^*v(0 \leq t \leq d) \). Then

\[ w_t^* = \frac{\langle v, v^* \rangle}{||v^*||^2} \cdot \frac{\eta_d(\theta_0)\eta_{d-t}(\theta_0)}{\phi_{t+1} \cdots \phi_d\eta_d(\theta_0)} w_{d-t} \quad (0 \leq t \leq d). \]

Proof. By (6), \( w_t^* \) is a scalar multiple of \( w_{d-t} \), and the scalar is found by looking at the coefficient of \( E_0^*v \) in \( w_{d-t} \) as given in Theorem 3.9 (iii), and by noting that \( \langle v, v^* \rangle ||v^*||^{-2} = ||E_0^*v||^2 = \phi_1 \cdots \phi_d\eta_d(\theta_0)^{-1}\eta_d^*(\theta_0^*)^{-1}||v||^2 \) in view of (5).

Our next goal is to compute the transition matrix from the EKR basis \( \{w_t\}_{t=0}^d \) to each of the three bases \( \{\tau_\ell(A)v^*\}_{\ell=0}^d \), \( \{E_j^*v\}_{j=0}^d \), and \( \{E_i^*v\}_{i=0}^d \). Let \( G_t : V \to W_t(0 \leq t \leq d) \) be the projection map onto \( W_t \) with respect to \( V = \bigoplus_{t=0}^d W_t \).

Lemma 3.11. The following hold.

(i) \( G_t E_i^* = 0 \) if \( t > d - i + 1 \), or \( t > 0 \) and \( i = 0 \) \((0 \leq i, t \leq d)\).

(ii) \( G_t E_j = 0 \) if \( t < j - 1 \), or \( t < d \) and \( j = 0 \) \((0 \leq j, t \leq d)\).

Proof. Immediate from Corollary 3.6 (i).
For the moment, we write \( u = u_\ell = \tau_\ell(A)v^{*\downarrow} \in U_\ell^\uparrow \). Then it follows that
\[
G_t u = \sum_{i=d-\ell}^{d} G_t E_i^* u = \sum_{j=\ell}^{d} G_t E_j u \quad (0 \leq t \leq d).
\]

Hence it follows from Lemma 3.11 that
\[
G_t u =
\begin{cases}
G_{\ell+1} E_{d-\ell}^* u & \text{if } t = \ell + 1, \\
G_{\ell} E_{\ell} u + G_{\ell} E_{\ell+1} u & \text{if } t = \ell, \\
G_{\ell-1} E_{\ell} u & \text{if } t = \ell - 1, \\
0 & \text{if } t \leq \ell - 2 \text{ or } t \geq \ell + 2.
\end{cases}
\]

In particular:
\[
(16) \quad u = G_{\ell-1} u + G_{\ell} u + G_{\ell+1} u.
\]

By Lemma 2.4 (iv) and (v), we have
\[
E_{\ell} u = \tau_\ell(\theta_\ell) E_\ell v^{*\downarrow} = \frac{\langle v, v^{*\downarrow} \rangle}{\langle v, v^* \rangle} \phi_{d-\ell+1} \ldots \phi_d \tau_\ell(\theta_\ell) E_\ell v^*,
\]
\[
E_{\ell+1} u = \tau_\ell(\theta_{\ell+1}) E_{\ell+1} v^{*\downarrow} = \frac{\langle v, v^{*\downarrow} \rangle}{\langle v, v^* \rangle} \phi_{d-\ell} \ldots \phi_d \tau_\ell(\theta_{\ell+1}) E_{\ell+1} v^*.
\]

Likewise, by Lemma 2.4 (ii) and Lemma 2.3 (ii),
\[
(19) \quad E_{d-\ell}^* u = E_{\ell}^{*\uparrow} u
\]
\[
= \langle v^\downarrow, v^{*\downarrow} \rangle \cdot \frac{\varphi_{\ell}^{\downarrow} \ldots \varphi_{\ell}^{\downarrow}}{\tau_{\ell}^{\downarrow}(\theta_{\ell}^{\downarrow}) ||E_{\ell}^{\uparrow}\rangle ||^2} E_{\ell}^{\uparrow} v^{\downarrow}
\]
\[
= \langle v, v^{*\downarrow} \rangle \cdot \frac{\phi_{d-\ell+1} \ldots \phi_d}{\eta_{\ell}^{\downarrow}(\theta_{d-\ell}^{\downarrow}) ||E_{d-\ell}^\uparrow||^2} E_{d-\ell}^\uparrow v.
\]

Notice that the transition matrix from the basis \( E_1 v^*, \ldots, E_d v^*, E_0 v^* \) to the EKR basis \( w_0, \ldots, w_d \) is lower triangular. Hence, for fixed \( t \) with \( 0 \leq t \leq d - 2 \), if we write
\[
(E_{t+1} + E_{t+2}) w_t = a E_{t+1} v^* + b E_{t+2} v^*,
\]
\[
(E_{t+1} + E_{t+2}) w_{t+1} = c E_{t+2} v^*,
\]
then it follows that
\[
(20) \quad (G_t + G_{t+1}) E_{t+1} v^* = a^{-1} w_t - a^{-1} c^{-1} b w_{t+1},
\]
\[
(21) \quad (G_t + G_{t+1}) E_{t+2} v^* = c^{-1} w_{t+1}.
\]
By Theorem 3.9 (ii), we routinely obtain

\[ a^{-1} = -\frac{\varphi_2 \cdots \varphi_{t+1} \eta_d(\theta_0)}{\phi_{d-t+1} \cdots \phi_d \tau_1(\theta_{t+1}) \eta_{d-t-1}(\theta_0) \vartheta_{t+1}}, \]

\[ c^{-1} = -\frac{\varphi_2 \cdots \varphi_{t+2} \eta_d(\theta_0)}{\phi_{d-t} \cdots \phi_{d \tau_2}(\theta_{t+2}) \eta_{d-t-2}(\theta_0) \vartheta_{t+2}}, \]

\[ -a^{-1} c^{-1} b = \frac{\varphi_2 \cdots \varphi_{t+1} \eta_d(\theta_0)(\theta_0 - \theta_{t+1})}{\phi_{d-t} \cdots \phi_{d \tau_1}(\theta_{t+1}) \eta_{d-t-1}(\theta_0)} \times \left( \frac{\phi_{d-t-1}}{(\theta_{t+2} - \theta_{t+1}) \vartheta_{t+2}} + \frac{\theta^*_0 - \theta^*_d}{\vartheta_{t+1}} \right). \]

From (15), (17), (18), and (20)–(24), it follows that

\[ G_{\ell-1}u = \frac{\langle v, v^* \rangle}{\langle v, v^* \rangle} \cdot \frac{\phi_{d-\ell+1} \cdots \phi_{d \tau_1}(\theta_{\ell})}{\varphi_1 \cdots \varphi_\ell} G_{\ell-1} E_{\ell} v^* \]

\[ = \frac{\langle v, v^* \rangle}{\langle v, v^* \rangle} \cdot \frac{\phi_{d-\ell+1} \eta_d(\theta_0)(\theta_{\ell} - \theta_0)}{\varphi_1 \eta_{d-\ell+1}(\theta_0) \vartheta_\ell} w_{\ell-1} \]

when \( 1 \leq \ell \leq d \), and that

\[ G_{\ell}u = \frac{\langle v, v^* \rangle}{\langle v, v^* \rangle} \left( \frac{\phi_{d-\ell+1} \cdots \phi_{d \tau_1}(\theta_{\ell})}{\varphi_1 \cdots \varphi_\ell} G_{\ell} E_{\ell+1} v^* \right) \]

\[ + \frac{\phi_{d-\ell} \cdots \phi_{d \tau_1}(\theta_{\ell+1})}{\varphi_1 \cdots \varphi_{\ell+1}} G_{\ell+1} E_{\ell+1} v^* \]

\[ = \frac{\langle v, v^* \rangle}{\langle v, v^* \rangle} \cdot \frac{\eta_d(\theta_0)}{\varphi_1 \eta_{d-\ell}(\theta_0)} \left( \frac{\phi_{d-\ell}}{\vartheta_{\ell+1}} + \frac{(\theta_0 - \theta_{\ell})(\theta^*_0 - \theta^*_{d-\ell+1})}{\vartheta_{\ell}} \right) w_{\ell} \]

\[ = \frac{\langle v, v^* \rangle}{\langle v, v^* \rangle} \cdot \frac{\eta_d(\theta_0)}{\varphi_1 \eta_{d-\ell}(\theta_0)} \left( \frac{\phi_{d-\ell}}{\vartheta_{\ell+1}} + \frac{\phi_{d-\ell+1}}{\vartheta_{\ell}} - \varphi_1 \right) w_{\ell} \]

when \( 1 \leq \ell \leq d - 1 \), where the last line follows from (3) and (4). When \( \ell = 0 \) or \( \ell = d \), we interpret \( \phi_0 / \vartheta_{d+1} = \phi_{d+1} / \vartheta_0 = \varphi_1 \) in (26). Indeed, when \( \ell = 0 \), since \( G_0 E_0 u_0 = 0 \) by Lemma 3.11 (ii), it follows from (15), (18), (20), and (22) that

\[ G_0u_0 = G_0 E_1 u_0 = \frac{\langle v, v^* \rangle}{\langle v, v^* \rangle} \cdot \frac{\phi_d}{\varphi_1} G_0 E_1 v^* = \frac{\langle v, v^* \rangle}{\langle v, v^* \rangle} \cdot \frac{\phi_d}{\varphi_1} w_0. \]

When \( \ell = d \), since

\[ (E_d + E_0) w_{d-1} = -\frac{\phi_2 \cdots \phi_d \tau_d(\theta_d)}{\varphi_2 \cdots \varphi_d \eta_d(\theta_0)} E_d v^* + E_0 v^*, \]

\[ (E_d + E_0) w_d = E_0 v^* \]

by Theorem 3.9 (ii), it follows that

\[ (G_{d-1} + G_d) E_d v^* = \frac{\varphi_2 \cdots \varphi_d \eta_d(\theta_0)}{\varphi_2 \cdots \varphi_d \tau_d(\theta_d)} (-w_{d-1} + w_d), \]
so that by (15) and (17) we have
\[ G_d u_d = \frac{\langle v, v^* \rangle}{\phi_1 \ldots \phi_d \theta_d(\theta_d)} G_d E_d v^* = \frac{\langle v, v^* \rangle}{\varphi_1} \cdot \phi_1 \eta_d(\theta_0) w_d. \]

Notice that the transition matrix from the basis \( E_0^* v, E_0^* v, \ldots, E_1^* v \) to the EKR basis \( w_0, \ldots, w_d \) is upper triangular. Hence, for \( 1 \leq t \leq d \), since
\[ E_{d-t+1}^* w_t = \frac{||v||^2}{\langle v, v^* \rangle} \langle \varphi_2 \ldots \varphi_{d-t+1}(\theta_{d-t+1}^* - \theta_0^*) \rangle w_t, \]
by Theorem 3.9 (iii) and (4), it follows that
\[ G_{t} E_{d-t+1}^* v = \frac{||v||^2}{\langle v, v^* \rangle} \frac{\eta_2(\theta_0)(\theta_{d-t} - \theta_0)}{\varphi_1 \eta_{d-t-1}(\theta_0) \partial_{t+1}} w_{t+1} \]
so that by (15), (19), and (5), we have
\[
G_{\ell+1} u = \frac{\langle v, v^* \rangle}{\langle v, v^* \rangle} \cdot \frac{\phi_{d-\ell+1} \ldots \phi_d}{\eta_d(\theta_0) \theta_{d-\ell}} G_{\ell} E_{d-\ell}^* v
= \frac{\langle v, v^* \rangle}{\langle v, v^* \rangle} \cdot \frac{\eta_d(\theta_0)(\theta_{d-\ell} - \theta_0)}{\varphi_1 \eta_{d-\ell-1}(\theta_0) \partial_{\ell+1}} w_{\ell+1}
\]
when \( 0 \leq \ell \leq d - 1 \).

**Theorem 3.12.** Setting \( w_{-1} = w_{d+1} = 0 \), the following hold.\(^8\)

(i) \( \tau_\ell(A)v^* = \frac{\langle v, v^* \rangle}{\varphi_1} \cdot \frac{\eta_d(\theta_0)}{\eta_{d-\ell}(\theta_0) \theta_{\ell}} w_{\ell-1} + \frac{1}{\eta_{d-\ell}(\theta_0) \theta_{\ell+1}} \left( \phi_{d-\ell} + \frac{t_{d-\ell+1}}{\theta_{d-\ell} - \theta_0} w_{\ell+1} \right) \]
for \( 0 \leq \ell \leq d \), where we interpret \( \phi_0/\theta_{d+1} = \phi_{d+1}/\theta_0 = \varphi_1 \).

(ii) \( E_j v^* = \frac{\varphi_2 \ldots \varphi_j \eta_d(\theta_0)}{\phi_{d-j+1} \ldots \phi_d \eta_{d-j}(\theta_j)} \left\{ \phi_{d-j+1} \eta_{d-j}(\theta_j) \frac{t_{d-j}}{\eta_{d-j}(\theta_j) \theta_{j}} w_{j-1} + \left( \theta_j - \theta_0 \right) \sum_{t=j}^{d-1} \eta_{d-t-1}(\theta_j) \frac{t_{d-t} \theta_{t+1}}{\eta_{d-t}(\theta_0) \theta_{t+1}} + \left( \theta_j - \theta_{t+1} \right) \theta_{\ell} \right\} \]
for \( 1 \leq j \leq d \), and \( E_0 v^* = w_d \).

---

\(^8\)We also interpret the coefficients of \( w_{-1} \) and \( w_{d+1} \) as zero (or indeterminates), whenever these terms appear.
\[ (iii) \quad E^*_i v = \frac{\langle v, v^* \rangle}{\|v^*\|^2} \cdot \varphi_2 \cdots \varphi_i \eta_d(\theta_0) \eta^*_d(\theta_0) \left\{ \varphi_1 + \left( (\theta_1 - \theta_0)(\theta^*_i - \theta_0) \right) \w_0 \\
+ (\theta^*_i - \theta_0) \sum_{t=1}^{d-i} \frac{\eta^*_t(\theta^*_t)}{\phi_{d-t+1} \cdots \phi_{d-t+1}} \left( \frac{\phi_{d-t+1}(\theta_t)}{\eta^*_t(\theta_t)} \right) \right. \\
+ \left. \frac{\eta^*_t(\theta^*_t)(\theta^*_t - \theta_0^*_t)}{\eta^*_t(\theta^*_t)} \w_{d-i+1} \right\} \\
\text{for } 1 \leq i \leq d, \text{ and } E^*_0 v = \langle v, v^* \rangle \|v^*\|^2 \w_0. \]

Proof. (i): Immediate from (16), (25), (26), and (27).

(ii): By (i) above, Lemma 2.4 (iii) and (v), and Lemma 2.3 (ii), we have

\[ E_j v^* = \frac{\langle v, v^* \rangle}{\langle v, v^* \rangle} \cdot \varphi_2 \cdots \varphi_j \eta_d(\theta_0) \eta^*_d(\theta_0) \sum_{\ell=j}^{d} \frac{\eta_d - \ell(\theta_j)}{\tau_j(\theta_j) \eta_d - \ell(\theta_j)} \tau_j(A) v^* \]

\[ = \frac{\varphi_2 \cdots \varphi_j \eta_d(\theta_0)}{\phi_{d-j+1} \cdots \phi_{d-j+1}} \sum_{\ell=j}^{d} \eta_d - \ell(\theta_j) \left( \frac{\phi_{d-\ell+1}(\theta_0 - \theta_0)}{\eta_d - \ell(\theta_0) \theta_0 \tau_\ell + \frac{\theta^*_d - \theta_0^*_d}{\eta_d - \ell(\theta_0)} w_{\ell+1} \right) \right) \]

\[ = 1 \sum_{\ell=j}^{d} \eta_d - \ell(\theta_0) \left( \frac{\phi_{d-\ell+1}(\theta_d - \theta_0)}{\eta_d - \ell(\theta_0) \theta_0 \tau_\ell + \frac{\theta^*_d - \theta_0^*_d}{\eta_d - \ell(\theta_0)} w_{\ell+1} \right) \right) \]

for \( 1 \leq j \leq d \). Now simplify the last line using (3) and (4).

(iii): Apply “*” to (ii) above with respect to the \( \Phi^*\)-EKR basis \( \{ w^*_t \}_{t=0}^d \) with \( E^*_0 w^*_t = E^*_0 v \) (0 ≤ t ≤ d), and then use Corollary 3.10, Lemma 2.3 (i), and (4).

Finally, we shall describe the matrices representing \( A \) and \( A^* \) with respect to the EKR basis \( \{ w_t \}_{t=0}^d \). We use the following notation:

\[ \Delta_s = \frac{\eta^*_s - \eta^*_s(\theta_0)(\theta^*_d - \theta_0^*_d - \theta^*_d - \theta_0^*_d)\theta_{s+1}}{\phi_{d-s+1} \cdots \phi_{d-\theta_{s+1}} \theta_{s+1}} \quad (1 \leq s \leq d - 1). \]

Notice that

\[ \Delta^*_s = \frac{\eta^*_s - \eta^*_s(\theta_0)(\theta^*_d - \theta_0^*_d - \theta^*_d - \theta_0^*_d)\theta_{s+1}}{\phi_{d-s+1} \cdots \phi_{d-\theta_{s+1}} \theta_{s+1}} \quad (1 \leq s \leq d - 1), \]

by virtue of Theorem 2.3 (i) and (4).

Theorem 3.13. With the above notation, the following hold.

(i) \( A w_t = \theta_{t+1} w_t + \left( \frac{\phi_{d-t+1} \cdots \phi_{d-\theta_{s+1}} \theta_{s+1}}{\eta^*_s(\theta_0)} \Delta_{t+1} - (\theta_{t+1} - \theta_0) \right) w_{t+1} \)

\[ + \frac{\phi_{d-t+1} \cdots \phi_{d-\theta_{s+1}}(\theta_0)}{\eta^*_s(\theta_0)} \left\{ \sum_{s=t+2}^{d-1} (\Delta_s - \Delta_{s-1}) w_s - \Delta_{d-1} w_d \right\} \]

for \( 0 \leq t \leq d - 2 \), \( A w_{d-1} = \theta_d w_{d-1} - (\theta_d - \theta_0) w_d \), and \( A w_d = \theta_0 w_d \).
(ii) \( A^*w_t = -\frac{\phi_1 \cdots \phi_d}{\eta_d(\theta_0)} \Delta_{d-1}^* w_0 \)
\[
+ \sum_{s=1}^{t-2} \frac{\phi_1 \cdots \phi_{d-s} \eta_s^*(\theta_0)}{\eta_{d-s}(\theta_0)} (\Delta_{d-s}^* - \Delta_{d-s-1}^*) w_s \\
+ \left( \frac{\phi_1 \cdots \phi_{d-t+1} \eta_{t-1}^*(\theta_0)}{\eta_{d-t+1}(\theta_0)} \Delta_{d-t+1}^* - \frac{\phi_{d-t+1}}{\theta_i - \theta_0} \right) w_{t-1} \\
+ \theta_{d-t+1}^* w_t
\]
for \(2 \leq t \leq d\), \(A^*w_1 = \theta_1^* w_1 - (\theta_1^* - \theta_0^*) w_0\), and \(A^*w_0 = \theta_0^* w_0\).

**Proof.** (i): By Theorem 3.9 (i), (3), (4), and since \(A\tau_\ell(A) = \tau_{\ell+1}(A) + \theta_\ell \tau_\ell(A)\), we obtain

\[
A w_t = \frac{\langle v, v^* \rangle}{\langle v, v^* \rangle} \left\{ \theta_0 \sum_{\ell=0}^{t} \frac{\eta_{d-\ell}(\theta_0)}{\eta_d(\theta_0)} \tau_\ell(A) v^* \right\}
\]
\[
+ \eta_{d-\ell}(\theta_0) \sum_{\ell=0}^{t} \frac{\eta_{\ell-1}(\theta_0)}{\eta_d(\theta_0)} \tau_\ell(A) v^* \right\}
\]
\[
= \frac{\langle v, v^* \rangle}{\langle v, v^* \rangle} \left\{ \theta_0 \sum_{\ell=0}^{t} \frac{\eta_{d-\ell}(\theta_0)}{\eta_d(\theta_0)} \tau_\ell(A) v^* \right\}
\]
\[
+ \eta_{d-\ell}(\theta_0) \sum_{\ell=0}^{t} \frac{\eta_{\ell-1}(\theta_0)}{\eta_d(\theta_0)} \tau_\ell(A) v^* \right\}
\]
\[
= \theta_0 w_t + \frac{\langle v, v^* \rangle}{\langle v, v^* \rangle} \cdot \frac{\phi_1 \eta_{d-\ell}(\theta_0)}{\eta_d(\theta_0) \eta_{\ell-1}(\theta_0)} \sum_{\ell=0}^{t} \frac{\eta_{\ell-1}(\theta_0)}{\eta_d(\theta_0) \eta_{\ell-1}(\theta_0)} \tau_\ell(A) v^* \right\}
\]

Now apply Theorem 3.12 (i) and simplify the result using (3) and (4).

(ii): Apply "\(\ast\)" to (i) above with respect to the \(\Phi^\ast\)-EKR basis \(\{w_0^\ast\}_{i=0}^d\) such that \(E_0^\ast w_t^\ast = E_0^\ast v (0 \leq t \leq d)\), and then use Corollary 3.10, Lemma 2.3 (i), and (4).

We end this section with an attractive formula for \(\Delta_s\).

**Lemma 3.14.** For \(1 \leq s \leq d-1\), we have

\[
(\theta_{d-s+1} - \theta_0) \partial_{s+1} - (\theta_{d-s} - \theta_0) \partial_s
\]
\[
= \frac{(\theta_{d-[s/2]} - \theta_{[s/2]}) (\theta_{d-[([s-1]/2]} - \theta_{[([s+1]/2]})}{\theta_d - \theta_0}. \]

Corollary 3.15. For $1 \leq s \leq d - 1$, we have
\[
\Delta_s = \eta_{s-1}(\theta_0^*) (\theta_{d-[s/2]}^* - \theta_{s/2}^*) (\theta_{d-[(s-1)/2]}^* - \theta_{(s+1)/2}^*) \phi_{d-s+1} \cdots \phi_{d-s-1} (\theta_0^* - \theta_0^*) \varphi_{s+1}.
\]

Proof. Immediate from Lemma 3.14 and (4). □

4. Applications to the Erdős–Ko–Rado theorems

The Erdős–Ko–Rado type theorems for various families of $Q$-polynomial distance-regular graphs provide one of the most successful applications of Delsarte’s linear programming method [4].

Let $\Gamma$ be a $Q$-polynomial distance-regular graph with vertex set $X$. (We refer the reader to [2, 3, 21] for background material.) Pick a “base vertex” $x \in X$ and let $\Phi = \Phi(\Gamma)$ be the Leonard system (over $K = \mathbb{R}$) afforded on the primary module of the Terwilliger algebra $T(x)$; cf. [19, Example (3.5)].

The second eigenmatrix $Q = (Q_{ij})_{i,j=0}^d$ of $\Gamma$ is defined by
\[
Q_{ij} = \langle v_i^*, v_j^* \rangle \sum_{i=0}^d Q_{ij} E_i^* v_i \quad (0 \leq j \leq d).
\]

As summarized in [20], every “$t$-intersecting family” $Y \subseteq X$ is associated with a vector $e = (e_0, \ldots, e_d)$ (called the inner distribution of $Y$) satisfying
\[
e_0 = 1, \quad e_1 \geq 0, \ldots, e_{d-t} \geq 0, \quad e_{d-t+1} = \cdots = e_d = 0, \quad |Y| = (eQ)_0, \quad \text{and} \quad (eQ)_1 \geq 0, \ldots, (eQ)_d \geq 0.
\]

Viewing these as forming a linear programming maximization problem with objective function $(eQ)_0$, we are then to construct a vector $f = (f_0, \ldots, f_d)$ such that
\[
f_0 = 1, \quad f_1 = \cdots = f_t = 0, \quad \text{and} \quad (fQ^T)_1 = \cdots = (fQ^T)_{d-t} = 0,
\]
which turns out to give a feasible solution to the dual problem with objective value $(fQ^T)_0$, provided that $f_{t+1} \geq 0, \ldots, f_d \geq 0$.

Set $w = \sum_{j=0}^d f_j E_j v^*$. Then
\[
w = \frac{\langle v, v^* \rangle}{||v||^2} \sum_{j=0}^d f_j \sum_{i=0}^d Q_{ij} E_i^* v = \frac{\langle v, v^* \rangle}{||v||^2} \sum_{i=0}^d (fQ^T)_i E_i^* v.
\]

Hence it follows that $f$ satisfies (28) if and only if $w = w_1$. In particular, such a vector $f$ is unique and is given by Theorem 3.9 (ii).

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9See, e.g., [5, 15] for more applications as well as extensions of this method.
10We remark that $\Phi$ is independent of $x \in X$ up to isomorphism.
11The matrix $Q$ is denoted $P^*$ in [26, p. 264].
We now give three examples. First, suppose $\Phi$ is of dual Hahn type \cite[Example 5.12]{27}, i.e.,

$$\theta_i = \theta_0 + hi(i + 1 + s), \quad \theta'_i = \theta'_0 + s'i$$

for $0 \leq i \leq d$, and

$$\varphi_i = hs^i(i - d - 1)(i + r), \quad \phi_i = hs^i(i - d - 1)(i + r - s - d - 1)$$

for $1 \leq i \leq d$, where $h, s$ are nonzero. Then it follows that

$$f_j = \frac{(1 - j)t(j + s + 2)t(s - r + 1)(-1)^{j-1}}{(t - r + s + 1)(s + 2)t!(r + 2)_{j-1}}$$

$$\times \left. 3F_2 \right|_{t - j + 1, t + j + s + 2, 1}^{t + 1, t - r + s + 2}$$

for $t + 1 \leq j \leq d$, and

$$fQ^T_0 = \frac{(-d - s - 1)d-t}{(r - s - d)d-t}.$$

If $\Gamma$ is the Johnson graph $J(v, d)$ \cite[Section 9.1]{3}, then $\Phi$ is of dual Hahn type with $r = d - v - 1, s = -v - 2$, and $s^* = -v(v - 1)/d(v - d)$; cf. \cite[pp. 191–192]{22}. In this case, the vector $f$ was essentially constructed by Wilson \cite{29} and was used to prove the original Erdős–Ko–Rado theorem \cite{6} in full generality.

Suppose $\Phi$ is of Krawtchouk type \cite[Example 5.13]{27}, i.e.,

$$\theta_i = \theta_0 + si, \quad \theta'_i = \theta'_0 + s'i$$

for $0 \leq i \leq d$, and

$$\varphi_i = ri(i - d - 1), \quad \phi_i = (r - ss^*)i(i - d - 1)$$

for $1 \leq i \leq d$, where $r, s, s^*$ are nonzero. Then it follows that

$$f_j = \frac{(1 - j)t(j + s + 2)t(s - r + 1)(-1)^{j-1}}{t! (r - ss^*)_{j-1}}$$

$$\times \left. 2F_1 \right|_{t - j + 1, t + j + s + 2, 1}^{t + 1, t - r + s + 2}$$

for $t + 1 \leq j \leq d$, and

$$(fQ^T)_0 = \left(\frac{ss^*}{ss^* - r}\right)^{d-t}.$$

If $\Gamma$ is the Hamming graph $H(d, n)$ \cite[Section 9.2]{3}, then $\Phi$ is of Krawtchouk type with $r = n(n - 1)$ and $s = s^* = -n$; cf. \cite[p. 195]{22}. In this case, the vector $f$ coincides (up to normalization) with the weight distribution of an MDS code \cite[Chapter 11]{14}, i.e., a code attaining the Singleton bound.\footnote{In this regard, one may also wish to call $\{w_t\}_{t=0}^d$ an MDS basis or a Singleton basis.}

Finally, suppose $\Phi$ is of the most general $q$-Racah type \cite[Example 5.3]{27}, i.e.,

$$\theta_i = \theta_0 + h(1 - q^i)(1 - sq^i+1)q^{-i}, \quad \theta'_i = \theta'_0 + h^*(1 - q^i)(1 - s^*q^{i+1})q^{-i}$$
for $0 \leq i \leq d$, and

\[ \varphi_i = hh^*q^{1-2i}(1-q^i)(1-q^{i-d})(1-r_1q^i)(1-r_2q^i), \]
\[ \phi_i = hh^*q^{1-2i}(1-q^i)(1-q^{i-d})(r_1 - s^*q^i)(r_2 - s^*q^i)/s^* \]

for $1 \leq i \leq d$, where $h, h^*, r_1, r_2, s, s^*, q$ are nonzero and $r_1r_2 = ss^*q^{d+1}$.

Then it follows that the $f_j$ are expressed as balanced $4\phi_3$ series:

\[
f_j = \frac{s^{j-1}q^{d+1}(j-1)^tt^{j-1}(q^{1-j}; q_t)(sq/r_1; q)j(sq/r_2; q)j}{(1 - sq^{1 + 1}/r_1)(1 - sq^{1 + 1}/r_2)(q; q_t)(sq^2; q_t)(r_1q^2; q_t)(r_2q^2; q_t)^{-1}} \\
\times \phi_3\left( q^{t-j+1}, sq^{t+j+2}, q^{t-d-1}/s^*, q^{q^t+1}, sq^{t+2}/r_1, sq^{t+2}/r_2 \middle| q; q \right)
\]

for $t + 1 \leq j \leq d$, and

\[
(fQ)^\top_0 = \frac{(sq^{t+2}; q)_{d-t}(s^*q^2; q)_{d-t}}{r_1^{d-t}q^{d-t}(sq^{t+1}/r_1; q)_{d-t}(s^*q/r_1; q)_{d-t}}.
\]

REFERENCES


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