DETERMINATION OF THE PRIME BOUND OF A GRAPH

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ABSTRACT. Given a graph \( G \), a subset \( M \) of \( V(G) \) is a module of \( G \) if for each \( v \in V(G) \setminus M \), \( v \) is adjacent to all the elements of \( M \) or adjacent to none of them. For instance, \( V(G) \), \( \emptyset \) and \( \{v\} \ (v \in V(G)) \) are modules of \( G \). Given a graph \( G \), \( \omega_M(G) \) (respectively \( \alpha_M(G) \)) denotes the largest cardinality of a clique (respectively a stable set) in \( G \). The largest cardinality of a clique (respectively a stable) set in \( G \) is a clique (respectively a stable) set in \( G \) is prime if \( |V(G)| \geq 4 \) and if all its modules are trivial. A graph \( G \) is prime if \( |V(G)| \geq 4 \) and if all its modules are trivial.

A graph \( G \) is prime if \( |V(G)| \geq 4 \) and if all its modules are trivial. The prime bound of \( G \) is a prime graph \( G \) with \( \max(\alpha_M(G), \omega_M(G)) \geq 2 \) and \( \log_2(\max(\alpha_M(G), \omega_M(G))) \) is not an integer, \( p(G) = \lceil \log_2(\max(\alpha_M(G), \omega_M(G))) \rceil \). Then, we prove that for every graph \( G \) such that \( \max(\alpha_M(G), \omega_M(G)) = 2^k \) where \( k \geq 1 \), \( p(G) = k \) if and only if \( G \) is not complete. Moreover \( p(G) = k+1 \) if and only if \( G \) is not complete. Lastly, we show that \( p(G) = 1 \) for every non prime graph \( G \) such that \( |V(G)| \geq 4 \) and \( \alpha_M(G) = \omega_M(G) = 1 \).

1. INTRODUCTION

A graph \( G = (V(G), E(G)) \) is constituted by a finite vertex set \( V(G) \) and an edge set \( E(G) \subseteq \binom{V(G)}{2} \). Given a set finite \( S \), \( K_S = (S, \binom{S}{2}) \) is the complete graph on \( S \) whereas \((S, \emptyset)\) is the empty graph. Let \( G \) be a graph. With each \( W \subseteq V(G) \) associate the subgraph \( G[W] = (W, \binom{W}{2} \cap E(G)) \) of \( G \) induced by \( W \). Given \( W \subseteq V(G) \), \( G[V(G) \setminus W] \) is also denoted by \( G - W \) and by \( G - w \) if \( W = \{w\} \). A graph \( H \) is an extension of \( G \) if \( V(H) \supseteq V(G) \) and \( H[V(G)] = G \). Given \( p \geq 0 \), a \( p \)-extension of \( G \) is an extension \( H \) of \( G \) such that \( |V(H) \setminus V(G)| = p \). The complement of \( G \) is the graph \( \overline{G} = (V(G), \binom{V(G)}{2} \setminus E(G)) \). A subset \( W \) of \( V(G) \) is a clique (respectively a stable set) in \( G \) if \( G[W] \) is complete (respectively empty). The largest cardinality of a clique (respectively a stable set) in \( G \) is the clique number (respectively the stability number) of \( G \), denoted by \( \omega(G) \) (respectively \( \alpha(G) \)). Given \( v \in V(G) \), the neighbourhood \( N_G(v) \) of \( v \) in \( G \) is the family \( \{w \in V(G) : \{v, w\} \in E(G)\} \). We consider \( N_G \) as the function

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from \( V(G) \) to \( 2^V(G) \) defined by \( v \mapsto N_G(v) \) for each \( v \in V(G) \). A vertex \( v \) of
\( G \) is isolated if \( N_G(v) = \emptyset \). The number of isolated vertices of \( G \) is denoted by \( i(G) \).

We use the following notation. Let \( G \) be a graph. For \( v \neq w \in V(G) \),
\[
(v, w)_G = \begin{cases} 0, & \text{if } \{v, w\} \notin E(G), \\ 1, & \text{if } \{v, w\} \in E(G). \end{cases}
\]

Given \( W \subseteq V(G) \), \( v \in V(G) \setminus W \) and \( i \in \{0, 1\} \), \( (v, W)_G = i \) means \( (v, w)_G = i \)
for every \( w \in W \). Given \( W, W' \subseteq V(G) \), with \( W \cap W' = \emptyset \), and \( i \in \{0, 1\} \),
\( (W, W')_G = i \) means \( (w, W')_G = i \) for every \( w \in W \). Given \( W \not\subseteq V(G) \) and
\( v \in V(G) \setminus W \), \( v \leftrightarrow_G W \) means that there is \( i \in \{0, 1\} \) such that \( (v, W)_G = i \).
The negation is denoted by \( v \leftrightarrow_G \neg W \).

Given a graph \( G \), a subset \( M \) of \( V(G) \) is a module of \( G \) if for each \( v \in V(G) \setminus M \), we have \( v \leftrightarrow_G M \). For instance, \( V(G) \), \( \emptyset \) and \( \{v \mid v \in V(G) \} \) are modules of \( G \) called trivial. Clearly, if \( |V(G)| \leq 2 \), then all the modules of \( G \) are trivial. On the other hand, if \( |V(G)| = 3 \), then \( G \) admits a nontrivial module. A graph \( G \) is then said to be prime if \( |V(G)| \geq 4 \) and if all its
modules are trivial. For instance, given \( n \geq 4 \), the path \( \{1, \ldots, n\}, \{\{p, q\} : |p - q| = 1\} \) is prime. Given a graph \( G \), \( G \) and \( \overline{G} \) share the same modules. Thus \( G \) is prime if and only if \( \overline{G} \) is.

Given a set \( S \) with \( |S| \geq 2 \), \( K_S \) admits a prime \( \lceil \log_2(|S| + 1) \rceil \)-extension (see Sumner [8, Theorem 2.45] or Lemma 3.2 below). This is extended to any graph in [3, Theorem 3.7] and [2, Theorem 3.2] as follows.

**Theorem 1.1.** A graph \( G \), with \( |V(G)| \geq 2 \), admits a prime \( \lceil \log_2(|V(G)| + 1) \rceil \)-extension.

We now introduce the notion of prime bound. Let \( G \) be a graph. The prime bound of \( G \) is the smallest integer \( p(G) \) such that \( G \) admits a prime \( p(G) \)-extension. Observe that \( p(G) = p(\overline{G}) \) for every graph \( G \). By Theorem 1.1, \( p(G) \leq \lceil \log_2(|V(G)| + 1) \rceil \). By considering the clique number and the stability number, Brignall [3, Conjecture 3.8] conjectured the following.

**Conjecture 1.2.** For a graph \( G \) with \( |V(G)| \geq 2 \),
\[
p(G) \leq \lceil \log_2(\max(\alpha(G), \omega(G)) + 1) \rceil.
\]

We answer the conjecture positively by refining the notions of clique number and of stability number as follows. Given a graph \( G \), the modular clique number \( \omega_M(G) \) of \( G \) is the largest cardinality of a clique in \( G \) which is also a module of \( G \). The modular stability number of \( G \) is \( \alpha_M(G) = \omega_M(\overline{G}) \). The following lower bound is simply obtained.

**Lemma 1.3.** For every graph \( G \) such that \( \max(\alpha_M(G), \omega_M(G)) \geq 2 \),
\[
p(G) \geq \lceil \log_2(\max(\alpha_M(G), \omega_M(G))) \rceil.
\]

Theorem 3.2 of [2] is proved by induction on the number of vertices. Using the main arguments of this proof, we improve Theorem 1.1 as follows.
Theorem 1.4. For every graph $G$ such that $\max(\alpha_M(G), \omega_M(G)) \geq 2$,  
\[ p(G) \leq \lceil \log_2(\max(\alpha_M(G), \omega_M(G)) + 1) \rceil. \]

Theorem 1.4 is proved using an induction argument as well. A direct construction of a suitable extension is provided in [1, Theorem 2]. The following is an immediate consequence of Lemma 1.3 and Theorem 1.4.

Corollary 1.5. For every graph $G$ such that $\max(\alpha_M(G), \omega_M(G)) \geq 2$,  
\[ \lceil \log_2(\max(\alpha_M(G), \omega_M(G))) \rceil \leq p(G) \leq \lceil \log_2(\max(\alpha_M(G), \omega_M(G)) + 1) \rceil. \]

Let $G$ be graph such that $\max(\alpha_M(G), \omega_M(G)) \geq 2$. On the one hand, it follows from Corollary 1.5 that  
\[ p(G) = \lceil \log_2(\max(\alpha_M(G), \omega_M(G))) \rceil \]
when  
\[ \max(\alpha_M(G), \omega_M(G)) \notin \{2^k : k \geq 1\}. \]
On the other, if $\max(\alpha_M(G), \omega_M(G)) = 2^k$, where $k \geq 1$, then $p(G) = k$ or $k + 1$. The next theorem allows us to determine this.

Theorem 1.6. For every graph $G$ such that $\max(\alpha_M(G), \omega_M(G)) = 2^k$ where $k \geq 1$,  
\[ p(G) = k + 1 \text{ if and only if } i(G) = 2^k \text{ or } i(\overline{G}) = 2^k. \]

Lastly, we show that $p(G) = 1$ for every non prime graph $G$ such that $|V(G)| \geq 4$ and $\alpha_M(G) = \omega_M(G) = 1$ (see Proposition 5.2).

2. Preliminaries

Given a graph $G$, the family of the modules of $G$ is denoted by $\mathcal{M}(G)$. Furthermore set $\mathcal{M}_{\geq 2}(G) = \{M \in \mathcal{M}(G) : |M| \geq 2\}$. We begin with the well known properties of the modules of a graph (for example, see [4, Theorem 3.2, Lemma 3.9]).

Proposition 2.1. Let $G$ be a graph.

1. Given $W \subseteq V(G)$,  
\[ \{M \cap W : M \in \mathcal{M}(G)\} \subseteq \mathcal{M}(G[W]). \]
2. Given a module $M \in \mathcal{M}(G)$,  
\[ \mathcal{M}(G[M]) = \{N \in \mathcal{M}(G) : N \subseteq M\}. \]
3. Given $M, N \in \mathcal{M}(G)$ with $M \cap N = \emptyset$, there is $i \in \{0, 1\}$ such that  
\[ (M, N)_G = i. \]

Given a graph $G$, a partition $P$ of $V(G)$ is a modular partition of $G$ if $P \subseteq \mathcal{M}(G)$. Let $P$ be such a partition. Given $M \neq N \in P$, there is $i \in \{0, 1\}$ such that $(M, N)_G = i$ by (3) of Proposition 2.1. This justifies the following definition: The quotient of $G$ by $P$ is the graph $G/P$ defined on $V(G/P) = P$ by $(M, N)_{G/P} = (M, N)_G$ for $M \neq N \in P$. We use the following properties of the quotient (for example, see [4, Theorems 4.1–4.3, Lemma 4.1]).

Proposition 2.2. Given a graph $G$, consider a modular partition $P$ of $G$.

1. Given $W \subseteq V(G)$, if $|W \cap X| = 1$ for each $X \in P$, then $G[W]$ and $G/P$ are isomorphic.
(2) For every $M \in \mathcal{M}(G)$, \{ $X \in P : M \cap X \neq \emptyset$ \} $\in \mathcal{M}(G/P)$.

(3) For every $Q \in \mathcal{M}(G/P)$, $\cup Q \in \mathcal{M}(G)$.

The following strengthening of the notion of module is introduced to present the modular decomposition theorem (see Theorem 2.4 below). Given a graph $G$, a module $M$ of $G$ is said to be strong provided that for every $N \in \mathcal{M}(G)$, if $M \cap N \neq \emptyset$, then $M \subseteq N$ or $N \subseteq M$. The family of the strong modules of $G$ is denoted by $\mathcal{S}(G)$. Furthermore set

$$S_{=2}(G) = \{ M \in \mathcal{S}(G) : |M| \geq 2 \}.$$ 

We recall the following well known properties of the strong modules of a graph (for example, see [4, Theorem 3.3]).

**Proposition 2.3.** Let $G$ be a graph. For every $M \in \mathcal{M}(G)$,

$$\mathcal{S}(G[M]) = \{ N \in \mathcal{S}(G) : N \not\subseteq M \} \cup \{ M \}.$$ 

With each graph $G$, we associate the family $\Pi(G)$ of the maximal proper and nonempty strong modules of $G$ under inclusion. For convenience set

$$\Pi_1(G) = \{ M \in \Pi(G) : |M| = 1 \} \text{ and } \Pi_{=2}(G) = \{ M \in \Pi(G) : |M| \geq 2 \}.$$ 

The modular decomposition theorem is stated as follows.

**Theorem 2.4** (Gallai [5, 6]). For a graph $G$ with $|V(G)| \geq 2$, the family $\Pi(G)$ realizes a modular partition of $G$. Moreover, the corresponding quotient $G/\Pi(G)$ is complete, empty or prime.

Let $G$ be a graph with $|V(G)| \geq 2$. As a direct consequence of the definition of a strong module, we obtain that the family $\mathcal{S}(G) \setminus \{ \emptyset \}$ endowed with inclusion is a tree called the modular decomposition tree [7] of $G$. Given $M \in \mathcal{S}_{=2}(G)$, it follows from Proposition 2.3 that $\Pi(G[M]) \subseteq \mathcal{S}(G)$. Furthermore, given $W \subseteq V(G)$, the family $\{ M \in \mathcal{S}(G) : M \supseteq W \}$ endowed with inclusion is a total order. Its smallest element is denoted by $\overline{W}$.

Let $G$ be a graph with $|V(G)| \geq 2$. Using Theorem 2.4, we label $S_{=2}(G)$ by the function $\lambda_G$ defined as follows. For each $M \in S_{=2}(G)$,

$$\lambda_G(M) = \begin{cases} 
\blacklozenge & \text{if } G[M]/\Pi(G[M]) \text{ is complete}, \\
\circ & \text{if } G[M]/\Pi(G[M]) \text{ is empty}, \\
\blacklozenge & \text{if } G[M]/\Pi(G[M]) \text{ is prime}.
\end{cases}$$

### 3. Some prime extensions

**Lemma 3.1.** Let $S$ and $S'$ be disjoint and finite sets such that $|S| \geq 2$ and $|S'| = \lceil \log_2(|S| + 1) \rceil$. There exists a prime graph $G$ defined on $V(G) = S \cup S'$ such that $S$ and $S'$ are stable sets in $G$.

**Proof.** If $|S| = 2$, then $|S'| = 2$ and we can choose a path on 4 vertices for $G$. Assume that $|S| \geq 3$. As $|S'| = \lceil \log_2(|S| + 1) \rceil$, $2^{|S'| - 1} \leq |S|$ and hence $|S'| \leq |S|$. Thus there exists a bijection $\psi_{S'}$ from $S'$ onto $S'' \subseteq S$. Consider the injection $f_{S''} : S'' \rightarrow 2^{|s'| - \{\emptyset\}}$ defined by $s'' \mapsto S' \setminus \{(\psi_{S'})^{-1}(s'')\}$. Since
\[ |S'| = \lfloor \log_2(|S| + 1) \rfloor, \quad |S| < 2^{|S'|} \] and there exists an injection \( f_S \) from \( S \) into \( 2^{S'} \setminus \{\emptyset\} \) such that \( (f_S)_{|S''} = f_{S''} \). Lastly, consider the graph \( G \) defined on \( V(G) = S \cup S' \) such that \( S' \) are stable sets in \( G \) and \( (N_G)_{|S} = f_S \). We prove that \( G \) is prime. If \( |S'| = 3 \), then \( |S'| = 2 \) and \( G \) is a path on 5 vertices which is prime. Assume that \( |S'| \geq 4 \) and hence \( |S'| \geq 3 \). Let \( M \in \mathcal{M}_2(G) \).

First, if \( M \subseteq S \), then we would have \( f_S(u) = f_S(v) \) for any \( u \neq v \in M \). Thus \( M \cap S' = \emptyset \).

Second, suppose that \( M \subseteq S' \). Recall that for each \( s \in S \), either \( M \cap N_G(s) = \emptyset \) or \( M \subseteq N_G(s) \). Given \( u \in M \), consider the function \( f : S \rightarrow 2^{(S \setminus M) \cup \{u\}} \setminus \{\emptyset\} \) defined by

\[
 f(s) = \begin{cases} 
 N_G(s), & \text{if } M \cap N_G(s) = \emptyset, \\
 (N_G(s) \setminus M) \cup \{u\}, & \text{if } M \subseteq N_G(s), 
\end{cases}
\]

for every \( s \in S \). Since \( (N_G)_{|S} \) is injective, \( f \) is also and we would obtain that \( |S| < 2^{|S'|} \). It follows that \( M \cap S = \emptyset \).

Third, suppose that \( S' \setminus M = \emptyset \). We have \( (S \cap M, S' \setminus M)_G = (S' \cap M, S' \setminus M)_G = 0 \). Given \( s' \in S' \cap M \), \( N_G(\psi_{S'}(s')) = S' \setminus \{s'\} \). In particular \( S' \cap M \subseteq N_G(\psi_{S'}(s')) \). Furthermore \( (\psi_{S'}(s'), S' \cap M)_G = (\psi_{S'}(s'), S \cap M)_G = 0 \). Therefore \( S' \cap M = \{s'\} \). Similarly, we prove that \( |S' \cap M| = 1 \) which would imply that \( |S'| = 2 \). It follows that \( S' \subseteq M \).

Lastly, suppose that \( S \setminus M = \emptyset \). For each \( s \in S \setminus M = \emptyset \), we have \( (s, S')_G = (s, S \cap M)_G = 0 \) and hence \( N_G(s) = \emptyset \). It follows that \( S \subseteq M \) and \( M = S \cup S' \).

**Lemma 3.2.** Let \( C \) and \( S' \) be disjoint and finite sets such that \( |C| \geq 2 \) and \( |S'| = \lfloor \log_2(|C| + 1) \rfloor \). There exists a prime graph \( G \) defined on \( V(G) = C \cup S' \) such that \( C \) is a clique and \( S' \) is a stable set in \( G \).

**Proof.** There exists a bijection \( \psi_{S'} \) from \( S' \) onto \( S'' \subseteq C \). Consider the injection \( f_{S''} : S'' \rightarrow 2^{S'} \setminus \{S'\} \) defined by \( s'' \mapsto \{(\psi_{S'})^{-1}(s'')\} \). Let \( f_C \) be any injection from \( C \) into \( 2^{S'} \setminus \{S'\} \) such that \( (f_C)_{|S''} = f_{S''} \). Lastly, consider the graph \( G \) defined on \( V(G) = C \cup S' \) such that \( C \) is a clique in \( G \), \( S' \) is a stable set in \( G \) and \( N_G(c) \cap S'' = f_C(c) \) for each \( c \in C \). We prove that \( G \) is prime. Let \( M \in \mathcal{M}_2(G) \). As in the proof of Lemma 3.1, we have \( M \cap C = \emptyset \) and \( M \cap S' = \emptyset \).

Now, suppose that \( C \setminus M \neq \emptyset \). We have \( (C \cap M, S' \setminus M)_G = (S' \cap M, S' \setminus M)_G = 0 \). Given \( t' \in S' \setminus M \), \( N_G(\psi_{S'}(t')) \cap S' = \{t'\} \). Thus \( \psi_{S'}(t') \in C \setminus M \). But \( (\psi_{S'}(t'), S' \setminus M)_G = (\psi_{S'}(t'), C \setminus M)_G = 1 \) which contradicts \( N_G(\psi_{S'}(t')) \cap S' = \{t'\} \). It follows that \( S'' \subseteq M \).

Lastly, suppose that \( C \setminus M = \emptyset \). For each \( c \in C \setminus M \neq \emptyset \), we have \( (c, S')_G = (c, C \cap M)_G = 1 \) and hence \( N_G(c) \cap S' = S' \). It follows that \( C \subseteq M \) and \( M = C \cup S' \).

The question of prime extensions of a prime graph is not detailed enough in [2]. For instance, the number of prime 1-extensions of a prime graph
given in [2] is not correct. Moreover, Corollary 3.4 below is used without a precise proof.

Lemma 3.3. Let $G$ be a prime graph. Given $a \notin V(G)$, there exist exactly

$$2^{|V(G)|} - 2|V(G)| - 2$$

distinct prime extensions of $G$ to $V(G) \cup \{a\}$.

Proof. Consider any graph $H$ defined on $V(H) = V(G) \cup \{a\}$ such that $H[V(G)] = G$. We prove that $H$ is not prime if and only if

$$N_H(a) \in \{\emptyset, V(G)\} \cup \{N_G(v) : v \in V(G)\} \cup \{N_G(v) \cup \{v\} : v \in V(G)\}.$$

To begin, assume that $N_H(a) \in \{\emptyset, V(G)\} \cup \{N_G(v) : v \in V(G)\} \cup \{N_G(v) \cup \{v\} : v \in V(G)\}$. If $N_H(a) = \emptyset$ or $V(G)$, then $V(G)$ is a nontrivial module of $H$. If there is $v \in V(G)$ such that $N_H(a) \setminus \{v\} = N_G(v)$, then $\{a, v\}$ is a nontrivial module of $H$.

Conversely, assume that $H$ admits a nontrivial module $M$. By Proposition 2.1,(1), $M \setminus \{a\} \in \mathcal{M}(G)$. As $G$ is prime, $M \setminus \{a\} \neq \emptyset$ and $M \not\subseteq V(H)$, either $|M \setminus \{a\}| = 1$ or $M = V(G)$. In the second instance, $N_H(a) = \emptyset$ or $V(G)$. In the first, there is $v \in V(G)$ such that $M = \{a, v\}$. Thus $N_H(a) = N_G(v)$ or $N_G(v) \cup \{v\}$. To conclude, observe that

$$|\emptyset, V(G)\} \cup \{N_G(v) : v \in V(G)\} \cup \{N_G(v) \cup \{v\} : v \in V(G)\}| = 2^{|V(G)|} - 2|V(G)| - 2$$

because $G$ is prime.

□

Corollary 3.4. Let $G$ be a prime graph. For any $a \neq b \notin V(G)$, there exists a prime extension $H$ of $G$ to $V(G) \cup \{a, b\}$ such that $(a, b)_H = 0$.

Proof. Since $|V(G)| \geq 4$, $2^{|V(G)|} - 2|V(G)| - 2 \geq 2$. Consequently there is an extension $H$ of $G$ to $V(G) \cup \{a, b\}$ such that $(a, b)_H = 0$, $N_H(a) \neq N_H(b)$ and

$$N_H(a), N_H(b) \notin \{\emptyset, V(G)\} \cup \{N_G(v) : v \in V(G)\} \cup \{N_G(v) \cup \{v\} : v \in V(G)\}.$$

By the proof of Lemma 3.3, $H - a$ and $H - b$ are prime. We show that $H$ is prime also. Let $M \in \mathcal{M}(H - a)$. By Proposition 2.1,(1), $M \setminus \{a\} \in \mathcal{M}(H - a)$. As $H - a$ is prime and $M \setminus \{a\} \neq \emptyset$, either $|M \setminus \{a\}| = 1$ or $M \setminus \{a\} = V(H) \setminus \{a\}$. In the first, there is $v \in V(G) \cup \{b\}$ such that $M = \{a, v\}$. If $v = b$, then $N_H(a) = N_H(b)$. If $v \in V(G)$, then $\{a, v\}$ would be a nontrivial module of $H - b$. Consequently $M \setminus \{a\} = V(H) \setminus \{a\}$. Since $H - b$ is prime, $a \leftrightarrow_H V(G)$ and hence $a \in M$. Thus $M = V(H)$. □

4. Proof of Theorem 1.4

Let $G$ be a graph with $|V(G)| \geq 2$. By [2, Theorem 3.2], there exists a prime extension $H$ of $G$ such that

$$2 \leq |V(H) \setminus V(G)| \leq \lceil \log_2(|V(G)| + 1) \rceil$$
and \( V(H) \setminus V(G) \) is a stable set in \( H \). We can consider the smallest integer \( q(G) \) such that \( q(G) \geq 2 \) and \( G \) admits a prime \( q(G) \)-extension \( H \) such that \( V(H) \setminus V(G) \) is a stable set in \( H \).

The results below, from Proposition 4.1 to Corollary 4.4, are suggested by the proof of [2, Theorem 3.2].

We introduce a basic construction. Consider a graph \( G \) and a modular partition \( P \) of \( G \) such that \( P \subseteq S(G) \) and \( P \cap S_{22}(G) \neq \emptyset \). Let \( X \in P \cap S_{22}(G) \) such that

\[
q(G[X]) = \max(\{q(G[Y]) : Y \in P \cap S_{22}(G)\}).
\]

Consider a set \( S \) such that \( S \cap V(G) = \emptyset \) and \( |S| = q(G[X]) \). There exists a prime \( q(G[X]) \)-extension \( H_X \) of \( G[X] \) to \( X \cup S \) such that \( S \) is a stable set in \( H_X \). Since \( X \) is not a module of \( H_X \), there is \( s_X \in S \) such that \( s_X \leftrightarrow H_X X \).

Furthermore, if there is \( v \in S \) such that \( (v, X)_{H_X} = 0 \), then \( V(H_X) \setminus \{v\} \) would be a nontrivial module of \( H_X \). Thus \( \{v \in S : v \leftrightarrow H_X X\} = \{v \in S : (v, X)_{H_X} = 1\} \). As \( S \) is a stable set in \( H_X \), \( \{v \in S : (v, X)_{H_X} = 1\} \) is a module of \( H_X \). It follows that

\[
\begin{align*}
\{v \in S : v \leftrightarrow H_X X\} &= \{v \in S : (v, X)_{H_X} = 1\}, \\
\{v \in S : v \leftrightarrow H_X X\} &\leq 1, \\
s_X &\in S \setminus \{v \in S : v \leftrightarrow H_X X\}.
\end{align*}
\]

Now, for each \( Y \in (P \cap S_{22}(G)) \setminus \{X\} \), there is a prime \( q(G[Y]) \)-extension \( H_Y \) of \( G[Y] \) to \( Y \cup S_Y \) such that \( \{v \in S : v \leftrightarrow H_X X\} \subseteq S_Y \subseteq S \) and \( S_Y \) is a stable set in \( H_Y \). Consider the extension \( H \) of \( G \) and of \( H_X \) to \( V(G) \cup S \) satisfying

- for each \( Y \in (P \cap S_{22}(G)) \setminus \{X\} \), \( H[Y \cup S_Y] = H_Y \);
- for each \( v \in V(G) \) such that \( \{v\} \subseteq P \), \( (v, S \setminus \{s_X\})_H = 0 \) and \( (v, s_X)_H = 1 \).

**Proposition 4.1.** Given a graph \( G \), consider a modular partition \( P \) of \( G \) such that \( P \subseteq S(G) \) and \( P \cap S_{22}(G) \neq \emptyset \). If the corresponding extension \( H \) is not prime, then all the nontrivial modules of \( H \) are included in \( \{v \in V(G) : \{v\} \subseteq P\} \).

**Proof.** Let \( M \) be a nontrivial module of \( H \). By Proposition 2.1.(1), \( M \cap (X \cup S) \in \mathcal{M}(H[X \cup S]) \). Since \( H[X \cup S] \) is prime, we have \( M \supseteq X \cup S \), \( |M \cap (X \cup S)| = 1 \), or \( M \cap (X \cup S) = \emptyset \).

For a first contradiction, suppose that \( M \supseteq X \cup S \). Given \( v \in V(G) \), if \( \{v\} \subseteq P \), then \( v \leftrightarrow H S \) so that \( v \notin M \). Thus \( \{v \in V(G) : \{v\} \subseteq P\} \subseteq M \). Let \( Y \in P \cap S_{22}(G) \). By Proposition 2.1.(1), \( M \cap (Y \cup S_Y) \in \mathcal{M}(H[Y \cup S_Y]) \).

Since \( H[Y \cup S_Y] \) is prime and since \( S_Y \subseteq M \cap (Y \cup S_Y) \), \( Y \subseteq M \). Therefore \( \bigcup(P \cap S_{22}(G)) \subseteq M \) and we would have \( M = V(H) \).

For a second contradiction, suppose that \( |M \cap (X \cup S)| = 1 \). Consider \( v \in S \setminus X \) such that \( M \cap (X \cup S) = \{v\} \). Suppose that \( v \in X \). We have \( M \subseteq V(G) \) and \( M \in \mathcal{M}(G) \) by Proposition 2.1.(1). As \( X \subseteq S(G) \) and \( v \in X \cap M, X \subseteq M \) or \( M \subseteq X \). In both cases, we would have \( |M \cap (X \cup S)| \geq 2 \).
Suppose that \( v \in S \). There is \( Y \in P \setminus \{X\} \) such that \( Y \cap M \neq \emptyset \). Let \( y \in Y \cap M \).

Since \( y \not\in_G X, u \not\in_{H_S} X \) and hence \( v \neq s_X \). If \( y \in P \cap \mathcal{S}_{G_2}(G) \), then \( v \in S_Y \) and \( M \cap (Y \cup S_Y) \) would be a nontrivial module of \( H[Y \cup S_Y] \). If \( Y = \{y\} \), then \( (y, s_X)_H = 1 \). Thus \( (v, s_X)_H = 1 \) and \( S \) would not be a stable set in \( H \).

It follows that \( M \cap (X \cup S) = \emptyset \). By Proposition 2.1.(1), \( M \in \mathcal{M}(G) \).

Suppose for a contradiction that there is \( Y \in (P \cap \mathcal{S}_{G_2}(G) \setminus \{X\}) \) such that \( Y \cap M \neq \emptyset \). As \( Y \in \mathcal{S}(G) \), \( Y \subseteq M \) or \( M \subseteq Y \). In both cases, \( M \cap (Y \cup S_Y) \) would be a nontrivial module of \( H[Y \cup S_Y] \). It follows that \( Y \cap M = \emptyset \).

Therefore \( M \subseteq \{v \in V(G) : \{v\} \in P\} \).

**Corollary 4.2.** Given a graph \( G \) such that \( G/\Pi(G) \) is prime, we have

\[
q(G) \leq \begin{cases} 
2, & \text{if } \Pi_{G_2}(G) = \emptyset, \\
\max\{q(G[X]) : X \in \Pi_{G_2}(G)\}, & \text{if } \Pi_{G_2}(G) \neq \emptyset.
\end{cases}
\]

**Proof.** If \( G \) is prime, then \( q(G) = 2 \) by Corollary 3.4. Assume that \( G \) is not prime, that is, \( \Pi_{G_2}(G) \neq \emptyset \). Let \( H \) be the extension of \( G \) associated with \( \Pi(G) \). Suppose that \( H \) admits a nontrivial module \( M \). By Proposition 4.1, \( \{\{u\} : u \in M\} \in \Pi_1(G) \). Thus \( M \in \mathcal{M}(G) \) by Proposition 2.1.(1). By Proposition 2.2.(2), \( \{\{u\} : u \in M\} \) would be a nontrivial module of \( G/\Pi(G) \).

**Proposition 4.3.** Given a graph \( G \) such that \( G/\Pi(G) \) is complete or empty, we have

\[
q(G) \leq \max(2, \lceil \log_2(|\Pi_1(G)| + 1) \rceil),
\]

or

\[
q(G) \leq \max\{q(G[X]) : X \in \Pi_{G_2}(G)\}.
\]

**Proof.** Assume that \( G/\Pi(G) \) is empty. If \( \Pi(G) = \Pi_1(G) \), then \( G \) is empty by Proposition 2.2.(1), and it suffices to apply Lemma 3.1. Assume that \( \Pi_{G_2}(G) \neq \emptyset \) and set

\[
W_2 = \bigcup \Pi_{G_2}(G).
\]

Let \( H \) be the extension of \( G \) associated with \( \Pi(G) \). Recall that \( V(H) = V(G) \cup S, V(G) \cap S = \emptyset \) and \( |S| = q(G[X]) \) where \( X \in \Pi_{G_2}(G) \) such that \( q(G[X]) = \max\{q(G[Y]) : Y \in \Pi_{G_2}(G)\} \). Moreover \( H[X \cup S] \) is prime.

If \( |\Pi_1(G)| \leq 1 \), then \( H \) is prime by Proposition 4.1 so that \( q(G) \leq \max(\{q(G[Y]) : Y \in \Pi_{G_2}(G)\}) \). Assume that \( |\Pi_1(G)| \geq 2 \) and set

\[
W_1 = V(G) \setminus W_2.
\]

By Lemma 3.1, there exists a prime extension \( H_1 \) of \( G[W_1] \) to \( W_1 \cup S_1 \) such that \( |S_1| = \lceil \log_2(|W_1| + 1) \rceil \) and \( S_1 \) is stable in \( H_1 \). As \( G/\Pi(G) \) is empty, \( \Pi_{G_2}(G) \in \mathcal{M}(G/\Pi(G)) \). By Proposition 2.2.(3), \( W_2 \in \mathcal{M}(G) \). Thus \( \Pi_{G_2}(G) \subseteq \mathcal{S}(G[W_2]) \) by Proposition 2.3. It follows from Proposition 4.1 that \( H[W_2 \cup S] \) is prime. We construct suitable extensions of \( G \) according to whether \( |S_1| \leq |S| \) or not.
To begin, suppose \(|S_1| \leq |S|\). We can assume that
\[ \{v \in S : v \leftrightarrow_{H[X \cup S]} X \} \subseteq S_1 \subseteq S \]
and consider an extension \(H'\) of \(H_1\) and \(H[W_2 \cup S]\) to \(V(G) \cup S\). We show that \(H'\) is prime. Let \(M \in \mathcal{M}_{22}(H')\). By Proposition 2.1.(1), \(M \cap (W_2 \cup S) \in \mathcal{M}(H[W_2 \cup S])\). Since \(H[W_2 \cup S]\) is prime, \(M \cap (W_2 \cup S) = \emptyset\), \(|M \cap (W_2 \cup S)| = 1\) or \(M \supseteq (W_2 \cup S)\).

- Suppose for a contradiction that \(M \cap (W_2 \cup S) = \emptyset\). By Proposition 2.1.(1), \(M\) would be a nontrivial module of \(H_1\).
- Suppose for a contradiction that \(|M \cap (W_2 \cup S)| = 1\) and consider \(w \in W_2 \cup S\) such that \(M \cap (W_2 \cup S) = \{w\}\). First, suppose that \(w \in W_2\) and consider \(Y \in \Pi_{22}(G)\) such that \(w \in Y\). By Proposition 2.1.(1), \(M \in \mathcal{M}(G)\). As \(Y \subseteq S(G)\) and \(w \in X \cap M, X \subseteq M\) or \(M \subseteq X\). In both cases, we would have \(|M \cap (W_2 \cup S)| \geq 2\). Second, suppose that \(w \in S\) and consider \(v \in W_1 \cap M\). Since \(v \leftrightarrow_{G} X, w \leftrightarrow_{H[W_2 \cup S]} X\) and hence \(w \in S_1\). It follows from Proposition 2.1.(1) that \(M\) would be a nontrivial module of \(H_1\).

Consequently \(M \supseteq (W_2 \cup S)\). By Proposition 2.1.(1), \(M \cap (W_1 \cup S_1) \in \mathcal{M}(H_1)\).

As \(H_1\) is prime and \(M \cap (W_1 \cup S_1) \subseteq S_1\), \(M \cap (W_1 \cup S_1) = (W_1 \cup S_1)\) so that \(M = V(H')\).

Now, assume that \(|S_1| > |S|\). We can assume that \(S \not\subseteq S_1\) and we consider the unique extension \(H''\) of \(H_1\) and \(H[W_2 \cup S]\) to \(V(G) \cup S_1\) such that
\[
(W_2, S_1 \setminus S)_{H''} = 0.
\]
We show that \(H''\) is prime. Let \(M \in \mathcal{M}_{22}(H'')\). We obtain \(M \cap (W_1 \cup S_1) = \emptyset\), \(|M \cap (W_1 \cup S_1)| = 1\) or \(M \supseteq (W_1 \cup S_1)\). If \(M \cap (W_1 \cup S_1) = \emptyset\), then \(M\) would be a nontrivial module of \(H[W_2 \cup S]\).

Suppose for a contradiction that \(|M \cap (W_1 \cup S_1)| = 1\) and consider \(w \in W_1 \cup S_1\) such that \(M \cap (W_1 \cup S_1) = \{w\}\). There is \(v \in W_2 \cap M\). Let \(Y \in \Pi_{22}(G)\) such that \(v \in Y\).

- Suppose that \(w \in W_1\). By Proposition 2.1.(1), \(M \in \mathcal{M}(G)\). Since \(Y \subseteq S(G)\) and since \(\cap M \neq \emptyset\) and \(w \in M \setminus Y, Y \subseteq M\). It follows from Proposition 2.1.(1) that \(M \cap (W_2 \cup S)\) would be a nontrivial module of \(H[W_2 \cup S]\).
- Suppose that \(w \in S_1\). By Proposition 2.1.(1), \(M \cap (W_2 \cup S) \in \mathcal{M}(H[W_2 \cup S])\). As \(H[W_2 \cup S]\) is prime, \(v \in M \cap W_2\) and \(M \cap S \subseteq \{w\}\), \(M \cap (W_2 \cup S) = \{v\}\) hence \(w \in S_1 \setminus S\). For every \(u \in W_2 \setminus \{v\}\), we have \((u,v)_{G} = (u,w)_{H''} = 0\) by (4.1). Since \((v,W_1)_{G} = 0\), we would have \(N_{G}(v) = \emptyset\) and hence \(\{v\} \in \Pi_{1}(G)\).

It follows that \(M \supseteq (W_1 \cup S_1)\). By Proposition 2.1.(1), \(M \cap (W_2 \cup S) \in \mathcal{M}(H[W_2 \cup S])\). As \(H[W_2 \cup S]\) is prime and \(M \cap (W_2 \cup S) \subseteq S\), \(M \cap (W_2 \cup S) = (W_2 \cup S)\) so that \(M = V(H'')\).

Finally, observe that when \(G/\Pi(G)\) is complete, we can proceed as previously by replacing (4.1) by \((W_2, S_1 \setminus S)_{H''} = 1\). \(\square\)
The next result follows from Corollary 4.2 and Proposition 4.3 by induction on the number of vertices.

**Corollary 4.4.** Given a graph $G$ with $|V(G)| \geq 2$,
- $q(G) = 2$ if for every $X \in \mathcal{S}_2(G)$ such that $\lambda_G(X) \in \{\circ, \bullet\}$, we have $|\Pi_1(G[X])| \leq 1$;
- $q(G) \leq \max\{|\log_2(|\Pi_1(G[Y])| + 1)| : Y \in \mathcal{S}_2(G), \lambda_G(Y) \in \{\circ, \bullet\}\}$ if there is $X \in \mathcal{S}_2(G)$ such that $\lambda_G(X) \in \{\circ, \bullet\}$ and $|\Pi_1(G[X])| \geq 2$.

Given the second assertion of Corollary 4.4, Theorem 1.4 follows from the next transcription in terms of the modular decomposition tree. Let $G$ be a graph. Denote by $\mathcal{M}(G)$ the family of the maximal elements of $\mathcal{M}_2(G)$ under inclusion which are cliques or stable sets in $G$.

**Proposition 4.5.** Let $G$ be a graph. Given $M \subseteq V(G)$, we have $M \in \mathcal{M}(G)$ if and only if $M \in \mathcal{M}_2(G)$, $\lambda_G(M) \in \{\circ, \bullet\}$ and $M = \{v \in \overline{M} : \{v\} \in \Pi(G[\overline{M}])\}$.

Proof. To begin, consider $M \in \mathcal{M}(G)$ and assume that $M$ is a stable set in $G$. By Proposition 2.1.(1), $M \in \mathcal{M}(G[\overline{M}])$. Set

$$Q = \{X \in \Pi(G[\overline{M}]) : X \cap M \neq \emptyset\}.$$

By definition of $\overline{M}$, $|Q| \geq 2$ and hence $M = \bigcup Q$ because $Q \subseteq \mathcal{S}(G[\overline{M}])$. Furthermore, $Q \subseteq \mathcal{S}(G[M])$ by Proposition 2.3. As all the strong modules of an empty graph are trivial, we obtain $|X| = 1$ for each $X \in Q$, that is,

$$M \subseteq \{v \in \overline{M} : \{v\} \in \Pi(G[\overline{M}])\}.$$

By Proposition 2.2.(2), $Q \in \mathcal{M}(G[\overline{M}]/\Pi(G[\overline{M}]))$. For a contradiction, suppose that $\lambda_G(\overline{M}) = \emptyset$. Since $Q \in \mathcal{M}_2(G[\overline{M}]/\Pi(G[\overline{M}]))$, $Q \subseteq \Pi(G[\overline{M}])$ and hence $M = \overline{M}$. As $|X| = 1$ for each $X \in Q$, $G[\overline{M}]/\Pi(G[\overline{M}])$ and $G[\overline{M}]$ are isomorphic by Proposition 2.2.(1). It would follow that $G[M]$ is prime. Consequently $\lambda_G(\overline{M}) \in \{\circ, \bullet\}$. Given $v \neq w \in M$, we have $(\{v\}, \{w\})_{G[\overline{M}]/\Pi(G[\overline{M}])} = (v, w)_G = 0$. Thus

$$\lambda_G(\overline{M}) = \emptyset.$$

Since $\lambda_G(\overline{M}) = \emptyset$, we have $\Pi_1(G[\overline{M}]) \notin \mathcal{M}(G[\overline{M}]/\Pi(G[\overline{M}]))$. By Proposition 2.2.(3), $\bigcup \Pi_1(G[\overline{M}]) \notin \mathcal{M}(G[\overline{M}]/\Pi(G[\overline{M}])$ and hence $\bigcup \Pi_1(G[\overline{M}]) \notin \mathcal{M}(G)$ by Proposition 2.1.(2). Given $v \neq w \in \bigcup \Pi_1(G[\overline{M}])$, we have

$$(v, w)_G = (\{v\}, \{w\})_{G[\overline{M}]/\Pi(G[\overline{M}])} = 0.$$

Therefore $\bigcup \Pi_1(G[\overline{M}])$ is a stable set of $G$. As $M \subseteq \bigcup \Pi_1(G[\overline{M}])$, $M = \bigcup \Pi_1(G[\overline{M}])$ by maximality of $M$. It follows that

$$M \subseteq \{v \in \overline{M} : \{v\} \in \Pi(G[\overline{M}])\}.$$

Conversely, consider $M \in \mathcal{M}_2(G)$ such that $\lambda_G(\overline{M}) = \emptyset$ and $M = \{v \in \overline{M} : \{v\} \in \Pi(G[\overline{M}])\}$. As $\lambda_G(\overline{M}) = \emptyset$, $\Pi_1(G[\overline{M}]) \notin \mathcal{M}(G[\overline{M}]/\Pi(G[\overline{M}]))$. However, $\bigcup \Pi_1(G[\overline{M}])$ is a stable set in $G$. As $M \subseteq \bigcup \Pi_1(G[\overline{M}])$, $M = \bigcup \Pi_1(G[\overline{M}])$ by maximality of $M$. It follows that

$$M \subseteq \{v \in \overline{M} : \{v\} \in \Pi(G[\overline{M}])\}.$$
By Proposition 2.2.(3), $M = \bigcup \Pi_1(G[M]) \in \mathcal{M}(G[M])$ and hence $M \in \mathcal{M}(G)$ by Proposition 2.1.(2). Since $(v, w)_G = (\{v\}, \{w\})_{G[M]/\Pi(G[M])} = 0$ for all $v \neq w \in M$, $M$ is a stable set in $G$. There exists $N \in \mathcal{M}(G)$ such that $N \supseteq M$. As $M$ is a stable set in $G$, $N$ is as well. By what precedes, $N = \{v \in \widehat{N} : \{v\} \in \Pi(G[\widehat{N}])\}$. We have $\widehat{M} \not\subseteq \widehat{N}$ because $M \not\subseteq N$. Furthermore $\widehat{M} \in \mathcal{S}(G[\widehat{N}])$ by Proposition 2.3. Given $v \in M$, we obtain $\{v\} \not\subseteq \widehat{M} \subseteq \widehat{N}$. Since $\{v\} \in \Pi(G[\widehat{N}])$, $\widehat{M} = \widehat{N}$. Therefore $M = N$ because $M = \{v \in \widehat{M} : \{v\} \in \Pi(G[\widehat{M}])\}$ and $N = \{v \in \widehat{N} : \{v\} \in \Pi(G[\widehat{N}])\}$.

Let $G$ be a graph such that $\max(\alpha_M(G), \omega_M(G)) \geq 2$. Consider $M \in \mathcal{M}(G)$. By Proposition 4.5, $\lambda_G(\widehat{M}) \in \{\circ, \bullet\}$ and $|\Pi_1(G[\widehat{M}])| = |M| \geq 2$. By Corollary 4.4, $p(G) \leq q(G) \leq \max(\{[\log_2(|\Pi_1(G[Y])| + 1)] : Y \in \mathcal{S}_{\geq 2}(G), \lambda_G(Y) \in \{\circ, \bullet\}\})$.

By Proposition 4.5,
\[
\max(\{[\log_2(|\Pi_1(G[Y])| + 1)] : Y \in \mathcal{S}_{\geq 2}(G), \lambda_G(Y) \in \{\circ, \bullet\}\})
\]
equals
\[
\max(\{[\log_2(|M| + 1)] : M \in \mathcal{M}(G)\}).
\]

Clearly
\[
\max(\{[\log_2(|M| + 1)] : M \in \mathcal{M}(G)\}) = [\log_2(\max(\alpha_M(G), \omega_M(G)) + 1)]
\]
and consequently we recover Theorem 1.4,
\[
p(G) \leq [\log_2(\max(\alpha_M(G), \omega_M(G)) + 1)].
\]

To obtain Corollary 1.5, we prove Lemma 1.3.

**Proof of Lemma 1.3.** Let $G$ be a graph such that $\max(\alpha_M(G), \omega_M(G)) \geq 2$. There exists $S \in \mathcal{M}(G)$ such that $|S| = \max(\alpha_M(G), \omega_M(G))$ and $S$ is a clique or a stable set in $G$. Given an integer $p < \log_2(\max(\alpha_M(G), \omega_M(G)))$, consider any $p$-extension $H$ of $G$. We must prove that $H$ is not prime. We have $2^{|V(H) \setminus V(G)|} < |S|$ so that the function $S \rightarrow 2^{|V(H) \setminus V(G)|}$, defined by $s \mapsto N_H(s) \cap (V(H) \setminus V(G))$ is not injective. There are $s \neq t \in S$ such that $v \leftrightarrow_H \{s, t\}$ for every $v \in V(H) \setminus V(G)$. As $S$ is a module of $G$, we have $v \leftrightarrow_H \{s, t\}$ for every $v \in V(G) \setminus S$. Since $S$ is a clique or a stable set in $G$, $\{s, t\}$ is a nontrivial module of $H$.

When a graph or its complement admits isolated vertices, we obtain the following.

**Lemma 4.6.** Given a graph $G$, if $\iota(G) \neq 0$ or $\iota(\overline{G}) \neq 0$, then
\[
p(G) \geq [\log_2(\max(\iota(G), \iota(\overline{G})) + 1)].
\]
For every graph $G$, assume that $\iota(G) \geq \iota(\overline{G})$. Given $p < \lceil \log_2(\iota(G) + 1) \rceil$, consider any $p$-extension $H$ of $G$. We have $2^{\lvert V(H) \setminus V(G) \rvert} \leq \iota(G)$ and we verify that $H$ is not prime.

For each $x \in V(G)$ such that $N_{G}(x) = \emptyset$, we have $N_{H}(x) \subseteq V(H) \setminus V(G)$. Thus $(N_{H})_{\{v \in V(G) : N_{G}(v) = \emptyset\}}$ is a function from $\{v \in V(G) : N_{G}(v) = \emptyset\}$ to $2^{V(H) \setminus V(G)}$. As observed in the proof of Lemma 3.1, if $(N_{H})_{\{v \in V(G) : N_{G}(v) = \emptyset\}}$ is not injective, then $\{x, y\}$ is a nontrivial module of $H$ when $x \neq y \in \{v \in V(G) : N_{G}(v) = \emptyset\}$ with $N_{H}(x) = N_{H}(y)$. So assume that $(N_{H})_{\{v \in V(G) : N_{G}(v) = \emptyset\}}$ is injective.

As $2^{\lvert V(H) \setminus V(G) \rvert} \leq \iota(G)$, we obtain that $(N_{H})_{\{v \in V(G) : N_{G}(v) = \emptyset\}}$ is bijective. Thus there is $x \in \{v \in V(G) : N_{G}(v) = \emptyset\}$ such that $N_{H}(x) = \emptyset$. Therefore $V(H) \setminus \{x\}$ is a nontrivial module of $H$ and $H$ is not prime. \hfill \Box

The next result is a simple consequence of Proposition 4.5 which is useful in proving Theorem 1.6.

**Corollary 4.7.** Given a graph $G$ such that $\max(\alpha_{M}(G), \omega_{M}(G)) \geq 2$, the elements of $M(G)$ are pairwise disjoint.

**Proof.** Consider $M, N \in M(G)$ such that $M \cap N \neq \emptyset$. Let $v \in M \cap N$. Since $M, N \in S(G)$ and $v \in M \cap N$, $M \subseteq N$ or $N \subseteq M$. For instance, assume that $M \subseteq N$. By Proposition 2.3, $M \in S(G[N])$. Furthermore $\{v\} \in \Pi(G[N])$ by Proposition 4.5. As $\{v\} \subsetneq M \subseteq N$, we obtain $M = N$.

Lastly, $M = \{w \in M : \{w\} \in \Pi(G[M])\}$ and $N = \{w \in N : \{w\} \in \Pi(G[N])\}$ by Proposition 4.5. Thus $M = N$. \hfill \Box

5. **Proof of Theorem 1.6**

Given a graph $G$, denote by $P(G)$ the family of $M \in M(G)$ such that $G[M]$ is prime. For every $M \in P(G)$, $M \in S(G)$ because $G[M]$ is prime. It follows that the elements of $P(G)$ are pairwise disjoint. Thus the elements of $M(G) \cup P(G)$ are also by Corollary 4.7. Set

$I(G) = V(G) \setminus ((\bigcup M(G)) \cup (\bigcup P(G)))$.

We prove Theorem 1.6 when $\max(\alpha_{M}(G), \omega_{M}(G)) = 2$.

**Proposition 5.1.** For every graph $G$ such that $\max(\alpha_{M}(G), \omega_{M}(G)) = 2$,

$$p(G) = 2 \text{ if and only if } \iota(G) = 2 \text{ or } \iota(\overline{G}) = 2.$$ 

**Proof.** It follows from Lemma 1.3 and Theorem 1.4 that $p(G) = 1$ or 2. To begin, assume that $\iota(G) = 2$ or $\iota(\overline{G}) = 2$. By Lemma 4.6, $p(G) \geq 2$ and hence $p(G) = 2$. Conversely, assume that $p(G) = 2$. Let $a \notin V(G)$. As $\max(\alpha_{M}(G), \omega_{M}(G)) = 2$, $[N] = 2$ for each $N \in M(G)$. Let $N_{0} \in M(G)$. For $N \in P(G)$, $G[N]$ is prime. By Lemma 3.3, $G[N]$ admits a prime extension $H_{N}$ defined on $N \cup \{a\}$. We consider any 1-extension $H$ of $G$ to $V(G) \cup \{a\}$ satisfying the following.
(1) For each \( N \in \mathcal{M}(G) \), \( a \leftrightarrow_H N \).
(2) For each \( N \in \mathcal{P}(G) \), \( H[N \cup \{ a \}] = H_N \).
(3) Let \( v \in I(G) \). There is \( i \in \{ 0, 1 \} \) such that \( (v, N_0)_{G} = i \). We require that \( (v, a)_{H} \neq i \).

To begin, we prove that \( S_2^2(G) \cap \mathcal{M}(H) = \emptyset \). Given \( M \in S_2^2(G) \), we have to verify that \( a \leftrightarrow_H M \). Let \( N \) be a minimal element under inclusion of \( \{ N' \in S_2^2(G) : N' \subseteq M \} \). By Proposition 2.3, \( \Pi(G[N]) \in S(G) \). By minimality of \( N \), \( \Pi(G[N]) = \Pi_1(G[N]) \) so that \( G[N] \) and \( G[N]/\Pi(G[N]) \) are isomorphic by Proposition 2.2.(1). We distinguish the following two cases.

- Assume that \( \lambda_{G}(N) = \emptyset \). We obtain that \( G[N] \) is prime, that is, \( N \in \mathcal{P}(G) \). As \( H[N \cup \{ a \}] \) is prime, \( a \leftrightarrow_H N \).
- Assume that \( \lambda_{G}(N) \in \{ \circ, \bullet \} \). By Proposition 4.5, \( N \in \mathcal{M}(G) \). Thus \( |N| = 2 \) and \( a \leftrightarrow_H N \) by definition of \( H \).

In both cases, \( a \leftrightarrow_H N \) and hence \( a \leftrightarrow_H M \).

Now we prove that \( M_2^2(G) \cap \mathcal{M}(H) = \emptyset \). Let \( M \in \mathcal{M}_2^2(G) \). Since \( S_2^2(G) \cap \mathcal{M}(H) = \emptyset \), assume that \( M \notin S_2^2(G) \). Set \( Q = \{ X \in \Pi(G[M]) : X \cap M \neq \emptyset \} \). By Proposition 2.1.(1), \( M \in \mathcal{M}(G[M]) \). By definition of \( M \), \( |Q| \geq 2 \). Thus \( M = \cup Q \) because \( \Pi(G[M]) \subseteq S(G[M]) \). Furthermore \( Q \neq \Pi(G[M]) \) because \( M \notin S_2^2(G) \). By Proposition 2.2.(2), \( Q \notin \mathcal{M}(G[M]) \). As \( 2 \leq |Q| < |\Pi(G[M])| \), \( \lambda_{G}(M) \notin \{ \circ, \bullet \} \). If there is \( X \in Q \cap \Pi_2^2(G[M]) \), then \( a \leftrightarrow_H X \) by what precedes and hence \( a \leftrightarrow_H M \). Assume that \( Q \notin \Pi_1(G[M]) \). We obtain that \( M \) is a clique or a stable set in \( G \). Since \( \max(\alpha_M(G), \omega_M(G)) = 2 \), \( M \in \mathcal{M}(G) \) and \( a \leftrightarrow_H M \) by definition of \( H \).

As \( p(G) = 2 \), \( H \) admits a nontrivial module \( M_H \). We have \( a \in M_H \) because \( M_2^2(G) \cap \mathcal{M}(H) = \emptyset \).

First, we show that \( N \subseteq M_H \) for each \( N \in \mathcal{P}(G) \). By Proposition 2.1.(1), \( M_H \cap (N \cup \{ a \}) \in \mathcal{M}(H[N \cup \{ a \}]) \). Since \( H[N \cup \{ a \}] \) is prime and \( a \in M_H \cap (N \cup \{ a \}) \), we obtain either \( (M_H \setminus \{ a \}) \cap N = \emptyset \) or \( N \subseteq M_H \setminus \{ a \} \). Suppose for a contradiction that \( (M_H \setminus \{ a \}) \cap N = \emptyset \). By Proposition 2.1.(1), \( M_H \setminus \{ a \} \in \mathcal{M}(G) \). There is \( i \in \{ 0, 1 \} \) such that \( (M_H \setminus \{ a \})_{G} = i \) by Proposition 2.1.(3). Therefore \( (a, N)_{H} = i \) which contradicts the fact that \( H[N \cup \{ a \}] \) is prime. It follows that \( N \subseteq M_H \). Thus \( \bigcup \mathcal{P}(G) \subseteq M_H \).

Second, we show that \( N \cap M_H \neq \emptyset \) for each \( N \in \mathcal{M}(G) \). Otherwise consider \( N \in \mathcal{M}(G) \) such that \( N \cap M_H = \emptyset \). There is \( i \in \{ 0, 1 \} \) such that \( (M_H \setminus \{ a \}, N)_{G} = i \). Thus \( (a, N)_{H} = i \) which contradicts \( a \leftrightarrow_H N \). Therefore \( N \cap M_H \neq \emptyset \) for each \( N \in \mathcal{M}(G) \).

Third, let \( v \in I(G) \). By (5.2), \( N_0 \cap M_H \neq \emptyset \). Since \( (v, N_0 \cap M_H)_{G} \neq (v, a)_{H} \), \( v \in M_H \). Hence

\[ I(G) \subseteq M_H. \]
By (5.1) and (5.3),

$$V(G) \setminus M_H \subseteq \mathcal{M}(G).$$

To conclude, consider \( v \in V(H) \setminus M_H \). By (5.4), there is \( N_v \in \mathcal{M}(G) \) such that \( v \in N_v \). By interchanging \( G \) and \( \overline{G} \), assume that \( N_v \) is a stable set in \( G \). Since \( v \in M_H \) and \( (v, N_v \cap M_H)_G = 0 \), we obtain \( (v, M_H)_H = 0 \). Let \( N \in \mathcal{M}(G) \setminus \{N_v\} \). By Corollary 4.7, \( N \cap N_v = \emptyset \). As \( N \cap M_H = \emptyset \), we have \( (v, N \cap M_H)_G = 0 \) and hence \( (v, N)_G = 0 \). It follows that \( N_G(v) = \emptyset \). Therefore \( (N_v, V(G) \setminus N_v)_G = 0 \) because \( N_v \in \mathcal{M}(G) \).

Since \( N_v \) is a stable set in \( G \), we obtain \( N_v \subseteq \{u \in V(G) : N_G(u) = \emptyset\} \).

Clearly \( \{u \in V(G) : N_G(u) = \emptyset\} \subseteq \mathcal{M}(G) \) and \( \{u \in V(G) : N_G(u) = \emptyset\} \) is a stable set in \( G \). Thus \( \iota(G) \leq \max(\alpha_M(G), \omega_M(G)) = 2 \). Consequently \( N_v = \{u \in V(G) : N_G(u) = \emptyset\} \).

**Proof of Theorem 1.6.** Consider a graph \( G \) such that

$$\max(\alpha_M(G), \omega_M(G)) = 2^k$$

where \( k \geq 1 \). It follows from Corollary 1.5 that \( p(G) = k \) or \( k + 1 \).

To begin, assume that \( \iota(G) = 2^k \) or \( \iota(\overline{G}) = 2^k \). By Lemma 4.6, \( p(G) \geq k + 1 \) and hence \( p(G) = k + 1 \).

Conversely, assume that \( p(G) = k + 1 \). If \( k = 1 \), then it suffices to apply Proposition 5.1. Assume that \( k \geq 2 \). For convenience set

$$\mathbb{M}_{\text{max}}(G) = \{N \in \mathcal{M}(G) : |N| = \max(\alpha_M(G), \omega_M(G))\}.$$

With each \( N \in \mathbb{M}_{\text{max}}(G) \) associate \( w_N \in N \). Set \( W = \{w_N : N \in \mathbb{M}_{\text{max}}(G)\} \).

We prove that \( \max(\alpha_M(G-W), \omega_M(G-W)) = 2^k - 1 \). Let \( N \in \mathbb{M}_{\text{max}}(G) \). By Corollary 4.7, the elements of \( \mathbb{M}_{\text{max}}(G) \) are pairwise disjoint. Thus \( N \setminus W = N \setminus \{w_N\} \).

Clearly \( N \setminus \{w_N\} \) is a clique or a stable set in \( G - W \). Furthermore \( N \setminus \{w_N\} \in \mathcal{M}(G - W) \). Therefore \( 2^k - 1 = |N \setminus \{w_N\}| \leq \max(\alpha_M(G-W), \omega_M(G-W)) \).

Now consider \( N' \in \mathbb{M}_G(G - W) \). We show that \( N' \in \mathcal{M}(G) \).

We have to verify that for each \( N \in \mathbb{M}_{\text{max}}(G) \), \( w_N \leftrightarrow_G N' \). Let \( N \in \mathbb{M}_{\text{max}}(G) \). First, assume that there is \( v \in (N \setminus \{w_N\}) \setminus N' \). We have \( v \leftrightarrow_G N' \). As \( N \) is a clique or a stable set in \( G \), \( \{v, w_N\} \in \mathcal{M}(G[N]) \). By Proposition 2.1.2, \( \{v, w_N\} \in \mathcal{M}(G) \). Thus \( w_N \leftrightarrow_G N' \). Second, assume that \( N \setminus \{w_N\} \subseteq N' \). Clearly \( w_N \leftrightarrow_G N' \) when \( N \setminus \{w_N\} = N' \).

Assume that \( N' \setminus (N \setminus \{w_N\}) \neq \emptyset \). By interchanging \( G \) and \( \overline{G} \), assume that \( N' \) is a clique in \( G - W \). As \( N \setminus \{w_N\} \subseteq N' \) and \( |N \setminus \{w_N\}| \geq 2 \), we obtain that \( N \) is a clique in \( G \). Since \( (N \setminus \{w_N\}, N' \setminus N)_G = 1 \) and since \( N \in \mathcal{M}(G) \), we have \( (w_N, N' \setminus N)_G = 1 \). Furthermore \( (w_N, N \setminus \{w_N\})_G = 1 \) because \( N \) is a clique in \( G \). Therefore \( (w_N, N')_G = 1 \). Consequently \( N' \in \mathcal{M}(G) \).

As \( N' \) is a clique in \( G \), there is \( M \in \mathcal{M}(G) \) such that \( M \supseteq N' \). If \( M \in \mathbb{M}_{\text{max}}(G) \), then \( |N'| \leq |M| < \max(\alpha_M(G), \omega_M(G)) \). If \( M \in \mathbb{M}_{\text{max}}(G) \), then \( N' \subseteq M \setminus \{w_M\} \) and hence \( |N'| < |M| = \max(\alpha_M(G), \omega_M(G)) \).

In both cases, we have \( |N'| = \max(\alpha_M(G-W), \omega_M(G-W)) < \max(\alpha_M(G), \omega_M(G)) \). It follows that \( \max(\alpha_M(G-W), \omega_M(G-W)) = 2^k - 1 \).
By Corollary 1.5, \( p(G - W) = k \) and hence there exists a prime \( k \)-extension \( H' \) of \( G - W \). We extend \( H' \) to \( V(H') \cup W \) as follows. Let \( N \in \mathbb{M}_{\text{max}}(G) \). Consider the function \( f_N : N \setminus \{ w_N \} \rightarrow 2^{V(H') \setminus V(G - W)} \) defined by \( v \mapsto N_H(v) \setminus V(G - W) \) for \( v \in N \setminus \{ w_N \} \). Since \( H' \) is prime, \( f_N \) is injective. As \( |N \setminus \{ w_N \}| = 2^{k - 1} \) and \( |2^{V(H') \setminus V(G - W)}| = 2^k \), there is a unique \( X_N \in V(H') \setminus V(G - W) \) such that \( f_N(v) = X_N \) for every \( v \in N \setminus \{ w_N \} \). Let \( H \) be the extension of \( H' \) to \( V(H') \cup W \) such that \( N_H(w_N) \cap (V(H') \setminus V(G - W)) = X_N \) for each \( N \in \mathbb{M}_{\text{max}}(G) \). As \( p(G) = k + 1 \), \( H \) is not prime. Consider a nontrivial module \( M_H \) of \( H \).

Observe the following. Given \( N \neq N' \in \mathbb{M}_{\text{max}}(G) \),

\[
\begin{align*}
N \cap M_H & \neq \emptyset \\
N' \cap M_H & \neq \emptyset
\end{align*}
\]

Indeed, by Proposition 2.1.1, \( M_H \cap V(G) \in \mathcal{M}(G) \). Since \( \mathcal{N}, \mathcal{N}' \in \mathcal{S}(G) \) and since \( (M_H \cap V(G)) \cap \mathcal{N} \neq \emptyset \) and \( (M_H \cap V(G)) \cap \mathcal{N}' \neq \emptyset \), \( M_H \cap V(G) \) is comparable to \( \mathcal{N} \) and \( \mathcal{N}' \) under inclusion. Suppose for a contradiction that \( M_H \cap V(G) \not\subseteq \mathcal{N} \) and \( M_H \cap V(G) \not\subseteq \mathcal{N}' \). It follows that \( N' \cap \mathcal{N} \neq \emptyset \) and \( N \cap \mathcal{N}' \neq \emptyset \). As \( \mathcal{N}' \in \mathcal{S}(G) \), \( \mathcal{N}' \not\subseteq N \) or \( N \not\subseteq \mathcal{N}' \). In the first instance, it follows from Proposition 2.3 that \( \mathcal{N}' \) would be a nontrivial strong module of \( G[N] \) which contradicts the fact that \( N \) is a clique or a stable set in \( G \). Thus \( N \subseteq \mathcal{N}' \) and hence \( \mathcal{N} \subseteq \mathcal{N}' \). Similarly \( N' \subseteq \mathcal{N} \) and \( \mathcal{N}' \subseteq \mathcal{N} \). Therefore \( \mathcal{N} = \mathcal{N}' \) and it would follow from Proposition 4.5 that \( N = N' \). Consequently \( \mathcal{N} \not\subseteq (M_H \cap V(G)) \) or \( \mathcal{N}' \not\subseteq (M_H \cap V(G)) \). For instance, assume that \( \mathcal{N} \not\subseteq (M_H \cap V(G)) \). By Proposition 2.1.1, \( M_H \cap V(H') \in \mathcal{M}(H') \). Furthermore \( (M_H \cap V(H')) \supseteq (N \setminus W) \) and \( N \setminus W = N \setminus \{ w_N \} \) by Corollary 4.7. Since \( H' \) is prime, we have \( V(H') \subseteq M_H \). It follows that (5.5) holds.

As \( H' \) is prime and \( M_H \cap V(H') \in \mathcal{M}(H') \), we have either \( |M_H \cap V(H')| \leq 1 \) or \( M_H \supseteq V(H') \). For a contradiction, suppose that \( |M_H \cap V(H')| \leq 1 \). There is \( N \in \mathbb{M}_{\text{max}}(G) \) such that \( w_N \in M_H \). It follows from (5.5) that \( M_H \cap W = \{ w_N \} \). Thus there is \( v \in V(H') \) such that \( M_H \cap V(H') = \{ v \} \). Clearly \( M_H = \{ v, w_N \} \) and we distinguish the following two cases to obtain a contradiction.

- Suppose that \( v \in V(G - W) \). By Proposition 2.1.1, \( \{ v, w_N \} \in \mathcal{M}(G) \). Therefore there is \( N' \in \mathbb{M}_{\text{max}}(G) \) such that \( w_N \in \mathcal{N}' \). By Corollary 4.7, \( N = N' \) and we would obtain \( N_H(w_N) \cap (V(H') \setminus V(G - W)) = f_N(v) \).

- Suppose that \( v \in V(H') \setminus V(G - W) \). There is \( i \in \{ 0, 1 \} \) such that \((w_N, N \setminus \{ w_N \})_G = i \). We obtain \( (v, N \setminus \{ w_N \})_H = i \) because \( \{ v, w_N \} \in \mathcal{M}(H) \). Since \( f_N \) is injective, the function \( g_N : N \setminus \{ w_N \} \rightarrow 2^{(V(H') \setminus V(G - W)) \setminus \{ v \}} \), defined by \( g_N(u) = f_N(u) \setminus \{ v \} \) for \( u \in N \setminus \{ w_N \} \), is injective as well. We would obtain \( 2^k - 1 \leq 2^{k - 1} \).
Consequently $V(H') \subseteq M_H$. As $M_H$ is a nontrivial module of $H$, there exists $N \in M_{\max}(G)$ such that $w_N \notin M$. By interchanging $G$ and $\overline{G}$, assume that $N$ is a stable set in $G$. We have $(w_N, N \setminus \{w_N\})_G = 0$ and hence $(w_N, V(H'))_G = 0$. In particular $(w_N, V(G - W))_G = 0$. Given $N' \in M_{\max}(G) \setminus \{N\}$, we obtain $(w_N, N' \setminus \{w_N\})_G = 0$. Since $N' \in M(G)$, $(w_N, w_{N'})_G = 0$. It follows that $N_G(w_N) = \emptyset$. As at the end of the proof of Proposition 5.1, we conclude by $N = \{u \in V(G) : N_G(u) = \emptyset\}$.

Lastly, we examine the non prime graphs $G$ such that

$$\alpha_M(G) = \omega_M(G) = 1.$$  

**Proposition 5.2.** For every non prime graph $G$ such that $|V(G)| \geq 4$ and $\alpha_M(G) = \omega_M(G) = 1$, we have $p(G) = 1$.

**Proof.** Consider a minimal element $N_{\min}$ of $S_{\max}(G)$. By Proposition 2.3, $\Pi(G[N_{\min}]) \subseteq S(G)$. By minimality of $N_{\min}$, $\Pi(G[N_{\min}]) = \Pi_1(G[N_{\min}])$. Thus $G[N_{\min}]$ and $G[N_{\min}]/{\Pi_1(G[N_{\min}]])}$ are isomorphic by Proposition 2.2.(1). If $\lambda_G(N_{\min}) \in \{\circ, \bullet\}$, then $N_{\min}$ is a clique or a stable set in $G$ and there would be $N \in M(G)$ such that $N \supseteq N_{\min}$. Therefore $\lambda_G(N_{\min}) = \emptyset$ and $N_{\min} \in P(G)$.

Let $a \notin V(G)$. For each $N \in P(G)$, $G[N]$ is prime. By Lemma 3.3, $G[N]$ admits a prime 1-extension $H_N$ to $N \cup \{a\}$. We consider the 1-extension $H$ of $G$ to $V(G) \cup \{a\}$ satisfying the following.

1. For each $N \in P(G)$, $H[N \cup \{a\}] = H_N$.
2. Let $v \in I(G)$. There is $i \in \{0, 1\}$ such that $(v, N_{\min})_G = i$. We require that $(v, a)_H \neq i$.

We proceed as in the proof of Proposition 5.1, to show that $M_{\max}(G) \cap M(H) = \emptyset$. To begin, we prove that $S_{\max}(G) \cap M(H) = \emptyset$. Given $M \in S_{\max}(G)$, we have to verify that $a \leftrightarrow_H M$. Let $N$ be a minimal element under inclusion of $\{N' \in S_{\max}(G) : N' \subseteq M\}$. We obtain that $\Pi(G[N]) = \Pi_1(G[N])$ so that $G[N]$ and $G[N]/{\Pi_1(G[N])}$ are isomorphic by Proposition 2.2.(1). If $\lambda_G(N) \in \{\circ, \bullet\}$, then $N$ is a clique or a stable set in $G$ and there would be $N' \in M(G)$ such that $N' \supseteq N$. Thus $\lambda_G(N) = \emptyset$. We obtain that $G[N]$ is prime, that is, $N \in P(G)$. Since $H[N \cup \{a\}]$ is prime, $a \leftrightarrow_H N$ and hence $a \leftrightarrow_H M$.

Now we prove that $M_{\max}(G) \cap M(H) = \emptyset$. Let $M \in M_{\max}(G)$. Since $S_{\max}(G) \cap M(H) = \emptyset$, assume that $M \notin S_{\max}(G)$. Set $Q = \{X \in \Pi(G[M]) : X \cap M \neq \emptyset\}$. We obtain that $M = \bigcup Q$, $|Q| \geq 2$ and $\lambda_G(M) \in \{\circ, \bullet\}$. If $|\Pi_1(G[M])| \geq 2$, then we would have $\{v \in M : \{v\} \in \Pi(G[M])\} \in M(G)$ by Proposition 4.5. Consequently $|\Pi_1(G[M])| \leq 1$ and there is $X \in Q \cap \Pi_{\max}(G[M])$. By what precedes $a \leftrightarrow_H X$ and hence $a \leftrightarrow_H M$.

Lastly, we establish that $H$ is prime. Let $M_H \in M_{\max}(H)$. As previously shown, $a \in M_H$. We show that $N \subseteq M_H$ for each $N \in P(G)$. By Proposition 2.1.(1), $M_H \cap (N \cup \{a\}) \in M(H[N \cup \{a\}])$. Since $H[N \cup \{a\}]$ is prime and $a \in M_H \cap (N \cup \{a\})$, we obtain either $(M_H \setminus \{a\}) \cap N = \emptyset$
or \( N \subseteq M_H \setminus \{a\} \). Suppose for a contradiction that \((M_H \setminus \{a\}) \cap N = \emptyset \).
By Proposition 2.1.(1), \( M_H \setminus \{a\} \in \mathcal{M}(G) \). There is \( i \in \{0,1\} \) such that \((M_H \setminus \{a\}, N)\) is \( i \) by Proposition 2.1.(3). Therefore \((a, N)_H = i\) which contradicts the fact that \( H[N \cup \{a\}] \) is prime. It follows that \( N \subseteq M_H \) for each \( N \in \mathcal{P}(G) \). In particular \( N_{\text{min}} \subseteq M_H \). Let \( v \in I(G) \). As \((v, N_{\text{min}})_G \neq (v, a)_H\), \( v \in M_H \). Consequently \( M_H = V(H) \). \qedhere

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