QUASI-HERMITIAN VARIETIES IN $PG(r, q^2)$, $q$ EVEN

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Abstract. In this paper a new example of quasi–Hermitian variety $V$ in $PG(r, q^2)$ is provided, where $q$ is an odd power of 2. In higher-dimensional spaces, $V$ can be viewed as a generalization of the Bukenhout-Tits unital in the desarguesian projective plane; see [9].

1. Introduction

In the $r$-dimensional projective space $PG(r, q^2)$ over a finite field $GF(q^2)$ of order $q^2$, a quasi–Hermitian variety is a set of points which has the same intersection numbers with hyperplanes as a (non–degenerate) Hermitian variety does. Therefore quasi-Hermitian varieties are two-character sets with respect to hyperplanes, where the characters, that is the intersection numbers, are

$$
\frac{(q^r + (-1)^{r-1})(q^{r-1} - (-1)^{r-1})}{q^2 - 1},
$$

and

$$
\frac{(q^r + (-1)^{r-1})(q^{r-1} - (-1)^{r-1})}{q^2 - 1} + (-1)^{r-1}q^{r-1}.
$$

Quasi–Hermitian varieties other than Hermitian varieties are known to exist; see [1] and [6]. The interest for quasi-Hermitian varieties arose from coding theory. Delsarte [8] proved indeed that a two–character set gives rise to a projective linear two weights code and a strongly regular graph. Recent papers on this subject are [4, 5, 7].

We construct a new family of non–trivial quasi–Hermitian varieties $V$ in $PG(r, q^2)$ with $q = 2^e$ and $e$ odd, using a procedure similar to that developed in [1]. The essential idea is to keep a Hermitian variety $\mathcal{H} = \mathcal{H}(r, q^2)$ invariant but modify the ambient space $PG(r, q^2)$ by a birational transformation so that $\mathcal{H}$ becomes a quasi–Hermitian variety of the $r$-dimensional projective space $PG(r, q^2)$ represented by a (non-standard) model $\Pi$ of $PG(r, q^2)$ where
(i) points of Π are those of $\text{PG}(r, q^2)$;
(ii) hyperplanes of Π are certain hyperplanes and hypersurfaces of $\text{PG}(r, q^2)$.

We give the equations of these hypersurfaces later in Section 2 where we also extend some results obtained in [2] from $r = 2$ to any $r > 2$. Interestingly, some planar sections of $\mathcal{V}$ are Buekenhout-Tits unitals, in particular $\mathcal{V}$ is a Buekenhout-Tits unital for $r = 2$.

For generalities on Hermitian varieties and unitals in projective spaces, the reader is referred to [14, 11, 13, 10, 3]. Basic facts on rational transformations of projective spaces are found in [12, Section 3.3].

2. A non-standard model of $\text{PG}(r, q^2)$

Fix a projective frame in $\text{PG}(r, q^2)$, where $q$ is an odd power of 2. Let $(X_0, X_1, \ldots, X_r)$ denote homogeneous coordinates, and consider the affine plane $\text{AG}(r, q^2)$ whose infinite hyperplane $\Sigma_\infty$ has equation $X_0 = 0$. Then $\text{AG}(r, q^2)$ has affine coordinates $(x_1, x_2, \ldots, x_r)$ where $x_i = X_i / X_0$ for $i \in \{1, \ldots, r\}$.

Take $\epsilon \in \text{GF}(q^2) \setminus \text{GF}(q)$ such that $\epsilon^2 + \epsilon + \delta = 0$ for some $\delta \in \text{GF}(q) \setminus \{1\}$ with $\text{Tr} (\delta) = 1$. Here, $\text{Tr}$ stands for the trace function $\text{GF}(q) \to \text{GF}(2)$. Then $\epsilon^{2q} + \epsilon^q + \delta = 0$. Therefore, $(\epsilon^{q} + \epsilon)^2 + (\epsilon^q + \epsilon) = 0$, whence $\epsilon^{q} + \epsilon + 1 = 0$.

Moreover, if $q = 2^e$, with $e$ an odd integer, then $\sigma : x \mapsto x^{2^{(e+1)/2}}$

is an automorphism of $\text{GF}(q)$. Set

$$\Delta_\epsilon(x) = \epsilon^{\sigma+2}x^{(\sigma+2)} + (\epsilon^{\sigma} + \epsilon^{\sigma+2})x^{q\sigma+2} + x^{\sigma} + (1 + \epsilon)x^2.$$  

For any $m = (m_1, \ldots, m_{r-1}, d) \in \text{GF}(q^2)^r$, let $\mathcal{D}(m)$ denote the algebraic hypersurface

$$x_r = \Delta_\epsilon(x_1) + \cdots + \Delta_\epsilon(x_{r-1}) + m_1x_1 + \cdots + m_{r-1}x_{r-1} + d.  \tag{2.1}$$

Consider the incidence structure $\Pi_\epsilon = (\mathcal{P}, \Sigma)$ whose points are the points of $\text{AG}(r, q^2)$ and whose hyperplanes are the hyperplanes through the point at infinity $P_\infty(0, 0, \ldots, 0, 1)$ together with the hypersurfaces $\mathcal{D}(m)$, where $m$ ranges over $\text{GF}(q^2)^r$.

**Lemma 2.1.** The incidence structure $\Pi_\epsilon = (\mathcal{P}, \Sigma)$ is an affine space isomorphic to $\text{AG}(r, q^2)$.

**Proof.** The birational transformation $\varphi$ given by

$$\varphi : (x_1, \ldots, x_{r-1}, x_r) \mapsto (x_1, \ldots, x_{r-1}, x_r + \Delta_\epsilon(x_1) + \cdots + \Delta_\epsilon(x_{r-1})),$$  

transforms the hyperplanes through $P_\infty(0, 0, \ldots, 0, 1)$ into themselves, whereas the hyperplane of equation $x_r = m_1x_1 + \cdots + m_{r-1}x_{r-1} + d$ is mapped into the hypersurface $\mathcal{D}(m)$. Therefore, $\varphi$ determines an isomorphism

$$\Pi_\epsilon \cong \text{AG}(r, q^2),$$
and the assertion is proven.

Completing $\Pi_\varepsilon$ with its points at infinity in the usual way gives a projective space isomorphic to $\text{PG}(r,q^2)$.

3. Main result

Let $H$ be the Hermitian variety of $\text{PG}(r,q^2)$. $H$ is assumed to be in an affine canonical form

$$x^q + x_r = x_1^{q+1} + \cdots + x_{r-1}^{q+1}. \tag{3.1}$$

The set of the infinity points of $H$ is

$$F = \{(0,x_1,\ldots,x_r) | x_1^{q+1} + \cdots + x_{r-1}^{q+1} = 0\} \tag{3.2}$$

and it can be viewed as a Hermitian cone of $\text{PG}(r-1,q^2)$ projecting a Hermitian variety of $\text{PG}(r-2,q^2)$. Set

$$\Gamma_\varepsilon(x) = [x + (x^q + x)\varepsilon]^{q+2} + (x^q + x)^\sigma + (x^{2q} + x^2)\varepsilon + x^{q+1} + x^2.$$

Theorem 3.1. The affine algebraic variety of equation

$$x^q + x_r = \Gamma_\varepsilon(x_1) + \cdots + \Gamma_\varepsilon(x_{r-1}), \tag{3.3}$$

together with the infinity points (3.2) of $H$ is a quasi-Hermitian variety $V$ of $\text{PG}(r,q^2)$.

Proof. Let $P = (\xi_1,\ldots,\xi_r)$ be an affine point in $\Pi_\varepsilon$. This point, viewed as an element of $\text{AG}(r,q^2)$, has coordinates $x_i = \xi_i$, for $i = 1,\ldots,r-1$, and $x_r = \xi_r + \Delta_\varepsilon(\xi_1) + \cdots + \Delta_\varepsilon(\xi_{r-1})$. Therefore, $H$ and $V$ coincide in the projective closure of $\Pi_\varepsilon$ thus, we just have to prove the following lemma. Let $D(m)$ be the hypersurface with equation (2.1).

Lemma 3.2. The hypersurfaces $D(m)$ and $H$ have either

$$N_1 = \frac{(q^r + (-1)^{r-1})(q^{r-1} - (-1)^{r-1})}{q^2 - 1} - |\text{H}(r-2,q^2)|$$

or

$$N_2 = \frac{(q^r + (-1)^{r-1})(q^{r-1} - (-1)^{r-1})}{q^2 - 1} + (-1)^{r-1}q^{r-1} - |\text{H}(r-2,q^2)|$$

common points in $\text{AG}(r,q^2)$.

Proof. The intersection size of $H$ and $D(m)$ in $\text{AG}(r,q^2)$ is the number of solutions $(x_1,\ldots,x_r) \in \text{GF}(q^2)^r$ of the following system

$$\begin{align*}
\begin{cases}
x^q + x_r = x_1^{q+1} + \cdots + x_{r-1}^{q+1}.

x_r = \Delta_\varepsilon(x_1) + \cdots + \Delta_\varepsilon(x_{r-1}) + m_1x_1 + \cdots + m_{r-1}x_{r-1} + d.
\end{cases}
\end{align*} \tag{3.4}
$$

Substituting the value of $x_r$ in the first equation gives
Consider now GF(q^2) as a vector space over GF(q). The set \{1, \varepsilon\} is a basis of GF(q^2), thus the elements in GF(q^2) can be written as linear combinations with respect to this basis, that is, \(x_i = x_i^0 + x_i^1 \varepsilon\), with \(x_i^0, x_i^1 \in GF(q)\). Hence, (3.5) becomes an equation over GF(q),

\[
0 = (x_0^0)^{\sigma+2} + x_0^0 x_1^1 + (x_1^1)^\sigma + \cdots + (x_{r-1}^0)^{\sigma+2} + x_{r-1}^0 x_{r-1}^1 + (x_{r-1}^1)^\sigma + m_1^1 x_0^1 + (m_0 + m_1^1) x_1^1 + \cdots + m_{r-1}^1 x_{r-1}^0 + (m_{r-1} + m_1^1) x_{r-1}^1 + d^1.
\]

The solutions \((x_0^0, x_1^1, \ldots, x_{r-1}^1)\) of (3.6) may be regarded as points of the affine space AG(2(r-1), q) over GF(q). In fact, (3.6) turns out to be the equation of a (possibly degenerate) affine hypersurface \(S\) of AG(2(r-1), q). The number \(N\) of points in AG(2(r-1), q) which lie on \(S\) is the number of points in AG(r, q^2) on \(H \cap D(m)\). We will show that \(N\) is either \(N_1\) or \(N_2\) by induction on \(r\).

First, suppose \(r = 2\). In this case \(S\) can be viewed as an affine planar section of the Tits ovoid \(O\) of affine equation \((x_1)^\sigma + x_0^0 x_1^1 + (x_0^1)^{\sigma+2} = z\)

Here \((x_0^0, x_1^1, z)\) denote affine coordinates for points in the affine 3-space in which AG(2, q) is embedded as a hyperplane. Therefore, \(S\) consists of 1 or \(q+1\) points according as our plane of equation \(z = m_1^1 x_0^0 + (m_0 + m_1^1) x_1^1 + d^1\) is tangent to \(O\) or not, and the assertion follows.

Now suppose \(r > 2\). Fix a 2(r-2)-tuple \((\bar{x}_0^0, \bar{x}_2^1, \ldots, \bar{x}_{r-3}^0, \bar{x}_{r-1}^1)\) of elements in GF(q). For each such tuple, the number of 2(r-1)-tuples

\[
(\alpha, \beta, \bar{x}_0^0, \bar{x}_2^1, \ldots, \bar{x}_{r-3}^0, \bar{x}_{r-1}^1)
\]

satisfying (3.6) is 1 or \(q+1\) according to whether

\[
0 = (\bar{x}_0^0)^{\sigma+2} + \bar{x}_0^0 \bar{x}_2^1 + (\bar{x}_2^1)^\sigma + \cdots + (\bar{x}_{r-3}^0)^{\sigma+2} + \bar{x}_{r-3}^0 \bar{x}_{r-1}^1 + (\bar{x}_{r-1}^1)^\sigma + m_2^1 \bar{x}_0^0 + (m_2^0 + m_1^1) \bar{x}_2^1 + \cdots + m_{r-3}^1 \bar{x}_{r-3}^0 + (m_{r-3} + m_1^1) \bar{x}_{r-1}^1 + (m_1^1)^{\sigma+2} + (m_0 + m_1^1) m_1^1 + (m_0 + m_1^1)^\sigma + d^1
\]

or not. The induction hypothesis applied to \(r-1\) yields that (3.7) has either

\[
n_1 = \frac{(q^{r-1} + (-1)^{r-2})(q^{r-2} - (-1)^{r-2})}{q^2 - 1} - |H(r-3, q^2)|
\]

or

\[
n_2 = \frac{(q^{r-1} + (-1)^{r-2})(q^{r-2} - (-1)^{r-2})}{q^2 - 1} + (-1)^{r-2}q^{r-2} - |H(r-3, q^2)|
\]
solutions. This implies that the number of solutions of (3.6) is either
\[ a = n_1 + (q^{2(r-2)} - n_1)(q + 1) \]
or
\[ b = n_2 + (q^{2(r-2)} - n_2)(q + 1). \]
A direct computation shows that \( a = N_1 \) and \( b = N_2 \) and our lemma follows
\[ \square \]
Since the points at infinity of a hyperplane of \( \text{AG}(r, q^2) \) are also the points at infinity of the corresponding hyperplane in the projective closure of \( \Pi_\varepsilon \), the assertion is proven. \[ \square \]

**Theorem 3.3.** The quasi–Hermitian variety \( \mathcal{V} \) defined in Theorem (3.1) is not projectively equivalent to the Hermitian variety \( \mathcal{H} \) of \( PG(r, q^2) \).

**Proof.** First assume \( r = 2 \). In this case \( \mathcal{V} \) consists of the infinity point \((0, 0, 1)\) together with the points \((1, x, y)\) such that
\[ y^q + y = [x + (x^q + x)\varepsilon]^{\sigma+2} + (x^q + x)^\sigma + (x^{2q} + x^2)\varepsilon + x^{q+1} + x^2. \]
Setting \( x^q + x = t \), and \( x + (x^q + x)\varepsilon = s \), we have
\[ x = s + t\varepsilon, \]
and
\[ y^q + y = s^{\sigma+2} + t^\sigma + ts, \]
that is,
\[ y = (s^{\sigma+2} + t^\sigma + t\sigma)\varepsilon + r, \]
where \( r \in GF(q) \). Therefore,
\[ \mathcal{V} = \{ (1, s + t\varepsilon, (s^{\sigma+2} + t^\sigma + t\sigma)\varepsilon + r) \mid r, s, t \in GF(q) \} \cup \{(0, 0, 1)\}, \]
namely, \( \mathcal{V} \) coincides with a Buekenhout-Tits unital which is not projectively equivalent to the hermitian curve of \( PG(2, q^2) \); see [2, 9].

In the case \( r > 2 \) let \( \pi \) be the plane of affine equations \( x_2 = \cdots = x_{r-1} = 0 \), and let \( \mathcal{U} \) denote the intersection of \( \mathcal{V} \) and \( \pi \). We can choose homogeneous coordinates in \( \pi \) in such a way that \( \mathcal{U} \) is the set of points
\[ \{ (1, s + t\varepsilon, (s^{\sigma+2} + t^\sigma + t\sigma)\varepsilon + r) \mid r, s, t \in GF(q) \} \cup \{(0, 0, 1)\} \]
that is, a Buekenhout-Tits unital of \( \pi \), and thus the assertion is proven. \[ \square \]
Remark. For each $\gamma = (\gamma_1, \ldots, \gamma_r) \in GF(q)^r$, let $\psi_\gamma$ be the collineation of $\text{PG}(r, q^2)$ induced by the non-singular matrix

$$
\begin{pmatrix}
1 & \gamma_1 \epsilon & \gamma_2 \epsilon & \cdots & \gamma_{r-1} \epsilon & \gamma_r + (\gamma_1 + \cdots + \gamma_{r-1})^\sigma \epsilon \\
0 & 1 & 0 & \cdots & 0 & \gamma_1 + \gamma_1 \epsilon \\
0 & 0 & 1 & \cdots & 0 & \gamma_2 + \gamma_2 \epsilon \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & \gamma_{r-1} + \gamma_{r-1} \epsilon \\
0 & 0 & 0 & \cdots & 0 & 1 
\end{pmatrix}.
$$

Let $G$ denote the following collineation group of order $q^r$,

$$
G = \{ \psi_\gamma \mid \gamma \in GF(q)^r \}.
$$

Straightforward computations show that $G$ is an abelian group which leaves $\mathcal{V}$ invariant; in particular, it fixes $P_\infty$ and has $q^r-1$ orbits of size $q^r$ on $\mathcal{V}\setminus\mathcal{F}$. Furthermore, for $r = 2$, it coincides with the stabilizer in $\text{PGL}(3, q^2)$ of $\mathcal{V}$; see [9].

References

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