MONOCHROMATIC EVEN CYCLES

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Abstract. We prove that any \( r \)-coloring of the edges of \( K_m \) contains a monochromatic even cycle, where \( m = 3r + 1 \) if \( r \) is odd and \( m = 3r \) if \( r \) is even. We also prove that \( K_{m-1} \) has an \( r \)-coloring without monochromatic even cycles.

An easy exercise, perhaps folkloristic, says that in any \( r \)-coloring of the edges of \( K_{2r+1} \) there is a monochromatic odd cycle (and this is not true for \( K_{2r} \)).

This note explores what happens if we ask the same question for even cycles. Let \( f(r) \) denote the smallest integer \( m \) for which there is a monochromatic even cycle in every edge coloring of \( K_m \).

**Theorem 1.** For odd \( r \), \( f(r) = 3r + 1 \) and for even \( r \), \( f(r) = 3r \).

Every graph with \( n \) vertices and with more than \( m = \lfloor 3(n-1)/2 \rfloor \) edges contains a \( \Theta \)-graph, i.e. three internally vertex disjoint paths connecting the same pair of vertices (see [1], Exercise 10.1). Since a \( \Theta \)-graph obviously contains an even cycle, any graph with \( n \) vertices and more than \( m \) edges contains an even cycle. This easily implies that the stated values are upper bounds of \( f(r) \) in Theorem 1. Indeed, considering the majority color, one can easily check that

\[
\left\lceil \frac{(3r+1)}{2} \right\rceil > \left\lfloor \frac{3(3r)}{2} \right\rfloor \quad \text{if } r \text{ is odd}
\]

and

\[
\left\lceil \frac{(3r)}{2} \right\rceil > \left\lfloor \frac{3(3r-1)}{2} \right\rfloor \quad \text{if } r \text{ is even}.
\]

Therefore to prove Theorem 1 we need a construction, a partition of the edge set of \( K_{3r} \) (\( K_{3r-1} \)) into \( r \) graphs, each without even cycles. Let \( H_1 \) be a triangle with vertices \( v_1, v_2, v_3 \). For odd \( r > 1 \) let \( H_r \) be the graph formed by three vertex disjoint copies of \( (r-1)/2 \) triangles sharing one common vertex \( v_i, i = 1, 2, 3 \) and the triangle \( v_1, v_2, v_3 \) which is called the central triangle of \( H_r \). Note that each block (maximal biconnected subgraph or cut-edge) of
$H_r$ is a triangle, so it has no even cycles. Thus for odd $r$ Theorem 1 follows from the next proposition.

**Proposition 2.** For odd $r$, $K_{3r}$ can be partitioned into $r$ copies of $H_r$.

**Proof.** The proof is based on a well-known construction of Steiner triple systems on $6t + 3$ vertices (see [2], Theorem 9.1). Set $r = 2t + 1$, then $3r = 6t + 3$. The vertex set of $K = K_{3r}$ is partitioned into $\{a_i, b_i, c_i\}$, for $i = 1, 2, \ldots, 2t + 1$. For $r = 1$, $\{a_i, b_i, c_i\}$ is an $H_1$, for $r > 1$ consider a near factorization of a complete graph $S_{2t+1}$ with vertex set $\{1, 2, \ldots, 2t+1\}$ into factors $F_i$, where $F_i$ avoids vertex $i$. To each factor $F_i$ we define a copy of $H_i^r$ as follows. Place the edges of the following triangles to $H_i^r$:

\[(1) \quad \{b_ia_k, a_la_l, a_ia_k, b_kb_l, c_kc_l : kl \in F_i\}, \{a_ib_ic_i\}.\]

One can easily see that $H_i^r$ is isomorphic to $H_r$ and for $i = 1, \ldots, 2t + 1$ they give a partition on the edge set of $K$ (in fact their blocks are triangles forming a Steiner triple system on $K$). $\square$

For $r = 2$ note that $K_5$ can be partitioned into two pentagons. However, $K_5$ can be also partitioned into two “bulls”, which is a triangle with two pendant edges (see Figure 1). This latter works well to reduce the even case to the odd one in Proposition 3.

For even $r$ define the graph $A_r$ from $H_{r-1}$ by removing the edges of its central triangle $v_1, v_2, v_3$ and adding two new vertices $u, w$ together with the five edges $v_1w_2, v_2w_1, wv_2$ (see Figure 2). Let $B_r$ be the graph with $r - 1$ triangles sharing a common vertex $x$ plus $r$ pendant edges, one from $x$ and one from each triangle (from a vertex different from $x$). Note that $A_r, B_r$
have $3r - 1$ vertices and their blocks are cut-edges and triangles so they do not have even cycles. The graphs $A_2, B_2$ are both bulls.

**Proposition 3.** For even $r$, $K_{3r-1}$ can be partitioned into $r - 1$ copies of $A_r$ and one copy of $B_r$.

**Proof.** Let $r$ be even and consider the construction of Proposition 2 for $r - 1$ colors. This gives a partition of $K_{3r-3}$ into $r - 1$ copies of $H_{r-1}$. Notice that the central triangles $T_i = \{a_i, b_i, c_i\}$ of the $i$-th copies are vertex disjoint ($i = 1, 2, \ldots, r - 1$). Adding two new vertices $d, e$ to $V(K_{3r-3})$ transform the $i$-th copy of $H_{r-1}$ as follows: remove the edges $a_ic_i, b_ic_i$ from $T_i$ and add $da_i, db_i, dc_i, eb_i$. This gives $r - 1$ copies of $A_r$ for ($i = 1, 2, \ldots, r - 1$). The “missing edges”, $de, ca_i, ec_i, a_ic_i, b_ic_i$ for $i = 1, 2, \ldots, r - 1$ define one copy of $B_r$. 

Proposition 3 shows that for even $r$, $f(r) \geq 3r$, thus completing the proof of Theorem 1.

**References**


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