CHARACTERIZING GRAPH CLASSES BY INTERSECTIONS OF NEIGHBORHOODS

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Abstract. The interplay between maxcliques (maximal cliques) and intersections of closed neighborhoods leads to new types of characterizations of several standard graph classes. For instance, being hereditary clique-Helly is equivalent to every nontrivial maxclique containing the intersection of closed neighborhoods of two vertices of Q, and also to, in all induced subgraphs, every nontrivial maxclique containing a simplicial edge (an edge in a unique maxclique). Similarly, being trivially perfect is equivalent to every maxclique containing the closed neighborhood of a vertex of Q, and also to, in all induced subgraphs, every maxclique containing a simplicial vertex. Maxcliques can be generalized to maximal cographs, yielding a new characterization of ptolemaic graphs.

1. Maximal cliques and closed neighborhoods

A clique of a graph G is a complete subgraph of G or, interchangeably, the vertex set of a complete subgraph. A clique Q is a p-clique if |Q| = p; is a maxclique if Q is not contained in a (|Q| + 1)-clique; and is a simplicial clique if Q is contained in a unique maxclique. Simplicial 1-cliques are traditionally called simplicial vertices, and so simplicial 2-cliques can be called simplicial edges. The open neighborhood \( N_G(v) \) of a vertex v is the set \( \{ w \in V(G) : wv \in E(G) \} \). The closed neighborhood \( N_G[v] \) of v is the set \( N_G(v) \cup \{ v \} \) or, interchangeably, the subgraph of G induced by the set \( N_G[v] \).

Lemma 1. For every graph G and every \( p \geq 1 \), a maxclique Q of G contains \( N_G[v_1] \cap \cdots \cap N_G[v_p] \) for distinct \( v_1, \ldots, v_p \in Q \) if and only if Q contains a simplicial p-clique of G.

For \( k \geq 2 \), define a k-ocular graph to consist of 2k vertices that are partitioned into \( W = \{ w_1, \ldots, w_k \} \) and \( U = \{ u_1, \ldots, u_k \} \) where W induces a k-clique and each \( N_G(u_i) = \{ w_j : j \neq i \} \) (the subgraph induced by U can contain any subset of edges \( u_iu_j \)); see Figure 1. This is the terminology of...
[2] when \( k \geq 3 \). The only 2-ocular graphs are the path \( u_1, w_2, w_1, u_2 \cong P_4 \) and the cycle \( u_1, w_2, w_1, u_2, u_1 \cong C_4 \).

For \( p \geq 2 \), define a graph \( G \) to be \( p \)-clique-Helly when, for every family \( \mathcal{F} \) of maxcliques of \( G \), if every \( p \) members of \( \mathcal{F} \) have an element in common, then all the members of \( \mathcal{F} \) have an element in common. If every induced subgraph of \( G \) is \( p \)-clique-Helly, then \( G \) is called hereditary \( p \)-clique-Helly. Reference [2] contains several characterizations of hereditary \( p \)-clique-Helly graphs, including condition (4) in Theorem 2.

**Theorem 2.** The following are equivalent for every graph \( G \) and \( p \geq 2 \):

1. If \( Q \) is a maxclique of an induced subgraph \( G' \) of \( G \) and \( |Q| \geq p \), then \( Q \) contains \( N_{G'}[v_1] \cap \cdots \cap N_{G'}[v_p] \) for distinct \( v_1, \ldots, v_p \in Q \).
2. If \( Q \) is a maxclique of an induced subgraph \( G' \) of \( G \) and \( |Q| \geq p \), then \( Q \) contains a simplicial \( p \)-clique of \( G' \).
3. \( G \) is hereditary \( p \)-clique-Helly.
4. \( G \) contains no induced \((p+1)\)-ocular subgraph.

**Proof.** Suppose \( p \geq 2 \). Lemma 1 implies the equivalence \((1) \Leftrightarrow (2)\). The equivalence \((3) \Leftrightarrow (4)\) is [2, Thm. 4].

To prove \((4) \Rightarrow (1)\), suppose \((4)\) holds, \( G' \) is an induced subgraph of \( G \), and \( Q \) is a maxclique of \( G' \) with \(|Q| > p\) (the \(|Q| = p\) case being immediate). Let \( Q' = \{w_1, \ldots, w_{p+1}\} \subseteq Q \) and, whenever \( 1 \leq i \leq p+1 \), let \( S_i = \bigcap_{j \neq i} N_{G'}[w_j] \). Thus \( Q' \subseteq S_i \) for each \( i \). If for each \( i \) there exists a vertex \( u_i \in S_i - Q \), then \( W = Q' \) and \( U = \{u_1, \ldots, u_{p+1}\} \) would induce a \((p+1)\)-ocular subgraph (noting that \( u_i \) and \( w_i \) are not adjacent, since \( Q \) is a maxclique). Therefore \((4)\) implies that some \( S_i = Q \); without loss of generality, say \( S_{p+1} = Q \). That makes \( Q = \bigcap_{i=1}^{p} N_{G'}[w_j] \) with distinct \( w_1, \ldots, w_p \in Q \), showing that \((1)\) holds.

To prove \((1) \Rightarrow (4)\), suppose \((4)\) fails because \( G \) contains an induced \((p+1)\)-ocular subgraph \( G' \) where \( V(G') \) is partitioned into \( W \cup U \) as in the definition of \((p+1)\)-ocular. Each \( N_{G'}[w_i] \) contains the \( p \) vertices \( u_j \) that have \( j \neq i \). Therefore, each \( u_j \in \bigcap_{i \neq j} N_{G'}[w_i] \), and so \( Q = W \) can never contain the intersection of \( p \) neighborhoods \( N_{G'}[w_j] \) with \( w_j \in Q \), showing that \((1)\) fails. \( \square \)
Hereditary 2-clique-Helly graphs are called hereditary clique-Helly graphs in [10] and clique reducible graphs in [11]. These papers contain several other characterizations, and [8] will contain more characterizations involving simplicial cliques.

Corollary 3. The following are equivalent for every graph $G$:

1. If $Q$ is a maxclique of an induced subgraph $G'$ of $G$ and $|Q| \geq 2$, then $Q$ contains $N_{G'}[v] \cap N_{G'}[v']$ for distinct $v, v' \in Q$.
2. Every nontrivial maxclique of an induced subgraph of $G$ contains a simplicial edge.
3. $G$ is hereditary clique-Helly.
4. $G$ contains no induced 3-ocular subgraph.

Proof. This follows from the $p = 2$ case of Theorem 2. (The equivalence (3) $\Leftrightarrow$ (4) also appears in [10, 11].) □

A graph $G$ is called trivially perfect if, for every induced subgraph $G'$ of $G$, the cardinality of the largest independent set in $G'$ equals the number of maxcliques in $G'$. Reference [4] contains several other characterizations, including condition (4) in Theorem 4. See [1, 9] for many additional characterizations—and names—for trivially perfect graphs; [8] will contain more characterizations involving simplicial cliques.

Theorem 4. The following are equivalent for every graph $G$:

1. If $Q$ is a maxclique of an induced subgraph $G'$ of $G$, then $Q$ contains $N_{G'}[v]$ for some $v \in Q$.
2. Every maxclique of an induced subgraph of $G$ contains a simplicial vertex.
3. $G$ is trivially perfect.
4. $G$ contains no induced $P_4$ or $C_4$ subgraph.

Proof. The equivalence (1) $\Leftrightarrow$ (2) follows from Lemma 1. The equivalence (3) $\Leftrightarrow$ (4) is [4, Thm. 2]. The equivalence (1) $\Leftrightarrow$ (4) can be proved by a simple modification of the proof of (1) $\Leftrightarrow$ (4) in Theorem 2, taking $p = 1$ and using that $P_4$ and $C_4$ are the two 2-ocular graphs. □

Corollary 5 will be a restriction of Corollary 3 to chordal graphs—the graphs in which every cycle of length four or more has a chord (see [1, 9] for other many characterizations and history). Note that the existence of an edge $u_iu_j$ in a $p$-ocular graph (with vertex set partitioned into $W$ and $U$ as in the definition) would produce a chordless 4-cycle $u_i, u_j, w_i, w_j, u_i$. Therefore a $p$-ocular graph is chordal if and only if the set $U$ is independent (meaning that there are no edges between vertices in $U$).

A disk $D_G[v, k]$ of a graph $G$ is the set $\{x : 0 \leq d(v, x) \leq k\} \subseteq V(G)$, where $d(v, x)$ denotes the $v$-to-$x$ distance in $G$. Define $G$ to be disk-Helly when, for every family $F$ of disks of $G$, if every two members of $F$ have an element in common, then all the members of $F$ have an element in common. If every induced subgraph of $G$ is disk-Helly, then $G$ is called hereditary
disk-Helly. References [3, 5] contain several characterizations of hereditary disk-Helly graphs, including the following two: (i) being both chordal and clique-Helly, and (ii) being chordal with no induced Hajós subgraph (where the Hajós graph—sometimes called the 3-sun—is the 3-ocular graph with \{u_1, u_2, u_3\} independent, shown as the center graph in Figure 1).

**Corollary 5.** The following are equivalent for every chordal graph \(G\):

1. If \(Q\) is a maxclique of an induced subgraph \(G'\) of \(G\) and \(|Q| \geq 2\), then \(Q\) contains \(N_{G'}[v] \cap N_{G'}[v']\) for distinct \(v, v' \in Q\).
2. Every nontrivial maxclique of an induced subgraph of \(G\) contains a simplicial edge.
3. \(G\) is hereditary disk-Helly.
4. \(G\) contains no induced Hajós subgraph.

**Proof.** Since the Hajós graph is the only chordal 3-ocular graph, the equivalences (1) \(\iff\) (2) and (1) \(\iff\) (4) follow from the \(p = 2\) case of Theorem 2. The equivalence (3) \(\iff\) (4) is [5, Thm. 1.2]. \[\Box\]

2. **Maximal cographs and closed neighborhoods**

In this section, cliques—which are simply the graphs that have no induced \(P_3\) subgraphs—will be generalized to the complement-reducible graphs (or cographs)—which are the graphs that have no induced \(P_4\) subgraphs (or, equivalently, the graphs in which every connected induced subgraph has diameter at most two). See [1, 9] for many additional characterizations. Echoing the \(CC(G)\) notation in [1] for the set of all inclusion-maximal subsets of \(V(G)\) that induce connected cographs, define a \(CC\)-subgraph of \(G\) to be a subgraph of \(G\) that is induced by a set in \(CC(G)\).

For each \(p \geq 1\), let \(K_{p+4} - P_4\) denote the graph on the vertex set \(\{w_1, \ldots, w_{p+2}, u_1, u_2\}\) that is complete except that edges \(w_1u_1, u_1u_2\) and \(u_2u_3\) do not occur. Equivalently, \(K_{p+4} - P_4\) is the chordal \((p + 2)\)-ocular graph with vertices \(u_3, \ldots, u_{p+2}\) deleted. The \(p\)-clique \(\{w_3, \ldots, w_{p+2}\}\) is the center of the \(K_{p+4} - P_4\) graph.

![Figure 2](image)

**Figure 2.** The graphs \(K_5 - P_4\) and \(K_6 - P_4\).

**Lemma 6.** For every graph \(G\) and \(p \geq 1\), a \(CC\)-subgraph \(H\) contains \(N_G[v_1] \cap \cdots \cap N_G[v_p]\) for distinct \(v_1, \ldots, v_p \in V(H)\) if and only if \(H\) does not contain the center of a \(K_{p+4} - P_4\) subgraph of \(G\).
Proof. Some $CC$-subgraph of $G$ contains $\bigcap_{i=1}^{p} N_G[v_i]$ for some set $S = \{v_1, \ldots, v_p\}$ of $p$ vertices if and only if $\bigcap_{i=1}^{p} N_G[v_i]$ contains no induced $P_4$ path $a, b, c, d$, which in turn is equivalent to no set $S \cup \{a, b, c, d\}$ inducing a $K_{p+4} - P_4$ subgraph of $G$ with $w_1 = b$, $w_2 = c$, $w_3 = v_1$, $\ldots$, $w_{p+2} = v_p$, $u_1 = d$, and $u_2 = a$ (in the notation in the definition of $K_{p+4} - P_4$).

Theorem 7. The following are equivalent for every graph $G$ and $p \geq 1$:

1. If $H$ is a $CC$-subgraph of an induced subgraph $G'$ of $G$, then $H$ contains $N_{G'}[v_1] \cap \cdots \cap N_{G'}[v_p]$ for distinct $v_1, \ldots, v_p \in V(H)$.

2. $G$ contains no induced $K_{p+4} - P_4$ subgraph.

Proof. To prove (1) $\Rightarrow$ (2), suppose $p \geq 1$ and condition (2) fails; specifically, suppose $G$ has an induced subgraph $G' \cong K_{p+4} - P_4$ on the vertex set \{w_1, \ldots, w_{p+2}, u_1, u_2\} as described in the definition of $K_{p+4} - P_4$. Take $H$ to be the $CC$-subgraph of $G'$ that is obtained by deleting $w_1$ from $G'$, and note that $H$ contains the center $\{w_3, \ldots, w_{p+2}\}$ of $G'$. Lemma 6 then implies that $H$ does not contain $\bigcap_{i=1}^{p} N_{G'}[v_i]$ for distinct vertices $v_1, \ldots, v_p$, and so condition (1) fails.

To prove (2) $\Rightarrow$ (1), suppose $p \geq 1$ and condition (1) fails; specifically, suppose $H$ is a $CC$-subgraph of an induced subgraph $G'$ of $G$ such that $H$ contains distinct vertices $v_1, \ldots, v_p$ without containing $\bigcap_{i=1}^{p} N_{G'}[v_i]$. Lemma 6 implies that $H$ contains the center of a $K_{p+4} - P_4$ subgraph of $G'$, and so condition (2) fails.

A graph $G$ is called ptolemaic if $G$ is both chordal and contains no induced $K_5 - P_4$ subgraph (often called a gem); see [1] for additional characterizations. Corollary 8 corresponds to Theorem 4.

Corollary 8. For every chordal graph $G$, every $CC$-subgraph $H$ of an induced subgraph $G'$ of $G$ contains $N_{G'}[v]$ for some $v \in V(H)$ if and only if $G$ is ptolemaic.

Proof. This follows from the $p = 1$ case of Theorem 7. 

References


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