STRONG $d$-COLLAPSIBILITY

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Abstract. We introduce a notion of strong $d$-collapsibility. Using this notion, we simplify the proof of Matoušek and the author [4] showing that the nerve of a family of sets of size at most $d$ is $d$-collapsible.

1. Introduction

Simplicial complexes and $d$-collapsibility. A finite simplicial complex $K$ is a collection of subsets (called faces or simplices) of a finite set $X$ which is downwards closed, i.e., if $\sigma \in K$ and $\tau \subset \sigma$ then $\tau \in K$. The dimension of a face $\sigma \in K$ is defined to be the value $|\sigma| - 1$. The dimension of $K$ is the maximum of the dimensions of faces contained in $K$. Zero-dimensional faces are called vertices. Often it is assumed that $X$ is the set of vertices; in particular we will work with this assumption.

Wegner, in his seminal 1975 paper [7], introduced $d$-collapsible simplicial complexes. To define this notion, we first introduce an elementary $d$-collapse. Let $K$ be a simplicial complex and let $\sigma, \tau \in K$ be faces (simplices) such that

(i) $\dim \sigma \leq d - 1$,
(ii) $\tau$ is an inclusion-maximal face of $K$,
(iii) $\sigma \subseteq \tau$, and
(iv) $\tau$ is the only face of $K$ satisfying (ii) and (iii).

Then we say that $\sigma$ is a $d$-collapsible face of $K$ and that the simplicial complex $K' := K \setminus \{\eta \in K : \sigma \subseteq \eta \subseteq \tau\}$ arises from $K$ by an elementary $d$-collapse. If we want to emphasize $\sigma$, we write $K \xrightarrow{\sigma} K'$ (note that $K'$ is uniquely determined by $\sigma$ and $K$). A simplicial complex $K$ is $d$-collapsible if there exists a sequence of elementary $d$-collapses that reduces $K$ to the empty complex $\emptyset$.

The motivation of introducing $d$-collapsibility comes from combinatorial geometry as a tool for studying intersection patterns of convex sets. Our
A nerve and its \(d\)-collapsibility. Given a finite collection \(C = \{C_1, \ldots, C_n\}\) of sets, the nerve \(N(C)\) of this collection is a simplicial complex where \(C\) is the (multi)set of its vertices and where its faces are collections \(C_{i_1}, \ldots, C_{i_k}\) of vertices such that \(C_{i_1} \cap \cdots \cap C_{i_k}\) is non-empty. We emphasize that it is allowed that \(C_i = C_j\) for \(i \neq j\); i.e., \(C\) is a multiset. In particular for such \(C_i\) and \(C_j\) there are two (twin) vertices in the nerve.

Matoušek and the author [4] studied how far is the notion of \(d\)-collapsibility from its geometrical motivation. As one of the main tools they proved the following proposition.

**Proposition 1.1.** Suppose that \(C\) is a collection of sets of size at most \(d\). Then \(N(C)\) is \(d\)-collapsible.

We will introduce a notion of strong \(d\)-collapsibility and using this notion we simplify the proof of Matoušek and the author. We also hope that this notion can be used in a different context as well.

**Strong \(d\)-collapsibility.** Assume that \(\eta\) is a face of a complex \(K\). The link of \(\eta\) in \(K\) is a simplicial complex defined by \(\text{lk}(\eta, K) = \{\varnothing \in K : \varnothing \cap \eta = \varnothing, \varnothing \cup \eta \in K\}\). Assume that \(v\) is a vertex of \(K\) such that \(\text{lk}\{\{v\}\}, K\) is \((d-1)\)-collapsible. By an elementary strong \(d\)-collapse of \(K\) we mean the simplicial complex \(K'\) obtained by removing all faces containing \(v\), i.e., \(K' = K - v = \{\varnothing \in K : v \notin \varnothing\}\). If we want to emphasize \(v\), we write \(K \xrightarrow{v} K'\). A simplicial complex is strongly \(d\)-collapsible if it can be vanished by a sequence of elementary strong \(d\)-collapses.

We will prove the following results.

**Proposition 1.2.** Let \(d\) be a non-negative integer. Assume that a simplicial complex \(K\) is strongly \(d\)-collapsible then it is \(d\)-collapsible as well.

**Theorem 1.3.** Let \(d\) be a positive integer. Suppose that \(C\) is a collection of sets of size at most \(d\). Then \(N(C)\) is strongly \(d\)-collapsible.

Proposition 1.1 is an obvious consequence of these two results.

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1 Coincidentally, during the review process, the author learnt that Eckhoff [2] uses the notion strongly \(d\)-collapsible complex for a different mathematical object. The author, however, wishes to keep this name for simplicial complexes defined in this note, since this definition is analogous to strong collapsibility in topology [1].

2 In an elementary strong \(d\)-collapse we could also use an inductive definition where \(\text{lk}\{\{v\}\}, K\) would be assumed to be strong \((d-1)\)-collapsible and strong 0-collapsible would mean being a simplex. Thus we would get a similar (but perhaps different) notion of strong \(d\)-collapsibility. The forthcoming results would remain unchanged.
2. Properties of Strong $d$-collapsibility

First, we prove Proposition 1.2.

**Proof.** It is sufficient to show that an elementary strong $d$-collapse $K \xrightarrow{v} K'$ can be simulated by a sequence of elementary $d$-collapses. Let $L = \text{lk}(\{v\}, K)$. We know that $L$ is $(d-1)$-collapsible. Let $L \xrightarrow{\sigma_1} L_2 \xrightarrow{\sigma_2} \cdots \xrightarrow{\sigma_k} \emptyset$ be a sequence of elementary $d$-collapses. Then it is routine to check that

$$K \xrightarrow{\sigma_1 \cup \{v\}} K_2 \xrightarrow{\sigma_2 \cup \{v\}} \cdots \xrightarrow{\sigma_k \cup \{v\}} K'$$

is a sequence of elementary $d$-collapses which indeed ends up with $K'$. (For this, we remark that $K_i = K' \cup \{\vartheta \cup \{v\} : \vartheta \in L_i\}$.)

We remark that there are complexes which are $d$-collapsible, but not strongly $d$-collapsible. An example of such a complex is drawn in Figure 1. The thick lines are identified according to the arrows. There are higher-dimensional analogues of this complex; see the construction of complex $C(\rho)$ in [5].

![Figure 1. A complex which is 2-collapsible, but not strongly 2-collapsible.](image)

3. Strong $d$-collapsibility of a nerve

Here we prove Theorem 1.3. Let $a$ be a point which is not contained in the vertex set of a given complex $K$. The *cone* of $K$ is a simplicial complex given by $aK = K \cup \{\sigma \cup \{a\} : \sigma \in K\}$.

**Lemma 3.1.** If $K$ is $d$-collapsible, then $aK$ is $d$-collapsible as well.

**Proof.** Let $K \xrightarrow{\sigma_1} K_2 \xrightarrow{\sigma_2} \cdots \xrightarrow{\sigma_k} \emptyset$ be a sequence of elementary $d$-collapses of $K$. Then $aK \xrightarrow{\sigma_1} aK_2 \xrightarrow{\sigma_2} \cdots \xrightarrow{\sigma_k} a\emptyset = \emptyset$ is a sequence of elementary $d$-collapses of $aK$.

\footnote{Purely formally, one has to be a bit careful here and distinguish a simplicial complex $\{\emptyset\}$ containing a single empty face from $\emptyset$ containing no face.}
Proof of Theorem 1.3. We proceed by induction on $d$ and on the size of $C$. Theorem 1.3 is surely true if $C$ contains a single set or if $d = 1$.

Let $C_1 \in C$ be a set of maximal size. We only want to show that

$$\text{N}(C) \xrightarrow{C_1} \text{N}(C \setminus \{C_1\}),$$

since $\text{N}(C \setminus \{C_1\})$ is strongly $d$-collapsible by induction.

It is sufficient to check that $\text{lk}(C_1, \text{N}(C))$ is $(d - 1)$-collapsible. Let us denote $C_{C_1} = \{C \cap C_1 : C \in C \setminus \{C_1\}\}$. Then $\text{lk}(C_1, \text{N}(C)) = \text{N}(C_{C_1})$. If there is no set of size $d$ in $C_{C_1}$, then $\text{lk}(C_1, \text{N}(C))$ is $(d - 1)$-collapsible by induction and we are done.

Otherwise, let $D = \{D_1, \ldots, D_m\} \subseteq C_{C_1}$ be the collection of all sets of size $d$ in $C_{C_1}$. For every $D \in D$ we thus have $D = C_1$. It means that $\text{lk}(C_1, \text{N}(C)) = D_1 D_2 \ldots D_m \text{N}(C_{C_1} \setminus D)$, where $D_1 D_2 \ldots D_m$ stands for (iterated) cone with vertices $D_1, \ldots, D_m$. By Lemma 3.1 and induction it follows that $\text{lk}(C_1, \text{N}(C))$ is $(d - 1)$-collapsible. \hfill \Box

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