FOLD AND MYCIELSKIAN ON HOMOMORPHISM COMPLEXES

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ABSTRACT. Homomorphism complexes were introduced by Lovász to study topological obstructions to graph colorings. We show that folding in the second parameter of the homomorphism complex yields a homotopy equivalence. We study how the Mycielski construction changes the homotopy type of the homomorphism complex. We construct graphs showing that the topological bound obtained by odd cycles can be arbitrarily worse than the bound provided by $\mathrm{Hom}(K_2, G)$.

1. INTRODUCTION

Basic topological concepts, definition of graphs, simplicial complexes, posets, and their properties can be found in [3, 10, 12]. Readers interested in homomorphism complexes can found further references in [10].

We assume that graphs $G = (V(G), E(G))$ are *simple*, i.e., without loops and parallel edges. A *graph homomorphism* is a map $\phi : V(G) \to V(H)$, such that the image of every edge of the graph G is an edge of the graph H. Let $\Delta^{V(H)}$ be the simplex whose set of vertices is $V(H)$. Let denote by $C(G, H)$ the direct product $\prod_{x \in V(G)} \Delta^{V(H)}$, i.e., the copies of $\Delta^{V(H)}$ are indexed by vertices of G. A cell of $C(G, H)$ is a direct product of simplices $\prod_{x \in V(G)} \sigma_x$.

For any pair of graphs G and H let the *homomorphism complex* $\mathrm{Hom}(G, H)$ be a subcomplex of $C(G, H)$ where

$$c = \prod_{x \in V(G)} \sigma_x \in \mathrm{Hom}(G, H)$$

if and only if for any $u, v \in V(G)$ if $\{u, v\} \in E(G)$, then $\{a, b\} \in E(H)$ for any $a \in \sigma_u, b \in \sigma_v$. $\mathrm{Hom}(G, H)$ is a polyhedral complex whose cells are products of simplices and are indexed by functions (multi-homomorphisms)

$\eta : V(G) \to 2^{V(H)} \backslash \{\varnothing\}$, such that if $\{\imath, \jmath\} \in E(G)$, then for every $\tilde{\imath} \in \eta(\imath)$ and $\tilde{\jmath} \in \eta(\jmath)$ it follows that $\{\tilde{\imath}, \tilde{\jmath}\} \in E(H)$.

A $\mathbb{Z}_2$-*space* is a pair (X, ν) where X is a topological space and $\nu \colon X \to X$, called the $\mathbb{Z}_2$-action, is a homeomorphism such that $\nu^2 = \nu \circ \nu = \mathrm{id}_X$. The sphere S^n is understood as a $\mathbb{Z}_2$-space with the antipodal homeomorphism $x \mapsto -x$. A $\mathbb{Z}_2$-*map* between $\mathbb{Z}_2$-spaces is a continuous map which commutes with the $\mathbb{Z}_2$-actions. The $\mathbb{Z}_2$-*index* of a $\mathbb{Z}_2$-space (X, ν) is

$$\mathrm{ind}(X) = \min \left\{ n \geq 0 \mid \text{there is a } \mathbb{Z}_2\text{-map } X \to S^n \right\}.$$

The Borsuk–Ulam Theorem can be re-stated as $\mathrm{ind}(S^n) = n$. Another index-like quantity of a $\mathbb{Z}_2$-space, the *coindex* can be defined by

$$\mathrm{coind}(X) = \max \left\{ n \geq 0 \mid \text{there is a } \mathbb{Z}_2\text{-map } S^n \to X \right\}.$$

Suppose that X and Y are topological spaces. Two maps $f, g \colon X \to Y$ are *homotopic* (written $f \simeq g$) if there is a map $F \colon X \times [0,1] \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$. X and Y are called *homotopy equivalent* if there are maps $f \colon X \to Y$ and $g \colon Y \to X$, such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$.

Similarly in the $\mathbb{Z}_2$ world, two $\mathbb{Z}_2$-maps f, g are $\mathbb{Z}_2$-homotopic if there is a $\mathbb{Z}_2$-map $F \colon X \times [0,1] \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$, where the $\mathbb{Z}_2$-action on $X \times [0,1]$ is just the action of X on each slice $X \times t$. A $\mathbb{Z}_2$-map $f \colon X \to Y$ is a $\mathbb{Z}_2$-*homotopy equivalence* if there exists a $\mathbb{Z}_2$-map $g \colon Y \to X$ such that $g \circ f$ is $\mathbb{Z}_2$-homotopic to id_X and $f \circ g$ is $\mathbb{Z}_2$-homotopic to id_Y. In this case we say that X and Y are $\mathbb{Z}_2$-*homotopy equivalent*. A general reference for group actions, $\mathbb{Z}_2$-spaces and related concepts and facts is the textbook of Bredon [4].

We will use the following Quillen-type Lemma. This version which turned out to be especially useful for dealing with homomorphism complexes was proven by Babson and Kozlov [1, Proposition 3.2].

Lemma 1.1. *Let* $\phi : P \to Q$ *be a map of finite posets. If* ϕ *satisfies*

(A) $\Delta(\phi^{-1}(q))$ *is contractible, for every* $q \in Q$, *and*
(B) *for every* $p \in P$ *and* $q \in Q$ *with* $\phi(p) \geq q$ *the poset* $\phi^{-1}(q) \cap P_{\leq p}$ *has a maximal element,*

then ϕ *is a homotopy equivalence.*

In Section 2 we will show that folding in the second parameter of the homomorphism complex yields a homotopy equivalence. In Section 3 we study how the generalized Mycielski construction changes the homotopy type of the homomorphism complex. As an application we show that the topological lower bound provided by odd cycles can be arbitrarily worse than the bound using $\mathrm{Hom}(K_2, G)$.

2. Folding

We will denote for a graph G the neighbors of $v \in V(G)$ by $\mathtt{N}(v)$; in other words, $\mathtt{N}(v) := \{u \in G \mid \{u, v\} \in E(G)\}$.

Definition 2.1. $G - v$ *is called a* fold *of a graph* G *if there exist* $u \in V(G)$, $u \neq v$ *such that* $N(u) \supseteq N(v)$.

It was proven in [1, Proposition 5.1] that fold in the first parameter of the homomorphism complex yields a homotopy equivalence. It was noticed in [7, Lemma 3.1] that one can fold in the second parameter if the deleted vertex is an identical twin. Now we will show that the fold in the second parameter is a homotopy equivalence in general. This was generalized by Kozlov [9] into simple homotopy equivalence. Note that our proof works for graphs with loops as well.

Theorem 2.2. *Let G and H be graphs and $u, v \in V(H)$ such that $N(u) \supseteq N(v)$. Also, let $i : H - v \hookrightarrow H$ be the inclusion and $\omega : H \to H - v$ the unique graph homomorphism which maps v to u and fixes other vertices. Then, these two maps induce homotopy equivalences*

$$i_H : \mathrm{Hom}(G, H - v) \to \mathrm{Hom}(G, H) \text{ and}$$

$$\omega_H : \mathrm{Hom}(G, H) \to \mathrm{Hom}(G, H - v),$$

respectively.

Proof. We will show that ω_H satisfies the conditions (A) and (B) of Lemma 1.1. Unfolding the definitions, we see that for a cell τ of $\mathrm{Hom}(G, H)$, $\tau : V(G) \to 2^{V(H)} \setminus \{\varnothing\}$, we have

$$\omega_H(\tau)(x) = \begin{cases} \tau(x) & \text{if } v \notin \tau(x), \\ (\tau(x) \cup \{u\}) \setminus \{v\} & \text{otherwise.} \end{cases}$$

Let η be a cell of $\mathrm{Hom}(G, H - v)$, $\eta : V(G) \to 2^{V(H)\setminus\{v\}} \setminus \{\varnothing\}$. Then $\omega_H^{-1}(\eta)$ is a set of all η' such that, for all $x \in V(G)$,

(1) $\eta'(x) = \eta(x)$, if $u \notin \eta(x)$; or
(2) if $u \in \eta(x)$ then (at least theoretically) we have the following possibilities:
  (a) $\eta'(x) = \eta(x)$,
  (b) $\eta'(x) = \eta(x) \setminus \{u\} \cup \{v\}$,
  (c) $\eta'(x) = \eta(x) \cup \{v\}$.

Because of the condition $N(u) \supseteq N(v)$, not all η' satisfying conditions (2)(b) and (2)(c) have to belong to $\mathrm{Hom}(G, H)$. Note that if H is simple, $u \in \eta(x)$ and $(x, y) \in E(G)$ then $u \notin \eta(y)$. But this is not true in general. This means that for any x it depends not only on $N(v)$ that we can use conditions (2)(b) and (2)(c) to get $\eta' \in \mathrm{Hom}(G, H)$. It depends on the choices of $\eta'(y)$ at the neighbors of x.

The map $\varphi \colon \omega_H^{-1}(\eta) \to \omega_H^{-1}(\eta)$ is defined by

$$\varphi(\zeta)(x) = \begin{cases} \zeta(x) & \text{if } u \in \zeta(x), \\ \zeta(x) & \text{if } u, v \notin \zeta(x), \\ \zeta(x) \cup \{u\} & \text{if } u \notin \eta(x) \text{ and } v \in \eta(x), \end{cases}$$

for all $x \in V(G)$. We show that φ is a homotopy equivalence by using Lemma 1.1.

$\varphi^{-1}(\zeta)$ is clearly a cone with apex ζ (it is the maximal element) so it is contractible and condition (A) satisfied for φ. Take now any

$$\tau \in \varphi^{-1}\left(\left(\omega_H^{-1}(\eta)\right)_{\geq \zeta}\right).$$

The maximal element ξ of the set $\varphi^{-1}(\zeta) \cap (\omega_H^{-1}(\eta))_{\leq \tau}$ is ζ.

Since φ satisfies conditions (A) and (B) it is a homotopy equivalence. The image of φ is a cone with apex η so contractible and condition (A) is satisfied for ω_H. Take now any $\tau \in \omega_H^{-1}(\mathrm{Hom}(G, H - v)_{\geq \eta})$. The maximal element ξ of the set $\omega_H^{-1}(\eta) \cap (\mathrm{Hom}(G, H))_{\leq \tau}$ is

$$\xi(x) = \begin{cases} \eta(x) & \text{if } u \notin \eta(x), \\ \tau(x) \cap (\eta(x) \cup \{u\}) & \text{otherwise.} \end{cases}$$

Since it satisfies conditions (A) and (B), we conclude that $\mathrm{sd}(\omega_H)$ and hence also ω_H are homotopy equivalences.

It is left to prove that i_H is also a homotopy equivalence. It is clear that $\omega_H \circ i_H = \mathrm{id}_{\mathrm{Hom}(G, H-v)}$. Let ϑ be the homotopy inverse of ω_H. Now we have that $i_H \circ \omega_H \simeq \vartheta \circ \omega_H \circ i_H \circ \omega_H \simeq \vartheta \circ \omega_H \simeq \mathrm{id}_{\mathrm{Hom}(G, H)}$. $\qquad\square$

3. Generalized Mycielski construction

Recall ([14] page 16) that the generalized Mycielskian $M_r(G)$ of a graph $G = (V, E)$ has vertex set $\{z\} \cup (V \times \{1, 2, \ldots, r\})$, z is connected to all vertices of $V \times \{1\}$, (v, i) is connected to $(u, i + 1)$ for all $(u, v) \in E$ and $i = 1, 2, \ldots, r - 1$, and a copy of G sits on $V \times \{r\}$.

We prove the following theorem, which was predicted in [14]. In [8] only the homotopy equivalence was proven.

Theorem 3.1. *For every graph G and every $r \geq 1$, the homomorphism complex $\mathrm{Hom}(K_2, M_r(G))$ is $\mathbb{Z}_2$-homotopy equivalent to the suspension*

$$\mathrm{susp}(\mathrm{Hom}(K_2, G)).$$

Our main tool is Bredon's theorem [4] which allows us to use standard topological combinatorics to prove $\mathbb{Z}_2$-homotopy equivalence (see [15] for other applications).

Theorem 3.2 (Bredon)**.** *Suppose that $f \colon X \to Y$ is a (simplicial) $\mathbb{Z}_2$-map of* free *simplicial $\mathbb{Z}_2$-complexes X and Y. The $\mathbb{Z}_2$-map $f \colon X \to Y$ is a $\mathbb{Z}_2$-homotopy equivalence if and only if it is an ordinary homotopy equivalence.*

Proof of Theorem 3.1. We will use induction on r. For $r = 1$ it was proven in [5]. Here we give a new proof.

BASE CASE: $r = 1$:

We extend the face poset of $\mathrm{Hom}(K_2, G)$ with two non-comparable maximal elements max_1, max_2 to obtain the face poset

$$\mathcal{F}(\mathrm{susp}(\mathrm{Hom}(K_2, G))).$$

We define the map

$$f\colon P := \mathcal{F}(\mathrm{Hom}(K_2, M_1(G))) \to \mathcal{F}(\mathrm{susp}(\mathrm{Hom}(K_2, G))) =: Q$$

by

$$
\begin{aligned}
f(A, B) &= (A, B), \\
f(A \cup \{z\}, B) &= \max_1, \\
f(z, B) &= \max_1,
\end{aligned}
$$

where (A, B) and $(A \cup \{z\}, B)$ denote the cells, and assuming that $z \notin A, B \subseteq V$ and $A, B \neq \varnothing$. We can do this since a cell η can be identified with $(\eta(1), \eta(2))$, where $V(K_2) = \{1, 2\}$.

Since we want a $\mathbb{Z}_2$-map, f is well defined. (For example $f(B, z) = \max_2$.) f is clearly monotone (simplicial), as are all maps we will introduce later. We will keep using Lemma 1.1.

$f^{-1}(A, B)$ is just (A, B) so in this case (A) and (B) are satisfied. If $f(p) = \max_1$ then $f^{-1}(\max_1) \cap P_{\leq p}$ has a maximal element p. To show that $R := f^{-1}(\max_1)$ is contractible we define $g\colon R \to \mathrm{im}(g)$ by $g(A \cup \{z\}, B) = (z, B)$ and $g(z, B) = (z, B)$. g is a homotopy equivalence since $g^{-1}(z, B)$ is a cone with apex (z, B) and let $q = (z, B)$ and $p = (\tilde{A} \cup \{z\}, \tilde{B})$ such that $g(p) \geq q$ $(B \subseteq \tilde{B})$. Now the maximal element of $g^{-1}(q) \cap R_{\leq p}$ is $(\tilde{A} \cup \{z\}, B)$. Moreover $\mathrm{im}(g)$ is a cone with apex (z, V).

INDUCTION STEP: $r \Rightarrow r + 1$:

The graph homomorphism

$$\phi\colon M_{r+1}(G) \to M_r(G)$$

defined by $\phi(z) = z$ and $\phi(v \times i) = v \times \min\{i, r\}$ gives a $\mathbb{Z}_2$-map

$$f\colon P := \mathcal{F}(\mathrm{Hom}(K_2, M_{r+1}(G))) \to \mathcal{F}(\mathrm{Hom}(K_2, M_r(G))) =: Q.$$

We will show that f is a homotopy equivalence. If

$$(A \cup B) \cap (\{z\} \cup V \times \{1, 2, \ldots, r - 1\}) \neq \varnothing,$$

then $|f^{-1}(A, B)| = 1$ so in Lemma 1.1 (A) and (B) are satisfied. In the case when $(A \cup B) \subseteq V \times r$, by slight abuse of notation, we will write $(A \times r, B \times r)$ instead of (A, B) to show which copy of V belongs to A and B in $M_i(G)$. Let $p = (\tilde{A}_1 \times r \cup \tilde{A}_2 \times (r + 1), \tilde{B} \times (r + 1))$ such that $f(p) \geq (A \times r, B \times r)$. Now the maximal element of $f^{-1}(A \times r, B \times r) \cap P_{\leq p}$ is $((\tilde{A}_1 \cap A) \times r \cup (\tilde{A}_2 \cap A) \times (r + 1), B \times (r + 1))$. We should show that $S := f^{-1}(A \times r, B \times r)$ is contractible as well.

We define $g\colon S \to \mathrm{im}(g)$ by $g(A \times r, B \times r) = (A \times r, B \times r)$,

$$g(A_1 \times r \cup A_2 \times (r + 1), B \times (r + 1))$$
$$= (A_1 \times r \cup A \times (r + 1), B \times (r + 1)),$$

where $A_1 \cup A_2 = A$, and symmetrically

$$g(A \times (r + 1), B_1 \times r \cup B_2 \times (r + 1))$$
$$= (A \times (r + 1), B_1 \times r \cup B \times (r + 1)),$$

where $B_1 \cup B_2 = B$. $\mathrm{im}(g)$ is a cone with apex $(A \times (r+1), B \times (r+1))$. $g^{-1}(q)$ is a cone with apex q. Let (without loss of generality) $q = (A_1 \times r \cup A \times (r+1), B \times (r+1))$ and $p = (\tilde{A}_1 \times r \cup \tilde{A}_2 \times (r+1), B \times (r+1))$ such that $f(p) \geq q$ $(\tilde{A}_1 \supseteq A_1)$. Now the maximal element of $f^{-1}(q) \cap S_{\leq p}$ is $(A_1 \times r \cup A \times (r+1), B \times (r+1))$.

This completes the proof. $\square$

Remark: There are interesting consequences of Theorem 3.1. As $M_1(K_n) = K_{n+1}$ and $\mathrm{Hom}(K_2, K_2)$ homeomorphic to S^0 we get that $\mathrm{Hom}(K_2, K_n)$ is $\mathbb{Z}_2$-homotopy equivalent[1] to S^{n-2}. This together with the functoriality of the Hom construction already implies Lovász's topological lower bound for the chromatic number [1, 10, 11, 12]:

$$\chi(G) \geq \mathrm{ind}(\mathrm{Hom}(K_2, G)) + 2.$$

In general about $\mathrm{Hom}(H, M_r(G))$ or $\mathrm{Hom}(M_r(H), G)$ one cannot expect something like Theorem 3.1, as shown by the following well known or easily computable examples:

$$\mathrm{Hom}(K_3, K_2) = \varnothing, \qquad\qquad \mathrm{Hom}(K_3, M_2(K_2) = C_5) = \varnothing,$$

$$\mathrm{Hom}(K_3, M_1(K_2)) \cong \bigvee^{5} S^0, \qquad \mathrm{Hom}(K_3, M_1(K_3)) \simeq \bigvee^{13} S^1,$$

$$\mathrm{Hom}(K_3, M_2(K_3)) \simeq \bigvee^{5} S^0, \qquad \mathrm{Hom}(C_5, K_2) = \varnothing,$$

$$\mathrm{Hom}(C_5, M_2(K_2)) \cong \bigvee^{9} S^0, \qquad \mathrm{Hom}(C_5, M_1(K_2)) \cong S^1 \cup S^1,$$

$$\mathrm{Hom}(C_5, M_2(C_5)) \simeq \bigvee^{41} S^1, \qquad \mathrm{Hom}(C_5, M_1(K_3)) \cong \mathbb{RP}^3.$$

But still something can be said.

Theorem 3.4. *If $n \geq 3$ and $r \geq 2$ then $\mathrm{Hom}(K_n, M_r(G))$ is homotopy equivalent to $\mathrm{Hom}(K_n, G)$.*

Proof. Let $\tilde{G}$ be a subgraph of $M_r(G)$ induced by the vertex set $V \times \{r, r-1\}$. Clearly $\mathrm{Hom}(K_n, M_r(G))$ is the same as $\mathrm{Hom}(K_n, \tilde{G})$. It is easy to see that $\tilde{G}$ folds down to G. Now Theorem 2.2 completes the proof. $\square$

Remark: Since $\chi(M_2(G)) = \chi(G) + 1$ we obtain graphs such that *no* topological lower bound using $\mathrm{Hom}(K_n, *)$ ($n \geq 3$) can give sharp bound on their chromatic number. On the other hand, for these graphs $\mathrm{Hom}(K_2, *)$ might provide a sharp bound.

It is interesting to mention that $\chi(M_r(G)) > \chi(G)$ does not hold in general if $r \geq 3$, e.g., if G is the graph from Figure 1 then $\chi(M_3(G)) = \chi(G) = 4$.

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[1]It is known [1] that $\mathrm{Hom}(K_2, K_n)$ is $\mathbb{Z}_2$-homeomorphic to S^{n-2}.

In the proof of Theorem 3.4 we used basically the following observation: $\mathrm{Hom}(K_n, G)$ ($n \geq 3$) is homeomorphic to $\mathrm{Hom}(K_n, G - v)$, if there is no triangle in G containing a vertex $v \in G$.
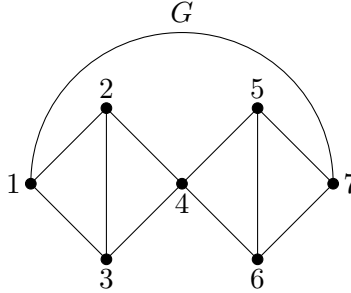

FIGURE 1. A graph G such that $\chi(M_3(G)) = \chi(G) = 4$.

Theorem 3.6. *If $2n + 1 \leq 2r$ and $r \geq 2$ then $\mathrm{Hom}(C_{2n+1}, M_r(G))$ is homotopy equivalent to $\mathrm{Hom}(C_{2n+1}, G)$.*

Proof. The same as the proof of Theorem 3.4, just now $\tilde{G}$ should be a subgraph of $M_r(G)$ induced by the vertex set $V(M_r(G)) \setminus \{z\}$. $\qquad\square$

Remark: The condition $2n+1 \leq 2r$ in Theorem 3.6 is the best possible since

$$\mathrm{Hom}(C_5, K_2) = \varnothing \ \text{ but } \ \mathrm{Hom}(C_5, M_2(K_2)) \cong \bigvee^{9} S^0.$$

As we already mentioned Lovász's original bound can be stated as

$$\chi(G) \geq \mathrm{ind}(\mathrm{Hom}(K_2, G)) + 2,$$

and it is known [6] that this bound can be arbitrarily bad. Babson and Kozlov [2, 10] solved the Lovász Conjecture, and showed that

$$\chi(G) \geq \mathrm{coind}(\mathrm{Hom}(C_{2n+1}, G)) + 3.$$

Surprisingly Schultz [13] discovered that these two bounds are closely related:

$$\mathrm{ind}(\mathrm{Hom}(K_2, G)) + 2 \geq \mathrm{coind}(\mathrm{Hom}(C_{2n+1}, G)) + 3.$$

Now the question is how big the gap can be in this last inequality.

Theorem 3.8. *The topological bound obtained by odd cycles (≥ 5) can be arbitrarily worse than the bound provided by $\mathrm{Hom}(K_2, *)$.*

Proof. Let G be a graph such that $\mathrm{Hom}(K_2, G)$ is $\mathbb{Z}_2$-homotopy equivalent to S^m. For G we have that

$$\mathrm{ind}(\mathrm{Hom}(K_2, G)) + 2 \geq \mathrm{coind}(\mathrm{Hom}(C_{2n+1}, G)) + 3.$$

We define

$$H := \underbrace{M_r(\ldots(M_r(G)))}_{k} \quad (2n+1 \leq 2r).$$

By Theorem 3.1 we get that $\mathrm{ind}(\mathrm{Hom}(K_2,H)) = \mathrm{ind}(\mathrm{Hom}(K_2,G)) + k$. Using Theorem 3.6 we have $\mathrm{coind}(\mathrm{Hom}(C_{2n+1},G)) = \mathrm{coind}(\mathrm{Hom}(C_{2n+1},H))$. So we showed that for any k one can construct a graph H such that

$$\mathrm{ind}(\mathrm{Hom}(K_2,H)) + 2 \geq \mathrm{coind}(\mathrm{Hom}(C_{2n+1},H)) + 3 + k.$$

Moreover, if $\chi(G) = \mathrm{ind}(\mathrm{Hom}(K_2,G)) + 2$, then for H we get that $\chi(H) = \mathrm{ind}(\mathrm{Hom}(K_2,H)) + 2$. Note that this construction works with the cohomological index used in [13] as well. $\square$

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