Contributions to Discrete Mathematics

Volume 21, Number 1, Pages 216–226
ISSN 1715-0868

STAR COLORING OF SOME TOROIDAL GRAPHS

KRISTINA AGO AND ANNA SLIVKOVÁ

Abstract. A proper coloring of a graph is a star coloring if there is no bicolored path on four vertices, or, equivalently, if every connected subgraph induced by any two color classes is a star. We investigate the star chromatic number χ_s of some well-known toroidal graphs. First, it is known that for the d-dimensional toroidal grid TG_d the star chromatic number is $\mathcal{O}(d^2)$. Some results published in the literature that are applicable to this family of graphs improve this bound to $\mathcal{O}(d^{\frac{3}{2}})$. In this article we show that $\chi_s(TG_d) = \mathcal{O}(d)$. Furthermore, we investigate the star chromatic number of the honeycomb torus $HT(n)$ of size n, and show that $\chi_s(HT(n)) = 4$.

1. Introduction

A proper coloring of a graph is an attaching of colors to the vertices of the graph such that there are no adjacent vertices with the same color. A proper coloring of a graph is a star coloring if there is no bicolored path on four vertices or, equivalently, if every connected subgraph induced by any two color classes is a star, which justifies the name. The star chromatic number, denoted by $\chi_s(G)$, is the least number of colors needed for a star coloring of the graph G. This coloring was introduced by Grünbaum in [6]. For the fundamental article on this topic, one can consider the work of Fertin, Raspaud, and Reed [4]. In this article, the authors provide the exact value of the star chromatic number for certain classes of graphs, such as trees, cycles, complete bipartite graphs, outerplanar graphs, and 2-dimensional grids. For other families of graphs, such as planar graphs, hypercubes, d-dimensional grids (where $d \geqslant 3$), d-dimensional tori (where $d \geqslant 2$), graphs with bounded treewidth, and cubic graphs, they establish bounds. They

Received by the editors January 14, 2025, and in revised form October 15, 2025.

2020 *Mathematics Subject Classification.* 05C15, 05C38.

Key words and phrases. star coloring, star chromatic number, d-dimensional toroidal grid, honeycomb torus.

The authors were supported by the Ministry of Science, Technological Development and Innovation of the Republic of Serbia (grants no. 451-03-137/2025-03/200125 & 451-03-136/2025-03/200125). The first author was also supported by the Hungarian Academy of Sciences (the *Domus* program).

also left numerous problems open, including obtaining optimal results for certain families of graphs or improving their nonoptimal bounds.

One of the classes of graphs in which we were interested is the toroidal grids. In the mentioned article the authors gave an upper bound on the star chromatic number of the d-dimensional toroidal grid TG_d; they proved that $\chi_s(TG_d) \leqslant 2d^2 + d + 1$ (in some special cases $\chi_s(TG_d) \leqslant 2d + 1$). Since then, some results have been published in the literature that are applicable to this family of graphs and they provide better bounds. Fertin et al. proved [4] that if G is a graph with maximum degree Δ, then $\chi_s(G) \leqslant \lceil 20\Delta^{\frac{3}{2}} \rceil$. This was improved by Fu and Xie [5], who proved that $\chi_s(G) \leqslant \lceil 7\Delta^{\frac{3}{2}} \rceil$. Another improvement was made by Ndreca, Procacci and Scoppola [9], who showed that $\chi_s(G) \leqslant \lceil 4.34\Delta^{\frac{3}{2}} + 1.5\Delta \rceil$. As a d-dimensional toroidal grid is a $2d$-regular graph the mentioned results show that $\chi_s(TG_d)$ is $\mathcal{O}(d^{\frac{3}{2}})$. In this article we improve these results and give an upper bound on the star chromatic number of TG_d which is $\mathcal{O}(d)$.

Another highly studied class of graphs similar to toroidal grids is honeycomb tori. Stojmenovic [12] introduced the honeycomb torus network on which he analyzed communication algorithms. Apart from computer science, honeycomb tori have interesting properties from the perspective of graph theory (see [2, 8, 13]). Honeycomb tori are 3-regular graphs. For these graphs there are also some results in the literature that we can apply. Chen, Raspaud and Wang proved [3] that the star chromatic number of a subcubic graph is at most 6. On the other hand, Xie, Xiao and Zhao proved [14] that the star chromatic number of a cubic graph is at least 4. This result was generalized recently by Shalu and Antony. We state their result on its own because we shall refer to it later.

Theorem 1.1 ([11])**.** *Let G be a d-regular graph, with $d \geqslant 2$. Then,* $\chi_s(G) \geqslant \lceil \frac{d+4}{2} \rceil$.

In this article we prove that the star chromatic number of the honeycomb torus of any size is 4. The technique we employ is conceptually similar to the so-called restricted star coloring—a star coloring with an additional condition for bicolored paths on three vertices (see, for example, [10]).

2. d-dimensional toroidal grids

The Cartesian product of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, denoted by $G_1 \square G_2$, is the graph such that the set of vertices is $V_1 \times V_2$ and two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if either ($u_1 = v_1$ and u_2 is adjacent to v_2) or ($u_2 = v_2$ and u_1 is adjacent to v_1).

The d-dimensional torus, $d \geqslant 2$, (or the toroidal d-dimensional grid) with n_i vertices in dimension i is denoted by $TG(n_1, n_2, \ldots, n_d)$, where $n_i \geqslant 3$ for all $1 \leqslant i \leqslant d$. We recall that $TG(n_1, n_2, \ldots, n_d)$ is the Cartesian product of d cycles of length n_i, $1 \leqslant i \leqslant d$, that is, $TG(n_1, n_2, \ldots, n_d) = C_{n_1} \square C_{n_2} \square \ldots \square C_{n_d}$. Sometimes we also use the notation TG_d when it is not important to emphasize the number of vertices in certain dimensions. Fertin et al. proved the following theorem.

Theorem 2.1 ([4]). *Let $TG_d = TG(n_1, n_2, \ldots, n_d)$ be any d-dimensional torus, where $d \geqslant 2$. Then*

$$d + 2 \leqslant \chi_s(TG_d) \leqslant \begin{cases} 2d+1, & \text{when } 2d+1 \text{ divides each } n_i; \\ 2d^2 + d + 1, & \text{otherwise.} \end{cases}$$

There are a few specific results in the literature for low-dimensional cases, for example, in [1] it is proved that

$$\chi_s(TG(m,n)) = \begin{cases} 6, & \text{when } (m,n) = (3,3) \text{ or } (m,n) = (3,5); \\ 5, & \text{otherwise.} \end{cases}$$

In [7] it is proved that $\chi_s(TG(3,3,k)) = 7$ for $k \geqslant 3$, $\chi_s(TG(\alpha i, \beta j, \gamma k)) = 6$, where $\alpha, \beta, \gamma \in \{4, 6\}$ and $i, j, k \geqslant 1$, as well as $\chi_s(TG(4i, 4j, 4l, 4k)) \leqslant 9$ for $i, j, k, l \geqslant 1$.

Now we introduce the notion of (almost) wavy coloring, which our proof is based on, and it might prove helpful for other classes of graphs in the future. Let G be a cycle or a path with n vertices, and $c : V(G) \to \{0, 1, \ldots, p-1\}$ a coloring function. The graph G is said to be colored *wavily with step* i, $1 \leqslant i \leqslant \lfloor \frac{p-1}{2} \rfloor$, if for every two adjacent vertices u and v it holds that $c(u) - c(v) \mod p$ is i or $p - i$. The graph G is said to be colored *almost wavily with step* i, $1 \leqslant i \leqslant \lfloor \frac{p-1}{4} \rfloor$, if for every two adjacent vertices u and v it holds that $c(u) - c(v) \mod p$ is i, $2i$, $p - i$ or $p - 2i$. The term *(almost) wavy coloring* will also be used. We are now ready to prove the following proposition.

Proposition 2.2. *Let $TG(n_1, n_2, \ldots, n_d)$ be any d-dimensional torus, $d \geqslant 2$. Then*

$$\chi_s(TG(n_1, n_2, \ldots, n_d)) \leqslant 8d - 3.$$

Proof. Let $TG(n_1, n_2, \ldots, n_d) = C_{n_1} \square C_{n_2} \square \ldots \square C_{n_d}$. First we prove the following claim.

Claim 2.3. *For each cycle C_{n_i} there exists an almost wavy star coloring $c_i : V(C_{n_i}) \to \{0, 1, \ldots, 8d-4\}$ with step $2i - 1$.*

Proof. Let $C_{n_i} = v_0 v_1 \ldots v_{n_i - 1} v_0$. Let us first discuss the case when the length of C_{n_i} is even. If $n_i = 4$, we define

$$c_i(v_j) = \begin{cases} j \cdot (2i-1), & \text{for } j = 0, 1, 2; \\ (2i-1), & \text{for } j = 3. \end{cases} \tag{2.1}$$

If $n_i = 6$, we define

$$c_i(v_j) = \begin{cases} j \cdot (2i-1), & \text{for } j = 0, 1, 2, 3; \\ (6-j) \cdot (2i-1), & \text{for } j = 4, 5. \end{cases} \tag{2.2}$$

It is easy to see that (2.1) and (2.2) are wavy star colorings with step $2i-1$ of the 4-length and the 6-length cycle, respectively. Now we shall define an adequate coloring for cycles of larger even lengths. Note that every even number greater than or equal to 8 can be represented as $k \cdot 4 + l \cdot 6$ for some nonnegative integers k and l. Thus, if $C_{n_i} = v_0 v_1 \dots v_{n_i-1} v_0$ is an even-length cycle, first, we split it into k successive paths with 4 vertices and the remaining into l successive paths with 6 vertices, then we color these paths according to the rules given in (2.1) and (2.2), respectively. Sometimes, this can be done in more than one way. One can note that the result is always a wavy coloring of C_{n_i} with step $2i-1$.

Let us move to the case when the length of C_{n_i} is odd, that is, $C_{n_i} = v_0 v_1 \dots v_{n_i-1} v_0$ for an odd number n_i. In that case we temporarily attach a vertex v_{n_i} between the vertices v_{n_i-1} and v_0, and then color the new cycle wavily like an even-length cycle by the rule from the previous paragraph. Let us write c'_i for this coloring and define a coloring c_i as $c_i(v_j) = c'_i(v_j)$ for $j = 0, 1, \dots, n_i - 1$. For adjacent vertices u and v, if $\{u, v\} \neq \{v_{n_i-1}, v_0\}$, $c_i(u) - c_i(v) \mod 8d-3$ is always $2i-1$ or $(8d-3)-(2i-1)$, while, if $\{u, v\} = \{v_{n_i-1}, v_0\}$, $c_i(u) - c_i(v) \mod 8d-3$ is $2(2i-1)$ or $(8d-3)-2(2i-1)$. Thus, c_i is an almost wavy coloring of C_{n_i}. The proof of the claim is now completed. □

Example 2.4. *An almost wavy coloring c_3 of $C_9 = v_0 v_1 \dots v_8 v_0$ from $TG(8,3,9)$ with step 5 is*

$$c_3 : \begin{pmatrix} v_0 & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 \\ 0 & 5 & 10 & 5 & 0 & 5 & 10 & 15 & 10 \end{pmatrix}.$$

Let us represent each vertex u of $TG(n_1, n_2, \dots, n_d)$ by its coordinates in each dimension, that is, $u = (x_1, \dots, x_d)$ where each x_i, $1 \leqslant i \leqslant d$ satisfies $0 \leqslant x_i \leqslant n_i - 1$. Note that if two vertices are adjacent, their representations will differ by one in exactly one of the d coordinates.

We assign a color $c(u)$ to u according to the following equation:

$$c((x_1, x_2, \dots, x_d)) = \sum_{i=1}^{d} c_i(v_{x_i}) \mod 8d-3,$$

where $c_i(v_{x_i})$ denotes the color of the vertex v_{x_i} in the i-th cycle in order colored by c_i according to the previous claim.

First, let us prove that c is a proper coloring. Let u and u' be two adjacent vertices. Without loss of generality let $u = (x_1, x_2, \dots, x_j, \dots, x_d)$ and $u' = (x_1, x_2, \dots, x_j \pm 1, \dots, x_d)$, $1 \leqslant j \leqslant d$. If $c(u) = c(u')$, then

$$\sum_{\substack{i=1 \\ i \neq j}}^{d} c_i(v_{x_i}) + c_j(v_{x_j}) \equiv \sum_{\substack{i=1 \\ i \neq j}}^{d} c_i(v_{x_i}) + c_j(v_{x_j \pm 1}) \pmod{8d-3},$$

that is $c_j(v_{x_j}) \equiv c_j(v_{x_j \pm 1}) \pmod{8d-3}$. Since $0 \leqslant c_j(v_{x_j}), c_j(v_{x_j \pm 1}) \leqslant 8d-4$, we have $c_j(v_{x_j}) = c_j(v_{x_j \pm 1})$. This is a contradiction, as c_j is a proper coloring of C_{n_j}.

Now, let us show that this coloring is a star coloring. We shall prove that if two vertices are of distance 2 and have the same color, then their coordinates have to differ in a single dimension. Let u' and u'' be two vertices such that $u' = (x_1, x_2, \ldots, x_j \pm 1, \ldots, x_k, \ldots, x_d)$ and $u'' = (x_1, x_2, \ldots, x_j, \ldots, x_k \pm 1, \ldots, x_d)$, and $c(u') = c(u'')$. Let $u = (x_1, x_2, \ldots, x_j, \ldots, x_k, \ldots, x_d)$ be the common neighbor of u' and u''. Then, $c(u') = c(u'')$ implies

$$c(u) + \alpha(2j-1) \equiv c(u) + \beta(2k-1) \pmod{8d-3},$$

which reduces to

$$\alpha(2j-1) \equiv \beta(2k-1) \pmod{8d-3}, \tag{2.3}$$

where $\alpha, \beta \in \{\pm 1, \pm 2\}$ (it follows from the fact that the differences between the colors of adjacent vertices along a cycle are limited—see the very last part of Claim 2.3). A straightforward calculation confirms that (2.3) cannot be true for $1 \leqslant j < k \leqslant d$. Thus, if two vertices are of distance 2 and have the same color, then their coordinates indeed have to differ in a single dimension j, which further implies that if we have a bicolored path with four vertices, all their coordinates have to differ in a single dimension j. However, it would imply the existence of a bicolored path with four vertices in C_{n_j} colored by c_j, which is impossible. Therefore, there is no bicolored path with four vertices, so c is a star coloring. $\square$

In Figure 1 we can see a star coloring of $C_5 \,\square\, C_7$ and $C_8 \,\square\, C_9$ with $8 \cdot 2 - 3 = 13$ colors, where in the left picture both cycles are colored almost wavily, while, in the right picture C_8 is colored wavily, and C_9 is colored almost wavily.

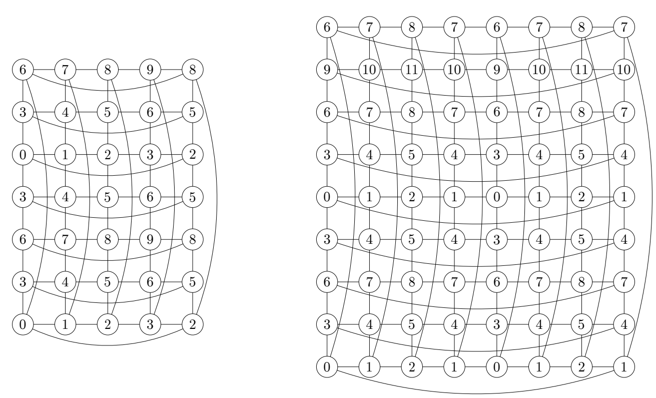

FIGURE 1. A star coloring of $TG(5,7)$ and $TG(8,9)$.

It should be noted that, as mentioned earlier, two-dimensional toroidal grids are always star-colorable with 5 or 6 colors. The strength of our method becomes apparent when dealing with high-dimensional cases; however, higher-dimensional instances are difficult (practically impossible) to represent visually. We provide a further three-dimensional example. In Figure 2 we can see a star coloring of $C_8 \,\square\, C_3 \,\square\, C_9$ with $8 \cdot 3 - 3 = 21$ colors, where C_8 is colored wavily, while C_3 and C_9 are colored almost wavily.

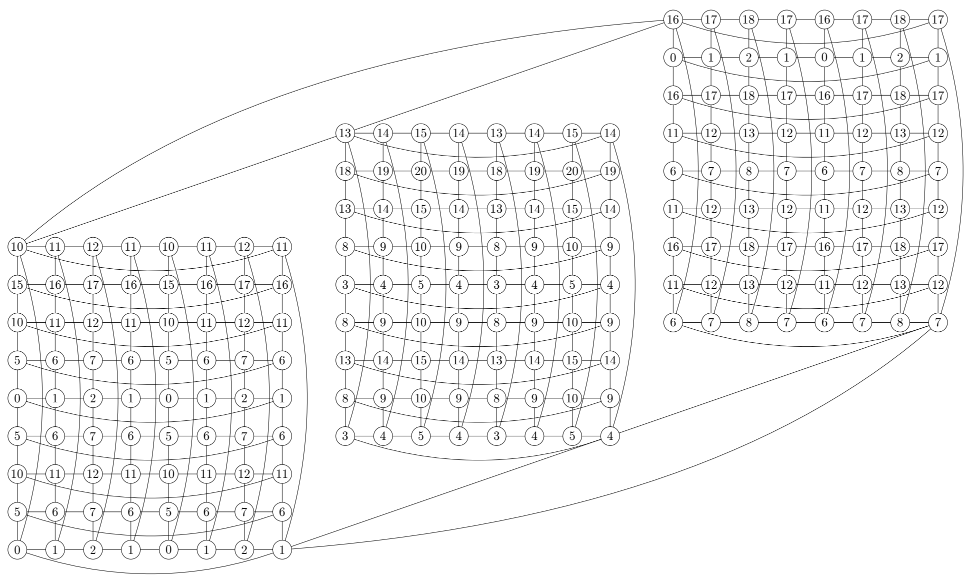

FIGURE 2. A star coloring of $TG(8, 3, 9)$. Most of the edges in the second dimension are omitted for visual clarity.

In [4] the authors present a class of cases where the upper bound is $2d+1$ (see Theorem 2.1). Our approach allows us to significantly extend the class of cases for which the upper bound remains $2d + 1$. This is proved in the following proposition.

Proposition 2.5. *Let $TG(n_1, n_2, \dots, n_d)$ be a d-dimensional torus, $d \geqslant 2$, such that each n_i is even or greater than $2d + 3$. Then*

$$\chi_s(TG(n_1, n_2, \dots, n_d)) \leqslant 2d + 1.$$

Proof. The proof is very similar to the proof of the previous proposition. The difference is that, in the general case, we 'exhaust' four different values for the differences between adjacent vertices, as we cannot guarantee whether a wavy coloring or just an almost wavy coloring of certain cycles can be ensured. In this case, we can color each cycle wavily, which is proved in the following claim.

Claim 2.6. *For each cycle C_{n_i} there exists a wavy star coloring $c_i : V(C_{n_i}) \to \{0, 1, \dots, 2d\}$ with step i.*

Proof. If $n_i = 4$, we define

$$c_i(v_j) = \begin{cases} j \cdot i, & \text{for } j = 0, 1, 2; \\ i, & \text{for } j = 3. \end{cases}$$

If $n_i = 6$, we define

$$c_i(v_j) = \begin{cases} j \cdot i, & \text{for } j = 0, 1, 2, 3; \\ (6 - j) \cdot i, & \text{for } j = 4, 5. \end{cases}$$

If $n_i = 2d + 1$, we define $c_i(v_j) = j \cdot i$.

The proof of the claim now directly follows from the fact that each number n_i which is even or greater than $2d + 3$ can be represented as $k \cdot 4 + l \cdot 6 + m \cdot (2d + 1)$ for some nonnegative integers k, l and m. □

Example 2.7. *A wavy coloring c_3 of $C_{11} = v_0 v_1 \dots v_{10} v_0$ from $TG(4, 4, 11)$ with step 3 is*

$$c_3 : \begin{pmatrix} v_0 & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_{10} \\ 0 & 3 & 6 & 3 & 0 & 3 & 6 & 2 & 5 & 1 & 4 \end{pmatrix}.$$

Let us represent each vertex u of $TG(n_1, n_2, \dots, n_d)$ by its coordinates in each dimension, that is, $u = (x_1, \dots, x_d)$ where each x_i, $1 \leqslant i \leqslant d$ satisfies $0 \leqslant x_i \leqslant n_i - 1$. We assign a color $c(u)$ to u according to the following equation:

$$c((x_1, x_2, \dots, x_d)) = \sum_{i=1}^{d} c_i(v_{x_i}) \mod 2d + 1,$$

where $c_i(v_{x_i})$ again denotes the color of the vertex v_{x_i} in the i-th cycle in order colored by c_i according to the previous claim. From this point onward, it can be shown in the same way as in the proof of the previous proposition that this is a proper coloring and also a star coloring. □

In Figure 3 we can see a star coloring of $C_4 \,\Box\, C_4 \,\Box\, C_{11}$ with $2 \cdot 3 + 1 = 7$ colors, where all the cycles can be colored wavily (not just almost wavily).

We can generate numerous intermediate results depending on the length of particular cycles. In any case, a linear upper bound exists. Let us now merge everything discussed so far into the main theorem of this section.

Theorem 2.8. *Let TG_d be a d-dimensional torus, $d \geqslant 2$. Then there is a constant C, $C < 8$, such that*

$$\chi_s(TG_d) \leqslant C \cdot d.$$

3. Honeycomb tori

Let n be a positive integer. The honeycomb torus of size n, denoted by $HT(n)$, is the graph with the vertex set

$$V(HT(n)) = \{(x_1, x_2, x_3) : -n + 1 \leqslant x_1, x_2, x_3 \leqslant n \text{ and } 1 \leqslant \sum_{i=1}^{3} x_i \leqslant 2\}.$$

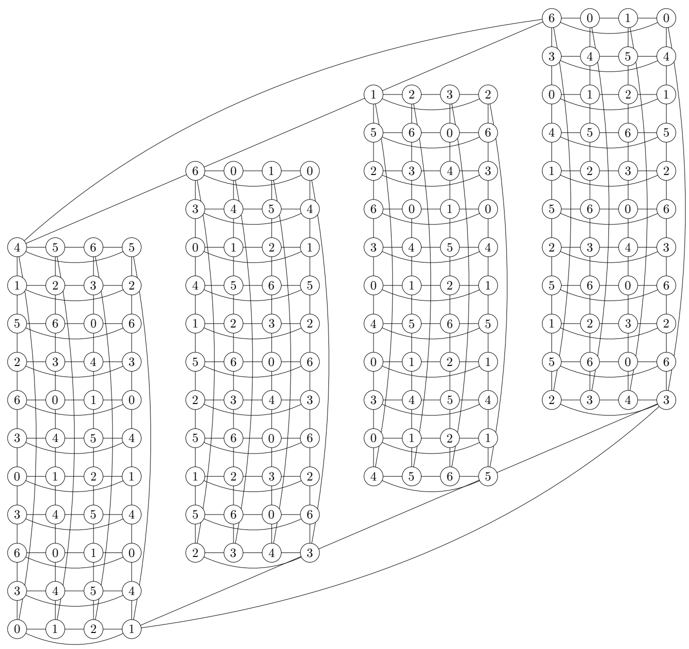

FIGURE 3. A star coloring of $TG(4,4,11)$. Again, most of the edges in the second dimension are omitted.

The edge set is the following

$$\begin{aligned} E(HT(n)) = &\{\{(x_1^1, x_2^1, x_3^1), (x_1^2, x_2^2, x_3^2)\} : \sum_{i=1}^{3} |x_i^1 - x_i^2| = 1\} \\ &\cup \{\{(i, n-i+1, 1-n), (i-n, 1-i, n)\} : 1 \leqslant i \leqslant n\} \\ &\cup \{\{(1-n, i, n-i+1), (n, i-n, 1-i)\} : 1 \leqslant i \leqslant n\} \\ &\cup \{\{(n-i+1, 1-n, i), (1-i, n, i-n)\} : 1 \leqslant i \leqslant n\}. \end{aligned}$$

We can see from the definition that if two vertices are adjacent, the difference between the sums of their coordinates is 1. This implies that honeycomb tori are bipartite graphs. They are also 3-regular. Figure 4 on the left shows the honeycomb torus $HT(3)$ with labeled vertices. On the right, it shows a star coloring of $HT(3)$ with four colors.

Theorem 3.1. *Let n be a positive integer. Then*

$$\chi_s(HT(n)) = 4.$$

Proof. Since the graph $HT(n)$ is a 3-regular graph, for any positive integer n, by Theorem 1.1, we have $\chi_s(HT(n)) \geqslant 4$. Therefore, it remains to construct a star coloring of $HT(n)$ with 4 colors.

Notice that we can divide the vertices into two classes based on the sum of their coordinates. Additionally, we can further divide one of these classes

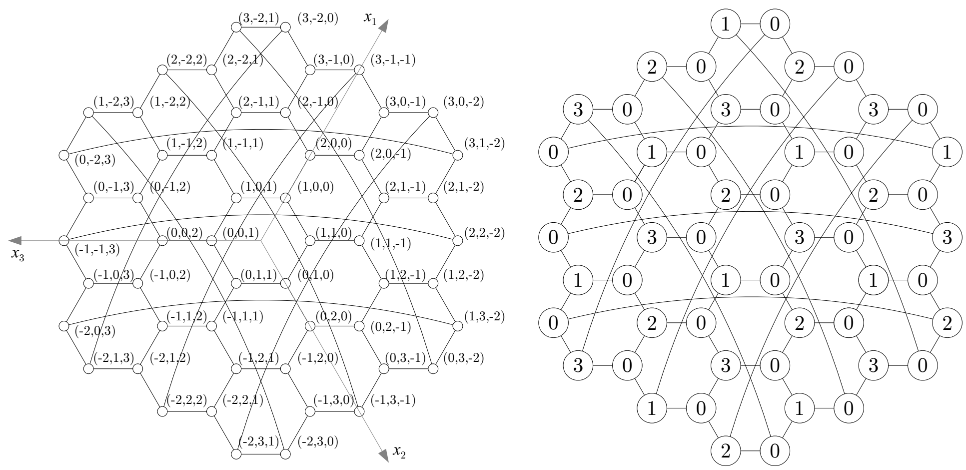

FIGURE 4. Honeycomb torus of size 3 and its star coloring.

into three sets as follows.

$S_0 = \{(x_1, x_2, x_3) \in V(HT(n)) : \sum_{i=1}^{3} x_i = 1\}$,
$S_1 = \{(x_1, x_2, x_3) \in V(HT(n)) : \sum_{i=1}^{3} x_i = 2 \text{ and } x_1 + 1 \equiv x_2 \pmod 3\}$,
$S_2 = \{(x_1, x_2, x_3) \in V(HT(n)) : \sum_{i=1}^{3} x_i = 2 \text{ and } x_2 + 1 \equiv x_3 \pmod 3\}$,
$S_3 = \{(x_1, x_2, x_3) \in V(HT(n)) : \sum_{i=1}^{3} x_i = 2 \text{ and } x_3 + 1 \equiv x_1 \pmod 3\}$.

It is not hard to check that $\{S_0, S_1, S_2, S_3\}$ makes a partition of the set of vertices of $HT(n)$.

Since we have that if two vertices are adjacent, then the sums of their coordinates are different, and we can define a coloring function $c : V(HT(n)) \to \{0, 1, 2, 3\}$ such that

$$c((x_1, x_2, x_3)) = i, \text{ for } (x_1, x_2, x_3) \in S_i \text{ and } i = 0, 1, 2, 3.$$

Now we show that c is also a star coloring. As we can see from the definition, all vertices adjacent to those colored with 1, 2, and 3 are colored with 0. Therefore, we need to show that no vertex colored with 0 has two adjacent vertices colored with the same color. We have two types of edges, so we observe two cases.

Let (i, j, k) be a vertex such that $i + j + k = 1$ and $i, j, k < n$. Its adjacent vertices are $(i + 1, j, k)$, $(i, j + 1, k)$, and $(i, j, k + 1)$. Let $(i + 1, j, k)$ and $(i, j + 1, k)$ have the same color (for the other two pairs is similar). They can have one of three colors.

(1) $c((i + 1, j, k)) = c((i, j + 1, k)) = 1$
This implies $i + 2 \equiv j \pmod 3$ and $i + 1 \equiv j + 1 \pmod 3$.
(2) $c((i + 1, j, k)) = c((i, j + 1, k)) = 2$
Now we have $j + 1 \equiv k \pmod 3$ and $j + 2 \equiv k \pmod 3$.
(3) $c((i + 1, j, k)) = c((i, j + 1, k)) = 3$
This gives $k + 1 \equiv i + 1 \pmod 3$ and $k + 1 \equiv i \pmod 3$.

Clearly, each of these cases leads to a contradiction.

Now, let (i, j, k) be such that one of the coordinates is n. There are three relevant options $(i-n, 1-i, n)$, $(n, i-n, 1-i)$, and $(1-i, n, i-n)$. Without loss of generality, let $(i-n, 1-i, n)$ be the vertex. Its adjacent vertices are $(i-n+1, 1-i, n)$, $(i-n, 2-i, n)$, and $(i, n-i+1, 1-n)$. If $(i-n+1, 1-i, n)$ and $(i-n, 2-i, n)$ have the same color, then we should consider three subcases depending on their color.

(1) $c((i-n+1, 1-i, n)) = c((i-n, 2-i, n)) = 1$.
We have $i-n+2 \equiv 1-i \pmod 3$ and $i-n+1 \equiv 2-i \pmod 3$. This implies $2i+1 \equiv n \pmod 3$ and $2i-1 \equiv n \pmod 3$, which gives a contradiction.

(2) $c((i-n+1, 1-i, n)) = c((i-n, 2-i, n)) = 2$.
Now we have $2-i \equiv n \pmod 3$ and $3-i \equiv n \pmod 3$. This gives a contradiction.

(3) $c((i-n+1, 1-i, n)) = c((i-n, 2-i, n)) = 3$.
This implies $n+1 \equiv i-n+1 \pmod 3$ and $n+1 \equiv i-n \pmod 3$, again a contradiction.

If $(i-n+1, 1-i, n)$ and $(i, n-i+1, 1-n)$ have the same color, then again we consider three subcases.

(1) $c((i-n+1, 1-i, n)) = c((i, n-i+1, 1-n)) = 1$.
Now it holds $i-n+2 \equiv 1-i \pmod 3$ and $i+1 \equiv n-i+1 \pmod 3$. Therefore $2i+1 \equiv n \pmod 3$ and $2i \equiv n \pmod 3$, which gives a contradiction.

(2) $c((i-n+1, 1-i, n)) = c((i, n-i+1, 1-n)) = 2$.
This gives $2-i \equiv n \pmod 3$ and $n-i+2 \equiv 1-n \pmod 3$. It follows that $3n \equiv 1 \pmod 3$, a contradiction.

(3) $c((i-n+1, 1-i, n)) = c((i, n-i+1, 1-n)) = 3$.
This implies $n+1 \equiv i-n+1 \pmod 3$ and $2-n \equiv i \pmod 3$, and then we get $3n \equiv 2 \pmod 3$, a contradiction.

Finally, the reasoning is similar when $(i-n, 2-i, n)$ and $(i, n-i+1, 1-n)$ have the same color.

(1) $c((i-n, 2-i, n)) = c((i, n-i+1, 1-n)) = 1$.
It holds $i-n+1 \equiv 2-i \pmod 3$ and $i+1 \equiv n-i+1 \pmod 3$. This implies $2i-1 \equiv n \pmod 3$ and $2i \equiv n \pmod 3$, a contradiction.

(2) $c((i-n, 2-i, n)) = c((i, n-i+1, 1-n)) = 2$.
Now we have $3-i \equiv n \pmod 3$ and $n-i+2 \equiv 1-n \pmod 3$. This implies $3n \equiv 2 \pmod 3$, a contradiction.

(3) $c((i-n, 2-i, n)) = c((i, n-i+1, 1-n)) = 3$.
In this subcase, we have $n+1 \equiv i-n \pmod 3$ and $2-n \equiv i \pmod 3$, what implies $3n \equiv 1 \pmod 3$, again a contradiction.

Now, we have proved that the vertex colored with 0 has no adjacent vertices colored with the same color. It follows that all three adjacent vertices are colored with distinct colors: 1, 2 and 3. This, together with the fact that there is no edge between vertices of color 1, 2 and 3, implies that there cannot exist a bicolored path on four vertices. Thus, c is indeed a star coloring. □

Acknowledgments

The authors would like to thank the two anonymous reviewers for their valuable comments, which helped to improve the content of the article.

The authors dedicate this article to students, teachers and professors who stood up against corruption and, despite facing immense pressure, devoted their 2024/25 academic year to the fight for institutional integrity and for a better future for all of us in Serbia.

References

1. S. Akbari, M. Chavooshi, M. Ghanbari, and S. Taghian, *Star chromatic number of some graphs*, Discrete Math. Algorithms Appl. **14** (2022), 2150089.
2. B. Alspach and M. Dean, *Honeycomb toroidal graphs are Cayley graphs*, Inform. Process. Lett. **109** (2009), 705–708.
3. M. Chen, A. Raspaud, and W. Wang, *6-star-coloring of subcubic graphs*, J. Graph Theory **72** (2013), 128–145.
4. G. Fertin, A. Raspaud, and B. Reed, *Star coloring of graphs*, J. Graph Theory **47** (2004), 163–182.
5. H. J. Fu and D. Z. Xie, *A note on star chromatic number of graphs*, J. Math. Res. Exposition **30** (2010), 841–844.
6. B. Grünbaum, *Acyclic colorings of planar graphs*, Israel J. Math. **14** (1973), 390–408.
7. T. Han, Z. Shao, E. Zhu, Z. Li, and F. Deng, *Star coloring of Cartesian product of paths and cycles*, Ars Combin. **124** (2016), 65–84.
8. G. M. Megson, X. Yang, and X. Liu, *Honeycomb tori are Hamiltonian*, Inform. Process. Lett. **72** (1999), 99–103.
9. S. Ndreca, A. Procacci, and B. Scoppola, *Improved bounds on coloring of graphs*, European J. Combin. **33** (2012), 592–609.
10. M. A. Shalu and C. Antony, *The complexity of restricted star colouring*, Discret. Appl. Math. **319** (2022) 327–350.
11. M. A. Shalu and C. Antony, *Star colouring of bounded degree graphs and regular graphs*, Discrete Math. **345** (2022), 112850.
12. I. Stojmenovic, *Honeycomb networks: Topological properties and communication algorithms*, IEEE Trans. Parallel Distributed Systems **8** (10) (1997) 1036–1042.
13. P. Šparl, *Symmetries of the honeycomb toroidal graphs*, J. Graph Theory. **99** (2022), 414–424.
14. D. Xie, H. Xiao, and Z. Zhao, *Star coloring of cubic graphs*, Inform. Process. Lett. **114** (2014), 689–691.

Department of Mathematics and Informatics, University of Novi Sad, Trg Dositeja Obradovića 4, 21000 Novi Sad, Serbia
E-mail address: kristina.ago@dmi.uns.ac.rs

Department of Mathematics and Informatics, University of Novi Sad, Trg Dositeja Obradovića 4, 21000 Novi Sad, Serbia
E-mail address: anna.slivkova@dmi.uns.ac.rs