

TWO-COLOR MULTIPARTITE RAMSEY NUMBERS OF
MATCHINGS AND CYCLES

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ABSTRACT. Let $K_{j \times n}$ denote a complete multipartite graph with j partite sets, each containing n vertices. For given graphs G_1 and G_2 , the multipartite Ramsey number (denoted $m_j(G_1, G_2)$) is the smallest integer t such that any subgraph G of $K_{j \times t}$ either contains a copy of G_1 , or its complement relative to $K_{j \times t}$ contains a copy of G_2 . Determining the exact value of Ramsey numbers is a challenging problem, with many cases providing only bounds rather than exact values. Due to the broad applications of these numbers, finding exact values for specific graph classes is highly valuable.

In recent years, researchers have studied the multipartite Ramsey numbers $m_j(nK_2, C_m)$. The exact values of these numbers have been determined for various specific cases of j and m . Notably, the problem is fully resolved for $m \leq 6$, and for $m = 7$, it has been solved when $j \leq 4$. In this article, we compute the M-R-number $m_j(nK_2, C_7)$ for all remaining unsolved cases, specifically for $j \geq 5$ and $n \geq 2$.

1. INTRODUCTION

Graph theory, one of the most active areas in combinatorics, explores the structural properties and behaviors of graphs. A graph G in this context refers to an undirected, simple, and finite graph. The *order* of a graph G , denoted $|V(G)|$, represents the number of vertices in G . A fundamental concept in graph theory is that of a *matching*, or in this paper, specifically an *n -stripe* of a graph, defined as a set of n edges that are pairwise disjoint, meaning they share no common vertex.

For any vertex $v \in V(G)$, we denote the *degree* of v as $\deg_G(v)$, representing the number of edges incident to v , while the set of *neighbors* of v is denoted by $N_G(v)$. In multipartite graphs, which are of particular interest in this study, for a vertex $v \in V(G)$, the neighborhood of v restricted to the partite set X_j is denoted $N_{X_j}(v)$, and it is defined as $N_{X_j}(v) = \{u \in V(X_j) \mid uv \in E(G)\}$. A *cycle* on n vertices is denoted by

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C_n , and we use the notation $[X_i, X_j]$ to represent the set of edges between partite sets X_i and X_j . Additionally, the *complement* of a graph G , denoted by $\overline{G}$, is the graph on the same vertex set as G where edges are present if and only if they are absent in G . The *join* of two graphs G and H , denoted by $G+H$, is a graph formed by connecting each vertex of G to every vertex of H .

One of the central themes in combinatorial graph theory is *Ramsey theory*, which concerns finding monochromatic subgraphs within edge-colored graphs [4]. More specifically, the *multipartite Ramsey number* (M-R number), denoted $m_j(G_1, G_2)$, generalizes the concept of the classical Ramsey number by considering complete multipartite graphs instead of complete graphs. In the classical case, the Ramsey number determines the smallest number of vertices required such that any edge coloring of a complete graph results in a monochromatic subgraph [4]. In contrast, the multipartite Ramsey number focuses on finding such structures within a complete multipartite graph, where the vertices are divided into disjoint sets (partite sets) and edges only occur between vertices from different sets. Let $K_{j \times n}$ denote a complete multipartite graph consisting of j partite sets, each with n vertices. The M-R number $m_j(G_1, G_2)$ is the smallest integer t such that any subgraph G of $K_{j \times t}$ either contains a copy of G_1 , or its complement (relative to $K_{j \times t}$) contains a copy of G_2 [2]. The problem of determining these numbers is challenging due to the vast combinatorial nature of multipartite graphs and the flexibility allowed in the structure of subgraphs.

In the seminal work of Burger et al. [2], the authors initiated the study of multipartite Ramsey numbers, focusing on cases where both G_1 and G_2 are complete multipartite graphs. Since then, interest in multipartite Ramsey theory has grown, with studies extending the work to various classes of graphs, including paths, cycles, and matchings. For further details on these developments, we refer the reader to [1, 3, 6–9, 11, 12, 14, 15]. These multipartite Ramsey numbers find applications in network theory, coding theory, and various fields where relational structures are represented as multipartite graphs [10].

The exact values of multipartite Ramsey numbers remain unknown for many graph classes. The problem of determining $m_j(nK_2, C_7)$ has garnered recent attention due to its challenging nature and wide applicability. Jayawardene et al. [5] made contributions to the study of multipartite Ramsey numbers by determining $m_j(nK_2, C_m)$ for small values of $m \in \{3, 4, 5, 6\}$ and $j \geq 2$. Their work primarily focused on matchings and cycles, providing exact values for multipartite Ramsey numbers in these specific cases. Building upon this, Rowshan et al. extended the study by considering larger cycles, specifically the cycle of length 7, and determined the values of the multipartite Ramsey number $m_j(nK_2, C_7)$ for $j \leq 4$ and $n \geq 2$ [13]. While their work improved the existence results of multipartite Ramsey numbers

for matchings and cycles, it primarily focused on small values of j , leaving open the question of determining these values for larger j .

Despite these advancements, the multipartite Ramsey numbers of matchings and cycles remain unsolved in many cases, even for specific cycles such as C_7 . In particular, for larger values of j and n , there is still much to be explored. This gap in the literature, especially in the case of cycle lengths beyond 7, remains unsolved. In this article, we aim to address these unsolved cases by completing the determination of the multipartite Ramsey numbers of matchings and cycle C_7 . Furthermore, we believe that the method we introduce here, with further refinement and effort, can be extended to compute $m_j(nK_2, C_m)$ for cycles of length greater than 7, thereby advancing the study of multipartite Ramsey numbers for larger cycles.

More precisely, in this article, we prove the following theorem:

Theorem 1.1. *For each positive integers j, n , where $j \geq 5$, $n \geq 2$, we have:*

$$m_j(nK_2, C_7) = \begin{cases} 1 & \text{if } n \geq 3 \text{ and } j \geq 2n + 3, \text{ or } (j \geq 8, n = 2), \\ 2 & \text{if } n \geq 3 \text{ and } n + 2 \leq j \leq 2n + 2, \\ & \text{or } (5 \leq j \leq 7, n = 2), \\ 3 & \text{if } j \geq 5 \text{ and } n = j - 1, \\ \left\lfloor \frac{2n-1}{j-1} \right\rfloor + 1 & \text{if } n \geq j \geq 5. \end{cases}$$

In the following section, we present the proof of Theorem 1.1. The proof is quite long, and we handle different values of n and j separately, using several theorems.

2. PROOF OF THEOREM 1.1

In this section, we derive the values of the M-R-number $m_j(nK_2, C_7)$. Specifically, we establish a formula for this M-R-number for each $j \geq 5$ and $n \geq 2$. The following theorem provides the foundational result:

Theorem 2.1. *For any positive integers j and n , where $n \geq 3$ and $j \geq 2n + 3$, we have:*

$$m_j(nK_2, C_7) = 1.$$

Proof. Consider a 2-edge-coloring of K_j , denoted by (G^r, G^b) . Let M be a maximum matching (M-M) in G^r , where $|M| \leq n - 1$. We aim to show that $C_7 \subseteq G^b$.

Given that $j \geq 2n + 3$ and $|M| \leq n - 1$, if $|M| \leq n - 2$, then it follows that $C_7 \subseteq K_7 \subseteq G^b$. Now, consider the case where $|M| = n - 1$. Since $j \geq 2n + 3$, we can conclude that $K' = K_5 \subseteq G^b$.

Since $n \geq 3$ and $|M| = n - 1$, the matching M contains at least two edges, say v_1v_2 and v_3v_4 .

Consider $e = v_1v_2 \in M$. We claim that $|N_{G^b}(v_i) \cap V(K')| \geq 4$ for at least one $i \in \{1, 2\}$. Assume, for contradiction, that $|N_{G^b}(v_i) \cap V(K')| \leq 3$ for both $i = 1, 2$. This implies that $|N_{G^r}(v_i) \cap V(K')| \geq 2$ for each $i \in \{1, 2\}$. Let $\{y_i, y'_i\} \subseteq N_{G^r}(v_i) \cap V(K')$ for $i = 1, 2$. In this case, we can construct a new matching $M' = M \setminus \{v_1v_2\} \cup \{v_1y, v_2y'\}$ for some $y, y' \in \{y_1, y'_1, y_2, y'_2\}$ in G^r . Clearly, $|M'| > |M|$, which contradicts the maximality of M . Hence, without loss of generality, we may assume that $|N_{G^b}(v_1) \cap V(K')| \geq 4$. By an analogous argument, for the edge $v_3v_4 \in M$, there exists at least one $j \in \{3, 4\}$ such that $|N_{G^b}(v_j) \cap V(K')| \geq 4$. Without loss of generality, let $|N_{G^b}(v_3) \cap V(K')| \geq 4$. Thus, we conclude that $C_7 \subseteq G^b[K' \cup \{v_1, v_3\}]$.

Therefore, we conclude that $m_j(nK_2, C_7) = 1$ for all j and n such that $n \geq 3$ and $j \geq 2n + 3$. $\square$

In the next theorem, we obtain the exact value of the M-R-number $m_j(nK_2, C_7)$ for each j, n where $j \geq 5$, $n \geq 3$, and $n + 2 \leq j \leq 2n + 2$.

Theorem 2.2. *For any positive integers j and n , where $n \geq 3$ and $n + 2 \leq j \leq 2n + 2$, we have:*

$$m_j(nK_2, C_7) = 2.$$

Proof. Consider a 2-edge-coloring of K_j , denoted by (G^r, G^b) , where $G^r \cong K_{j-3}$ and $G^b = \overline{G^r}$. Since $|V(G^r)| = j - 3 \leq 2n - 1$, it follows that $nK_2 \not\subseteq G^r$. Additionally, as $G^b \cong K_3 + (j - 3)K_1$, Figure 1 shows that $C_7 \not\subseteq G^b$. Thus, $m_j(nK_2, C_7) \geq 2$ for each j, n where $n \geq 3$ and $n + 2 \leq j \leq 2n + 2$.

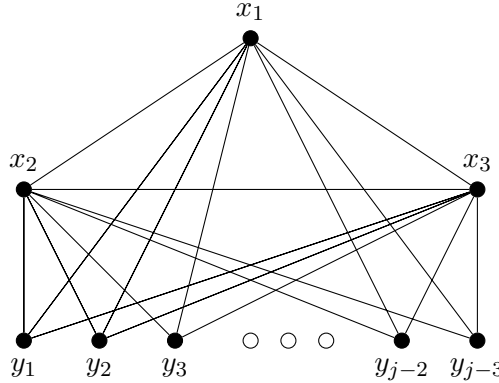


FIGURE 1. $C_7 \not\subseteq G^b \cong K_3 + (j - 3)K_1$

To establish the upper bound, let $K = K_{j \times 2}$ with partite sets X_i for $i = 1, 2, \dots, j$. Consider a 2-edge-coloring (G^r, G^b) of $K_{j \times 2}$. Suppose M is a maximum matching (M-M) in G^r where $|M| \leq n - 1$. If $|M| \leq n - 2$ or $j \geq n + 3$, then $K_{4 \times 2} \subseteq G^b$, which implies $C_7 \subseteq G^b$. Hence, we may assume $|M| = n - 1$ and $j = n + 2$.

Since $|V(K)| = 2n + 4$ and $|M| = n - 1$, there exist six vertices in $V(K)$, denoted by $Y = \{y_1, y_2, \dots, y_6\}$, that do not belong to $V(M)$. As $|X_i| = 2$

and $|Y| = 6$, it follows that $C_6 \subseteq G^b[Y] \subseteq G^b$. Without loss of generality, assume $C_6 = y_1 y_2 y_3 \dots y_6 y_1 \subseteq G^b$. Let $e = v_1 v_2 \in M$.

Define $N_i = |N_{G^r}(v_i) \cap Y|$ for $i = 1, 2$. We observe that $\min\{N_1, N_2\} \leq 1$. Otherwise, if $\{y_i, y_j\} \subseteq N_{G^r}(v_1) \cap Y$ and $\{y_k, y_l\} \subseteq N_{G^r}(v_2) \cap Y$, we can set $M' = M \setminus \{v_1 v_2\} \cup \{v_1 y_i, v_2 y_k\}$, with $y_i \neq y_k$, which contradicts the maximality of M . Thus, we have $\min\{N_1, N_2\} \leq 1$. Let assume that $\min\{N_1, N_2\} = N_1$

Consequently, $|N_{G^b}(v_1) \cap Y| \geq 4$. In this configuration, there exists at least one index i , $i \in [5]$, such that $y_i, y_{i+1} \in N_{G^b}(v_1) \cap Y$. Hence, we obtain $C_7 = y_1 \dots y_i v_1 y_{i+1} \dots y_6 y_1 \subseteq G^b$.

This implies that $m_j(nK_2, C_7) = 2$ for each j, n where $j \geq 5$, $n \geq 3$, and $n + 2 \leq j \leq 2n + 2$. $\square$

In the next theorem, we obtain the exact values of the M-R-number $m_j(nK_2, C_7)$ for each j where $j \geq 5$ and $j - 1 \leq n \leq j$.

Theorem 2.3. *For each $j \geq 5$, we have:*

$$m_j((j-1)K_2, C_7) = m_j(jK_2, C_7) = 3.$$

Proof. Consider $K_{j \times 2}$ with partite sets $X_i = \{x_1^i, x_2^i\}$ for $i = 1, 2, \dots, j$. Take a 2-edge-coloring (G^r, G^b) of $K_{j \times 2}$, where $G^r \cong K_1 + K_{(j-2) \times 2}$ and $G^b = \overline{G^r}$, as illustrated in Figure 2. Without loss of generality, assume $G^r \cong G[\{x_2^1\}, X_3, X_4, \dots, X_j]$.

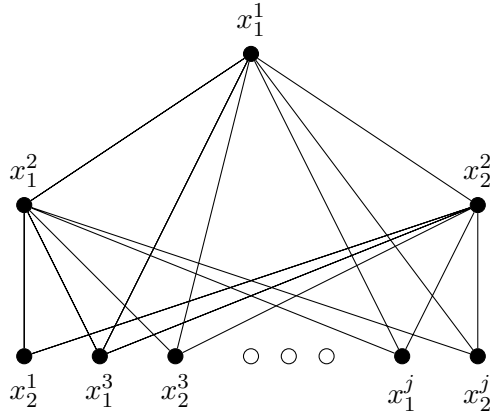


FIGURE 2. Graph $G^b = \overline{G^r}$ in 2-edge coloring of $K_{j \times 2}$

Since $|V(G^r)| = 2j - 3$, we conclude that $(j-1)K_2 \not\subseteq G^r$. By Figure 2, it follows that $G^b \cong P_3 + (2j-3)K_1 - e$, and we can verify that $C_7 \not\subseteq G^b$. Thus, for each $j \geq 5$, we have $m_j(nK_2, C_7) \geq 3$ for $n \in \{j-1, j\}$.

To prove the upper bound, consider $K = K_{j \times 3}$ with partite sets X_i , where $i \in [j]$. Take a 2-edge-coloring (G^r, G^b) of $K_{j \times 3}$, where $nK_2 \not\subseteq G^r$ for each $n \in \{j-1, j\}$. Suppose M is a maximum matching (M-M) in G^r with

$|M| \leq n - 1$. Then, there exist at least $j + 2$ vertices in $K_{j \times 3}$, denoted $Y = \{y_1, y_2, \dots, y_{j+2}\}$, such that the vertices of Y do not belong to $V(M)$. Since $j \geq 5$, we have $|Y| \geq 7$.

Because $|X_i| = 3$ and $|Y| \geq 7$, it follows that in any configuration, $C_7 \subseteq G^b[Y] \subseteq G^b$. This implies that $m_j(nK_2, C_7) = 3$ for each $j \geq 5$ and $n \in \{j - 1, j\}$. $\square$

In the following theorem, we determine the exact values of the M-R-number $m_j(nK_2, C_7)$ for each $j \geq 5$ and smaller values of n . By applying a similar approach to the arguments used in the proofs of Theorems 2.1 and 2.2, we obtain the following results:

Theorem 2.4. *For the M-R-number $m_j(nK_2, C_7)$, we have:*

- If $j \geq 8$, then $m_j(2K_2, C_7) = 1$.
- If $5 \leq j \leq 7$, then $m_j(2K_2, C_7) = 2$.

In the following results, we establish the exact values of the M-R-number $m_j(nK_2, C_7)$ for each j, n where $n \geq j \geq 5$. We begin with the following lemma:

Lemma 2.5. *For each n, j where $n \geq j \geq 5$, we have:*

$$m_j(nK_2, C_7) \geq \left\lfloor \frac{2n-1}{j-1} \right\rfloor + 1.$$

Proof. Consider a 2-edge-coloring (G^r, G^b) of the complete bipartite graph $K_{j \times t_0}$, where $t_0 = \left\lfloor \frac{2n-1}{j-1} \right\rfloor$, $G^r \cong K_{(j-1) \times t_0}$, and $G^b \cong K_{t_0, (j-1)t_0}$. Since G^b is a bipartite graph, it follows that $C_7 \not\subseteq G^b$. Now, we compute $|V(G^r)|$ as follows:

$$|V(G^r)| = (j-1) \times t_0 = (j-1) \left(\left\lfloor \frac{2n-1}{j-1} \right\rfloor \right).$$

Applying the floor function properties, we obtain:

$$|V(G^r)| \leq 2n - 1.$$

So, we conclude that $|V(G^r)| < 2n$, which implies that $nK_2 \not\subseteq G^r$. Therefore, we have $m_j(nK_2, C_7) \geq \left\lfloor \frac{2n-1}{j-1} \right\rfloor + 1$ for each $n \geq j \geq 5$. $\square$

To compute the exact value of $m_j(nK_2, C_7)$, we first establish the following fact.

Fact 2.6. *Given that $n \geq j \geq 5$, then:*

$$j \left(\left\lfloor \frac{2n-1}{j-1} \right\rfloor + 1 \right) - 2(n-1) \geq 5,$$

and

$$(j-1) \left(\left\lfloor \frac{2n-1}{j-1} \right\rfloor + 1 \right) - 2(n-1) \geq 2.$$

Proof. Since $n \geq j$, it implies that $\frac{2n-1}{j-1} > 2$. We consider two cases to prove the inequality. If $\frac{2n-1}{j-1} \geq 3$, then

$$j \left(\left\lfloor \frac{2n-1}{j-1} \right\rfloor + 1 \right) - 2(n-1) > j \left(\frac{2n-1}{j-1} \right) - 2n + 2 = \frac{2n-1}{j-1} + 1,$$

however since $\frac{2n-1}{j-1} \geq 3$ by assumption, we have $j \left(\left\lfloor \frac{2n-1}{j-1} \right\rfloor + 1 \right) - 2(n-1) > 4$.

If $2 < \frac{2n-1}{j-1} < 3$, then $\left\lfloor \frac{2n-1}{j-1} \right\rfloor = 2$ and $j \left(\left\lfloor \frac{2n-1}{j-1} \right\rfloor + 1 \right) - 2(n-1) > 3j - 3j + 4$, which gives the proof. The second inequality then follows directly. $\square$

Theorem 2.7. *Let n and j be two integers satisfying $n \geq j \geq 5$. Then, the M - R -number $m_j(nK_2, C_7)$ is given by:*

$$m_j(nK_2, C_7) = \left\lfloor \frac{2n-1}{j-1} \right\rfloor + 1$$

Proof. By Lemma 2.5, the lower bound holds. To establish the upper bound, consider $K_{j \times t}$ with partite sets $X_i = \{x_1^i, x_2^i, \dots, x_t^i\}$ for $i = 1, 2, \dots, j$, where $t = \left\lfloor \frac{2n-1}{j-1} \right\rfloor + 1$. We will prove this result by induction on n . Since $n \geq j \geq 5$, we begin with the base case $n = j$. By Theorem 2.3, we have $m_j(jK_2, C_7) = 3$, which matches the expression $\left\lfloor \frac{2n-1}{j-1} \right\rfloor + 1$ when $n = j$. Thus, the theorem holds for the base case.

Now Suppose that for any fixed j , we have $m_j(n'K_2, C_7) \leq \left\lfloor \frac{2n'-1}{j-1} \right\rfloor + 1$ for each $j \leq n' < n$. We will prove that:

$$m_j(nK_2, C_7) \leq \left\lfloor \frac{2n-1}{j-1} \right\rfloor + 1$$

We start with the following case.

CASE 1: $\left\lfloor \frac{2n-1}{j-1} \right\rfloor = \left\lfloor \frac{2n-3}{j-1} \right\rfloor + 1$.

Assume, by contradiction, that $m_j(nK_2, C_7) > \left\lfloor \frac{2n-1}{j-1} \right\rfloor + 1$. This implies $K = K_{j \times t}$ is 2-colorable to (nK_2, C_7) , meaning $nK_2 \not\subseteq G$ and $C_7 \not\subseteq \overline{G}$ for some $G \subseteq K_{j \times t}$ where $t = \left\lfloor \frac{2n-1}{j-1} \right\rfloor + 1$. Let $X'_i = X_i \setminus \{x_1^i\}$ for each $i \in [j]$.

By the induction hypothesis, $m_j((n-1)K_2, C_7) \leq \left\lfloor \frac{2n-3}{j-1} \right\rfloor + 1 = t - 1$.

On the other hand, since $|X'_i| = t - 1$ and $C_7 \not\subseteq \overline{G}[X'_1, \dots, X'_j]$, we have $M = (n-1)K_2 \subseteq G[X'_1, \dots, X'_j]$. If there exist $i, i' \in [j]$ such that $x_1^i x_1^{i'} \in E(G)$, then $nK_2 \subseteq G$, a contradiction. Therefore, $K_j \cong K[x_1^1, x_2^1, \dots, x_j^1] \subseteq \overline{G}$. If $j \geq 7$, we have $K_7 \subseteq \overline{G}$ and thus $C_7 \subseteq \overline{G}$ and the proof is complete. So let $j \in \{5, 6\}$ and define $S = \{x_1^1, x_2^1, \dots, x_j^1\}$.

Since $n \geq 5$ and $|E(M)| = (n-1)K_2$, the matching M has at least two edges. Consider two edges $(x_a^b, x_{a'}^{b'})$, and $(x_c^d, x_{c'}^{d'})$ in the matching M .

The following occur for the neighbors of x_a^b and $x_{a'}^{b'}$ in S :
 Either $|N_G(x_a^b) \cap S| < 2$ or $|N_G(x_{a'}^{b'}) \cap S| < 2$. Otherwise, if $|N_G(x_a^b) \cap S| \geq 2$ and $|N_G(x_{a'}^{b'}) \cap S| \geq 2$, then $M' = \left(M \setminus \{(x_a^b, x_{a'}^{b'})\}\right) \cup \{(x_a^b, y), (x_{a'}^{b'}, y')\}$ for some $y, y' \in S$, increasing the size of the matching, which contradicts the assumption. The same analysis applies for the vertices x_c^d and $x_{c'}^{d'}$.
 Hence, in all cases, we conclude that:

$$C_7 \subseteq \overline{G}[\{x_1^1, x_1^2, \dots, x_1^j, x, x'\}],$$

for some $x, x' \in \{x_a^b, x_{a'}^{b'}, x_c^d, x_{c'}^{d'}\}$. which contradicts the assumption that $\overline{G}$ is C_7 -free.

Hence, the assumption $m_j(nK_2, C_7) > \left\lfloor \frac{2n-3}{j-1} \right\rfloor + 1$ is invalid. Thus, we conclude that $m_j(nK_2, C_7) \leq \left\lfloor \frac{2n-1}{j-1} \right\rfloor + 1$, where $\left\lfloor \frac{2n-1}{j-1} \right\rfloor = \left\lfloor \frac{2n-3}{j-1} \right\rfloor + 1$.

Therefore, we proceed to the next case.

CASE 2: $\left\lfloor \frac{2n-3}{j-1} \right\rfloor = \left\lfloor \frac{2n-1}{j-1} \right\rfloor$.

Assume, for the sake of contradiction, that $m_j(nK_2, C_7) > \left\lfloor \frac{2n-1}{j-1} \right\rfloor + 1 = t$. This implies that $K_{j \times t}$ is 2-colorable to (nK_2, C_7) . That is, there exists a 2-edge coloring (G^r, G^b) of $K_{j \times t}$, where $nK_2 \not\subseteq G^r$, $C_7 \not\subseteq G^b$, and $G^r = \overline{G^b}$.

Since $m_j((n-1)K_2, C_7) = \left\lfloor \frac{2n-3}{j-1} \right\rfloor + 1 = t$ and $C_7 \not\subseteq G^b$, it follows that $(n-1)K_2 \subseteq G^r$. Let $M = (n-1)K_2 \subseteq G^r$.

Due to Fact 2.6, there exist at least 5 vertices in $V(K_{j \times t})$, say $Y = \{y_1, \dots, y_5\}$, that do not belong to $V(M)$.

We now state the following claim:

Claim 2.8. *Let $e = v_1v_2 \in E(M)$. The following statements hold:*

- *If $|N_{G^r}(v_1) \cap Y| \geq 2$, then $|N_{G^r}(v_2) \cap Y| = 0$.*
- *If $|N_{G^r}(v_1) \cap Y| = 1$, then $|N_{G^r}(v_2) \cap Y| \leq 1$.*
- *If $|N_{G^r}(v_1) \cap Y| = |N_{G^r}(v_2) \cap Y| = 1$, then v_1 and v_2 share the same neighbor in Y .*

Proof. Assume, for contradiction that $\{v, v'\} \subseteq N_{G^r}(v_1) \cap Y$ and $v'' \in N_{G^r}(v_2) \cap Y$. Set $M' = (M \setminus \{v_1v_2\}) \cup \{v_1v, v_2v''\} \subseteq G^r$, where $v \neq v''$. This contradicts the assumption that $nK_2 \not\subseteq G^r$. Similarly, if $|N_{G^r}(v_i) \cap Y| = 1$ for $i = 1, 2$ and v_1 has a different neighbor from v_2 , the proof follows the same reasoning. $\square$

Now, define $\mathcal{M}$ as the set of all matchings with size $n-1$ in G^r . That is,

$$\mathcal{M} = \{M \mid |M| = n-1, M \subseteq G^r\}.$$

For each $M \in \mathcal{M}$ we define Y_M as follows.

$$Y_M = \{V(K_{j \times t}) \setminus V(M)\}.$$

For each $M \in \mathcal{M}$, by Fact 2.6 we have $|Y_M| \geq 5$. Define $M' \in \mathcal{M}$ as a set that $Y_{M'}$ intersects the greatest number of partite sets among all elements of

$\mathcal{M}$. Formally, for $M \in \mathcal{M}$, and $i \in \{1, 2, \dots, j\}$, let $I_M = \{X_i \mid Y_M \cap X_i \neq \emptyset\}$, then M' is chosen in $\mathcal{M}$ satisfying:

$$|I_M| \leq |I_{M'}|, \quad \text{for all } M \in \mathcal{M}.$$

If $Y_{M'} \cap X_i \neq \emptyset$ for all $i \in [j]$, then we have $K_j \subseteq G^b[Y_{M'}]$ and the proof is the same as Case 1. Therefore we may assume that $Y_{M'} \cap X_i \neq \emptyset$ for at most $j - 1$ indices $i \in [j]$ and w.l.g assume that $Y_{M'} \cap X_j = \emptyset$. Moreover, due to Fact 2.6 it can be said that $|Y_{M'} \cap (V(G) \setminus X_i)| \geq 2$ for each $i \in [j]$. Now by considering $Y_{M'} \cap X_i$, we have the following claim.

Claim 2.9. *We have $|Y_{M'} \cap X_i| \neq 0$ for at least three distinct indices $i \in [j]$.*

Proof. We prove this by contradiction. Suppose that $Y_{M'} \cap X_i \neq \emptyset$ for at most two values of $i \in [j]$. Recall that we already assumed $Y_{M'} \cap X_j = \emptyset$.

CASE 1: $Y_{M'}$ intersects only one X_i , where $1 \leq i \leq j - 1$.

Without loss of generality, let $Y_{M'} \subseteq X_1$. Then for every $i \in \{2, 3, \dots, j\}$, we must have $X_i \subseteq V(M')$. This implies $|V(M')| \geq (j - 1)|X_i| = (j - 1)t \geq 2n$, contradicting the assumption that $|E(M')| = n - 1$. Thus, $Y_{M'}$ cannot be contained entirely in a single partite set X_i .

CASE 2: $Y_{M'} \cap X_i \neq \emptyset$ for exactly two values of $i \in [j]$.

Without loss of generality, assume these are $i = 1$ and $i = 2$. So $Y_{M'} \cap X_i = \emptyset$ for $i \in \{3, \dots, j\}$. Also, suppose $|Y_{M'} \cap X_1| \geq |Y_{M'} \cap X_2|$, and since $|Y_{M'}| \geq 5$, assume:

$$\{x_1^1, x_2^1, x_3^1\} \subseteq Y_{M'} \cap X_1, \quad \text{and} \quad \{x_1^2, x_2^2\} \subseteq Y_{M'} \cap X_2.$$

The latter follows from Fact 2.6, which guarantees $|Y_{M'} \cap X_2| \geq 2$.

Since $|M'| = n - 1$, at least one edge of M' must lie entirely within $G^r[X_3 \cup X_4 \cup \dots \cup X_j]$. Let $e = vv' \in E(M') \cap G^r[X_3 \cup X_4 \cup \dots \cup X_j]$, meaning $v, v' \in X_3 \cup X_4 \cup \dots \cup X_j$.

Now consider the structure of M' . Since M' was chosen such that the corresponding $Y_{M'}$ intersects the maximum number of partite sets among all matchings in $\mathcal{M}$, we must have $Y_{M'} \subseteq N_{G^b}(v) \cap N_{G^b}(v')$. Otherwise, suppose there exists $y' \in Y_{M'}$ such that $vy' \in E(G^r)$. Then replacing e with vy' gives a new matching $M'' = (M' \setminus \{vv'\}) \cup \{vy'\}$, which belongs to $\mathcal{M}$ and satisfies $|I_{M''}| > |I_{M'}|$, contradicting the structure of M' .

Therefore, $Y_{M'} \subseteq N_{G^b}(v) \cap N_{G^b}(v')$, and the subgraph G^b induced by $\{x_1^1, x_2^1, x_3^1, x_1^2, x_2^2, v, v'\}$, contains a cycle $C_7 \subseteq G^b[Y_{M'} \cup \{v, v'\}]$, contradicting the assumption that G^b is C_7 -free.

This contradiction completes the proof, and we conclude that $Y_{M'}$ must intersect at least three partite sets among $X_1, \dots, X_{j-1}$. $\square$

Now we consider $Y_{M'}$ and analyze two subcases as follows:

SUBCASE 2-1: $|Y_{M'} \cap X_i| \leq 2$ for each $i \in [j - 1]$.

As already discussed, $Y_{M'}$ must intersect at least three of the sets $X_1, \dots, X_{j-1}$.

First, assume that $Y_{M'} \cap X_i \neq \emptyset$ for exactly three indices $i \in [j - 1]$. Since $|Y_{M'}| \geq 5$, without loss of generality, suppose $|Y_{M'} \cap X_1| = |Y_{M'} \cap X_2| = 2$,

and $1 \leq |Y_{M'} \cap X_3| \leq 2$. Therefore we have $Y_{M'} \cap X_i = \emptyset$ for $i \in \{4, \dots, j\}$. Let $Y_{M'} \cap X_1 = \{x_1^1, x_2^1\}$ and $Y_{M'} \cap X_2 = \{x_1^2, x_2^2\}$.

Since $|E(M')| = n - 1$ and $Y_{M'} \cap X_i = \emptyset$ for $i \in \{4, \dots, j\}$, there must be at least one edge in $E(M') \cap G^r[X_4 \cup \dots \cup X_j]$.

So, let $e = vv' \in E(M') \cap G^r[X_4 \cup \dots \cup X_j]$, where $v, v' \in X_4 \cup \dots \cup X_j$.

We now claim that $Y_{M'} \cap X_i \subseteq N_{G^b}(v) \cap N_{G^b}(v')$ for $i = 1, 2$. Otherwise, suppose there exists $y' \in Y_{M'} \cap X_i$ such that $vy' \in E(G^r)$. Then, replacing vv' by vy' , we define $M'' = (M' \setminus \{vv'\}) \cup \{vy'\}$. This matching satisfies $M'' \in \mathcal{M}$ and $|I_{M''}| > |I_{M'}|$, contradicting the definition of M' . Thus, the induced subgraph $G^b[Y_{M'} \cup \{v, v'\}]$ contains a C_7 , which is a contradiction.

Now assume that $Y_{M'} \cap X_i \neq \emptyset$ for exactly four indices $i \in [j - 1]$. Since $|Y_{M'}| \geq 5$, this implies $K_{2,1,1,1} \subseteq G^b$. Without loss of generality, suppose $|Y_{M'} \cap X_1| = 2$, and $|Y_{M'} \cap X_i| \geq 1$ for $i = 2, 3, 4$, with

$$\{x_1^1, x_2^1\} = Y_{M'} \cap X_1, \quad \{x_i^i\} \subseteq Y_{M'} \cap X_i \text{ for } i = 2, 3, 4.$$

Consider X_j , and define $X'_j = \{x \in X_j \mid xv \in E(M') \text{ for some } v \notin X_1\}$. Since $|X_j| = |X_1|$ and two vertices from X_1 belong to $Y_{M'}$, we have $|X'_j| \geq 2$.

Without loss of generality, assume $x_1^j, x_2^j \in X'_j$, with $x_1^j v_1, x_2^j v_2 \in E(M')$ for some $v_1, v_2 \in X_2 \cup X_3 \cup \dots \cup X_{j-1}$.

We claim that $Y_{M'} \cap X_1 \subseteq N_{G^b}(v_1) \cap N_{G^b}(v_2)$. If not, suppose $v_1 y' \in E(G^r)$ for some $y' \in Y_{M'} \cap X_1$. Replacing edge $x_1^j v_1$ with $v_1 y'$ gives a matching $M'' = (M' \setminus \{x_1^j v_1\}) \cup \{v_1 y'\}$, with $|I_{M''}| > |I_{M'}|$, contradicting the choice of M' .

Now, if $N_{G^b}(v_i) \cap \{x_1^2, x_1^3, x_1^4\} \neq \emptyset$ for some $i \in \{1, 2\}$, then since $Y_{M'} \cap X_1 \subseteq N_{G^b}(v_i)$, it follows that $C_7 \subseteq G^b[Y_{M'} \cup \{v_1, v_2\}]$, again a contradiction. Therefore, assume $N_{G^b}(v_i) \cap \{x_1^2, x_1^3, x_1^4\} = \emptyset$ for both $i = 1, 2$. By Claim 2.8, this forces $|N_{G^b}(x_i^j) \cap \{x_1^2, x_1^3, x_1^4\}| = 3$ for each $i = 1, 2$. Hence, the subgraph $G^b[Y_{M'} \cup \{v_1, x_1^j\}]$ contains a C_7 , a contradiction.

If $Y_{M'} \cap X_i \neq \emptyset$ for at least five indices $i \in [j - 1]$, then $K_5 \subseteq G^b[Y_{M'}]$. Since $n \geq 5$ and $|E(M)| = (n - 1)K_2$, the matching M contains at least two edges. Consider two edges $v_1 v_2$ and $v_3 v_4$ in M . Using Claim 2.8, we obtain that $|N_{G^b}(v_i) \cap Y_{M'}| \geq 2$ for at least one $i \in \{1, 2\}$ and $|N_{G^b}(v_j) \cap Y_{M'}| \geq 2$ for at least one $j \in \{3, 4\}$. Therefore, $C_7 \subseteq G^b[Y_{M'} \cup \{v, v'\}]$ for some $v, v' \in \{v_1, v_2, v_3, v_4\}$.

Thus, in all cases, we reach a contradiction to the assumption that G^b is C_7 -free, completing the proof of Subcase 2-1.

SUBCASE 2-2: $|Y_{M'} \cap X_i| \geq 3$ for at least one index $i \in [j - 1]$.

Without loss of generality, suppose $|Y_{M'} \cap X_1| \geq 3$, and let $\{x_1^1, x_2^1, x_3^1\} \subseteq Y_{M'} \cap X_1$. By Claim 2.9, $Y_{M'}$ intersects at least three different partite sets among $X_1, X_2, \dots, X_{j-1}$, and we assume that $|Y_{M'} \cap X_i| \geq 1$ for $i = 2, 3$, with $x_1^i \in Y_{M'} \cap X_i$ for $i = 2, 3$. This implies that $K_{3,1,1} \subseteq G^b[Y_{M'}]$.

Now consider X_j , and define

$$X'_j = \{x \in X_j \mid xv \in E(M') \text{ for some } v \notin X_1\}.$$

Since $X_j \subseteq V(M')$, $|X_j| = |X_1|$, and $|Y_{M'} \cap X_1| \geq 3$, we have $|X'_j| \geq 3$. Without loss of generality, suppose $x_1^j, x_2^j \in X'_j$, and that $x_1^j v_1, x_2^j v_2 \in E(M')$ for some $v_1, v_2 \in X_2 \cup \dots \cup X_{j-1}$.

We claim that $\{x_1^1, x_2^1, x_3^1\} \subseteq N_{G^b}(v_1) \cap N_{G^b}(v_2)$. Otherwise, suppose $vy' \in E(G^r)$ for some $y' \in Y_{M'} \cap X_1$ and some $v \in \{v_1, v_2\}$, say $v = v_1$. Then replacing edge $x_1^j v_1$ with $v_1 y'$ gives the matching $M'' = (M' \setminus \{x_1^j v_1\}) \cup \{v_1 y'\}$, which belongs to $\mathcal{M}$ and satisfies $|I_{M''}| > |I_{M'}|$, which is a contradiction.

Thus, for each $v \in \{v_1, v_2\}$, we have $Y_{M'} \cap X_1 \subseteq N_{G^b}(v)$. Since $x_1^2 \in Y_{M'} \cap X_2$ and $x_1^3 \in Y_{M'} \cap X_3$, it follows that $C_7 \subseteq G^b[Y_{M'} \cup \{v_1, v_2\}]$, which contradicts the assumption that G^b is C_7 -free.

Therefore, in both subcases 2-1 and 2-2, we have shown that the structure of M' would force a copy of C_7 in G^b , contradicting our assumptions. Thus, Case 2 is fully resolved.

Combining the results of case 1 and case 2, we conclude that for any $j \geq 5$, if $n \geq j$, the following holds:

$$m_j(nK_2, C_7) = \left\lfloor \frac{2n-1}{j-1} \right\rfloor + 1.$$

This completes the proof of the theorem. $\square$

By combining Theorems 2.1, 2.2, 2.3, 2.4, and 2.7, we derive Theorem 1.1 that fully characterizes the exact value of the M-R-number $m_j(nK_2, C_7)$ for all pairs (j, n) , where $j \geq 5$ and $n \geq 2$.

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