Contributions to Discrete Mathematics

Volume 21, Number 1, Pages 49–59
ISSN 1715-0868

THE LOGARITHM OF THE EXPONENTIAL GENERATING FUNCTION OF EULERIAN POLYNOMIALS

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ABSTRACT. In this note we determine the series expansion of the logarithm of the exponential generating function of Eulerian polynomials, which results in a new identity on Eulerian polynomials. We also obtain similar results for general Eulerian polynomials introduced by Xiong, Tsao, and Hall. As consequences, we derive some relations between classical Eulerian polynomials and their variations.

1. INTRODUCTION

Eulerian polynomials are classical objects in combinatorics, which have been thoroughly studied, extended and applied since they were introduced by Euler [6]. These polynomials appear in the following expression (see [7, (3.3)])

$$\sum_{i=1}^{m} i^n y^i = -y^{m+1} \sum_{k=0}^{n} \binom{n}{k} \frac{m^{n-k}}{(1-y)^{k+1}} A_k(y) + \frac{y}{(1-y)^{n+1}} A_n(y), \tag{1.1}$$

where $n \geq 0$, $m \geq 1$ and the Eulerian polynomials $A_n(y)$ $(n = 0, 1, 2, \ldots)$ are recursively defined by

$$A_0(y) = 1, \quad A_n(y) = \sum_{k=0}^{n-1} \binom{n}{k} A_k(y)(y-1)^{n-1-k}. \tag{1.2}$$

The first few Eulerian polynomials are given by

$$\begin{aligned} A_0(y) &= 1 \\ A_1(y) &= 1 \\ A_2(y) &= 1 + y \\ A_3(y) &= 1 + 4y + y^2 \\ A_4(y) &= 1 + 11y + 11y^2 + y^3. \end{aligned}$$

Received by the editors November 20, 2024, and in revised form February 3, 2025.
2010 *Mathematics Subject Classification.* 05A19.
Key words and phrases. exponential generating polynomials, Eulerian numbers, Eulerian polynomials, general Eulerian polynomials, arithmetic progression.

Letting m go to infinity in (1.1), we obtain (see [7, (3.2)])

$$\frac{A_n(y)}{(1-y)^{n+1}} = \sum_{j\geq 0} y^j (j+1)^n, \tag{1.3}$$

which is commonly taken as the definition of Eulerian polynomials nowadays.

Forming the exponential generating function

$$A(x,y) = \sum_{n\geq 0} A_n(y)\frac{x^n}{n!},$$

one can derive that (see [7, (3.1)])

$$A(x,y) = \frac{y-1}{y-\exp(x(y-1))}. \tag{1.4}$$

Motivated by the exponential formula of combinatorics, it is natural to ask whether the coefficients in the series expansion of $\exp(A(x,y))$ and $\ln(A(x,y))$ have some combinatorial meaning. It is possible to give a combinatorial interpretation for the coefficients of $\exp(A(x,y))$ by directly using the exponential formula. The purpose of this note is to study the series expansion of the logarithm of $A(x,y)$. In Section 2 we determine the coefficients for $\ln(A(x,y))$. In Section 3 we propose some open problems for further study.

Let $A_{n,k}$ denote the coefficient of x^k in the Eulerian polynomial $A_n(y)$, and these coefficients are known as Eulerian numbers. A combinatorial interpretation of Eulerian numbers in terms of descents of permutations was given by Riordan [8], who proved the following recurrence relation:

$$A_{n,k} = (k+1)A_{n-1,k} + (n-k)A_{n-1,k-1} \quad (1 \leq k \leq n-1), \tag{1.5}$$

with boundary values $A_{n,0} = 1$ $(n \geq 0)$ and $A_{n,k} = 0$ $(k \geq n)$. Xiong, Tsao and Hall [12] defined general Eulerian numbers $A_{n,k}(a,d)$ associated with an arithmetic progression $\{a, a+d, a+2d, \ldots\}$ by the recurrence

$$\begin{aligned} A_{n,k}(a,d) = {} & (-a+(k+2)d)A_{n-1,k}(a,d) \\ & + (a+(n-k-1)d)A_{n-1,k-1}(a,d) \quad (0 \leq k \leq n-1), \end{aligned}$$

with $A_{0,-1} = 1, A_{n,k} = 0$ $(k \geq n$ or $k \leq -2)$. They also defined the general Eulerian polynomials associated with an arithmetic progression as

$$T_n(y,a,d) = \sum_{k=-1}^{n-1} A_{n,k}(a,d) y^{k+1}. \tag{1.6}$$

Generally, let

$$T(x,y,a,d) = \sum_{n\geq 0} T_n(y,a,d)\frac{x^n}{n!}.$$

Xiong, Tsao and Hall [12, Lemma 12] proved that

$$T(x,y,a,d)=\frac{(y-1)\exp(ax(y-1))}{y-\exp(dx(y-1))}. \tag{1.7}$$

When $a=1$ and $d=2$, the polynomial $T_n(y,a,d)$ coincides with the type B Eulerian polynomials $B_n(y)$ introduced by Brenti [2]. Explicitly, we have

$$B(x,y)=\sum_{n\geq 0}B_n(y)\frac{x^n}{n!}=\frac{(y-1)\exp(x(y-1))}{y-\exp(2x(y-1))}.. \tag{1.8}$$

In Section 2 we also determine the coefficients for $\ln(T(x,y,a,d))$ and hence for $\ln(B(x,y))$.

2. The main results

Our first result is as follows, which shows that Eulerian polynomials also appear in the series expansion of the logarithm of $A(x,y)$.

Theorem 2.1. *We have*

$$\ln(A(x,y))=x+\sum_{n\geq 2}yA_{n-1}(y)\frac{x^n}{n!}. \tag{2.1}$$

Proof. Rewrite (1.3) as

$$A_n(y)=\left(\sum_{j\geq 0}y^j(j+1)^n\right)(1-y)^{n+1}. \tag{2.2}$$

Set

$$D(x,y)=x+\sum_{n\geq 2}yA_{n-1}(y)\frac{x^n}{n!}. \tag{2.3}$$

Substituting (2.2) into the right-hand side leads to

$$D(x,y)=x+\sum_{n\geq 2}y\left(\left(\sum_{j\geq 0}y^j(j+1)^{n-1}\right)(1-y)^n\right)\frac{x^n}{n!}.$$

Then exchanging the order of summation, we get

$$\begin{aligned}
D(x,y)&=x+\sum_{j\geq 0}\frac{y^{j+1}}{j+1}\sum_{n\geq 2}((j+1)^n(1-y)^n)\frac{x^n}{n!}\\
&=x+\sum_{j\geq 0}\frac{y^{j+1}}{j+1}(\exp((j+1)(1-y)x)-1-(j+1)(1-y)x)\\
&=x+\sum_{j\geq 0}\frac{(y\exp((1-y)x))^{j+1}}{j+1}-\sum_{j\geq 0}\frac{y^{j+1}}{j+1}-\sum_{j\geq 0}y^{j+1}(1-y)x.
\end{aligned}$$

Now using the following standard series expansions

$$\ln\left(\frac{1}{1-z}\right)=\sum_{j\geq 0}\frac{z^{j+1}}{j+1},\qquad \frac{1}{1-z}=\sum_{j\geq 0}z^j, \tag{2.4}$$

we obtain

$$\begin{aligned}
D(x,y)&=x+\ln\left(\frac{1}{1-y\exp(x(1-y))}\right)-\ln\left(\frac{1}{1-y}\right)-xy\\
&=x+\ln\left(\frac{\exp(x(y-1))}{\exp(x(y-1))-y}\right)-\ln\left(\frac{1}{1-y}\right)-xy\\
&=x+x(y-1)-\ln(\exp(x(y-1))-y)+\ln(1-y)-xy.
\end{aligned}$$

By further simplification, we have

$$\begin{aligned}
D(x,y)&=\ln(1-y)-\ln(\exp(x(y-1))-y)\\
&=\ln\left(\frac{y-1}{y-\exp(x(y-1))}\right),
\end{aligned}$$

which is just $\ln A(x,y)$ by (1.4). This completes the proof. □

In view of (1.4), we see that (2.1) is equivalent to

$$\sum_{n\geq 0}A_n(y)\frac{x^n}{n!}=\exp\left(x+\sum_{n\geq 2}yA_{n-1}(y)\frac{x^n}{n!}\right). \tag{2.5}$$

We wish to give a combinatorial interpretation of this identity, and hence that of Theorem 2.1. Let $[n]=\{1,2,\ldots,n\}$ and let $\mathfrak{S}_n$ denote the set of permutations π of $[n]$. Given a permutation $\pi=\pi_1\pi_2\ldots\pi_n\in\mathfrak{S}_n$, the number i $(1\leq i\leq n-1)$ is called a descent of π if $\pi_i>\pi_{i+1}$. The number of descents of π is denoted by $\mathrm{des}(\pi)$. It is well known that

$$A_n(y)=\sum_{\pi\in\mathfrak{S}_n}y^{\mathrm{des}(\pi)}.$$

Note that the concept of descents can be naturally defined for permutations of any totally ordered elements. We proceed to give a combinatorial proof of Theorem 2.1.

Second proof of Theorem 2.1. Note that the left-hand side of (2.5) is the exponential generating function for the following structure on $[n]$: Choose a permutation π of $[n]$ and weight it by $y^{\mathrm{des}(\pi)}$, where $\mathrm{des}(\pi)$ denotes the number of descents of π. Moreover, the right-hand side of (2.5) is the exponential generating function for the following structure on $[n]$: Choose a set partition $\{S_1,S_2,\ldots,S_k\}$ of $[n]$, and then for each j choose a permutation $\pi^{(j)}$ of $S_j\backslash\{\min S_j\}$, where $\min S_j$ denotes the smallest number of S_j; weight each $\pi^{(j)}$ by $y^{\mathrm{des}(\pi^{(j)})+1}$ (adopting the convention that the unique permutation of the empty set has -1 descents), and define the total weight $\{\pi^{(1)},\pi^{(2)},\ldots,\pi^{(k)}\}$ to be the products of the weights of each $\pi^{(j)}$. Let

$\mathcal{C}_n$ represent the set of pairs $(\{S_1, S_2, \ldots, S_k\}, \{\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(k)}\})$ where $\{S_1, S_2, \ldots, S_k\}$ is a set partition of $[n]$ and each $\pi^{(j)}$ is a permutation of $S_j \backslash \{\min S_j\}$. In order to prove (2.5) bijectively, we need to find a bijection $\phi : \mathfrak{S}_n \to \mathcal{C}_n$ such that if $\phi(\pi) = (\{S_1, S_2, \ldots, S_k\}, \{\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(k)}\})$ then

$$\operatorname{des}(\pi) = \operatorname{des}(\pi^{(1)}) + \cdots + \operatorname{des}(\pi^{(k)}) + k. \tag{2.6}$$

To describe ϕ, for a set partition $\{S_1, S_2, \ldots, S_k\}$ of $[n]$ we impose some linear ordering on the S_j's such that $\min S_1 < \min S_2 < \cdots < \min S_k$. Given a permutation $\pi = \pi_1\pi_2\cdots\pi_n$, suppose that $\pi_{i_1} = 1$. Let

$$\begin{aligned} S_1 &= \{\pi_1, \pi_2, \ldots, \pi_{i_1-1}, \pi_{i_1}\}, \\ \pi^{(1)} &= \pi_1\pi_2\cdots\pi_{i_1-1}. \end{aligned}$$

If $S_1, S_2, \ldots, S_j$ has been defined and $\pi_{i_{j+1}} = \min [n] \backslash (S_1 \uplus \cdots \uplus S_j)$, then let

$$\begin{aligned} S_{j+1} &= \{\pi_{i_j+1}, \pi_{i_j+2}, \ldots, \pi_{i_{j+1}-1}, \pi_{i_{j+1}}\}, \\ \pi^{(j+1)} &= \pi_{i_j+1}\pi_{i_j+2}\cdots\pi_{i_{j+1}-1}. \end{aligned}$$

Continue until the process stops, say at the kth step. Set

$$\phi(\pi) = (\{S_1, S_2, \ldots, S_k\}, \{\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(k)}\}),$$

which clearly satisfies (2.6). For instance if $n = 9$ and $\pi = 856127493$, then $\phi(\pi) = (\{\{1, 5, 6, 8\}, \{2\}, \{3, 4, 7, 9\}\}, \{856, \emptyset, 749\})$ and

$$\operatorname{des}(856) + \operatorname{des}(\emptyset) + \operatorname{des}(749) + 3 = 1 + (-1) + 1 + 3 = 4 = \operatorname{des}(856127493).$$

Moreover, from $(\{S_1, S_2, \ldots, S_k\}, \{\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(k)}\})$ one can uniquely recover π by setting π to be the concatenation $\pi^{(1)}a_1\pi^{(2)}a_2\cdots\pi^{(k)}a_k$ where $a_j = \min[n] \backslash (S_1 \uplus \cdots \uplus S_{j-1})$. The reason that $\pi^{(1)}a_1\pi^{(2)}a_2\cdots\pi^{(k)}a_k$ is a valid permutation of $[n]$ is as follows. By the construction of ϕ, we see that $\pi_{i_1} = 1$ and $i_1 < i_2 < \cdots < i_k = n$. Note that $\{\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(k)}\}$ uniquely determines $\{S_1, S_2, \ldots, S_k\}$ in the following way: Set $S_1 = \{1\} \cup \pi^{(1)}$; if $S_1, S_2, \ldots, S_j$ are defined then set $S_{j+1} = \{\min [n] \backslash (S_1 \uplus \cdots \uplus S_j)\} \cup \pi^{(j+1)}$, where each $\pi^{(i)}$ is considered as a set by abuse of notation. This completes the proof. □

As a consequence of (1.4) and Theorem 2.1, we obtain the following recurrence formula for Eulerian polynomials, which seems to be new.

Corollary 2.2. *For any $n \geq 1$, we have*

$$A_n(y) = \sum_{m_1+2m_2+\cdots+nm_n=n} \frac{n!}{\prod_{i=1}^n i^{m_i} m_i!} \cdot \prod_{i=1}^{n-1} \left(\frac{yA_i(y)}{i!}\right)^{m_{i+1}}.$$

Proof. Set

$$D_1(y) = 1$$
$$D_n(y) = yA_{n-1}(y) \quad (n \geq 2).$$

From (2.5) it follows that

$$\sum_{n\geq 0} A_n(y)\frac{x^n}{n!} = \exp\left(\sum_{i\geq 1} D_i(y)\frac{x^i}{i!}\right) = \sum_{k\geq 0} \frac{\left(\sum_{i\geq 1} D_i(y)\frac{x^i}{i!}\right)^k}{k!}.$$

Comparing the coefficients of x^n on both sides, we readily find the desired result. □

If we form the exponential generating function

$$\hat{A}(x,y) = \sum_{n\geq 0} \frac{A_n(y)}{(1-y)^{n+1}}\frac{x^n}{n!} = \frac{\exp(x)}{1-y\exp(x)},$$

then the following result can be derived from Theorem 2.1, whose proof will be omitted here.

Corollary 2.3. *We have*

$$\ln(\hat{A}(x,y)) = -\ln(1-y) + \frac{x}{1-y} + \sum_{n\geq 2} \frac{yA_{n-1}(y)}{(1-y)^n}\cdot\frac{x^n}{n!}.$$

Finally, we would like to point out Theorem 2.1 can also be used to prove the following identity.

Corollary 2.4 ([7, p. 18])**.** *We have*

$$1 + \sum_{n\geq 1} yA_n(y)\frac{x^n}{n!} = \frac{1-y}{1-y\exp((1-y)x)}.$$

Proof. Taking the partial derivative of (2.1) with respect to x, we get that

$$\frac{\partial \ln A(x,y)}{\partial x} = 1 + \sum_{n\geq 2} yA_{n-1}(y)\frac{x^{n-1}}{(n-1)!}.$$

Recalling (1.4) we obtain

$$1 + \sum_{n\geq 1} yA_n(y)\frac{x^n}{n!} = \frac{\frac{\partial A(x,y)}{\partial x}}{A(x,y)} = \frac{(y-1)^2\exp(x(y-1))}{(y-\exp(x(y-1)))^2}\cdot\frac{y-\exp(x(y-1))}{y-1},$$
$$= \frac{(y-1)\exp(x(y-1))}{y-\exp(x(y-1))},$$

which is equal to $\frac{1-y}{1-y\exp((1-y)x)}$, as desired. □

We proceed to determine the series expansion of $\ln(T(x,y,a,d))$.

Theorem 2.5. *We have*

$$\ln(T(x,y,a,d)) = (a(y-1)+d)x + \sum_{n\geq 2} yA_{n-1}(y)d^n \frac{x^n}{n!}.$$

Proof. By (1.4) and (1.7), we observe that

$$T(x,y,a,d) = A(dx,y)\exp(ax(y-1)).$$

Thus

$$\ln(T(x,y,a,d)) = \ln(A(dx,y)) + ax(y-1).$$

Then using (2.1), we obtain the desired result. □

In the same way that Corollary 2.2 is proved, we obtain the following expression of general Eulerian polynomials in terms of classical Eulerian polynomials.

Corollary 2.6. *For any $n \geq 1$, we have*

$$T_n(y,a,d) = \sum_{m_1+2m_2+\cdots+nm_n=n} \frac{n!(a(y-1)+d)^{m_1}}{\prod_{i=1}^n i^{m_i} m_i!} \cdot \prod_{i=1}^{n-1} \left(\frac{yA_i(y)d^{i+1}}{i!}\right)^{m_{i+1}}.$$

Proof. Set

$$\bar{D}_1(y) = a(y-1)+d$$
$$\bar{D}_n(y) = yA_{n-1}(y)d^n \quad (n \geq 2)$$

and

$$\bar{D}(x,y,a,d) = (a(y-1)+d)x + \sum_{n\geq 2} yA_{n-1}(y)d^n \frac{x^n}{n!}.$$

From $T(x,y,a,d) = \exp(\bar{D}(x,y,a,d))$ it follows that

$$\sum_{n\geq 0} T_n(y,a,d)\frac{x^n}{n!} = \exp\sum_{i\geq 1} \bar{D}_i(y)\frac{x^i}{i!} = \sum_{k\geq 0} \frac{\left(\sum_{i\geq 1} \bar{D}_i(y)\frac{x^i}{i!}\right)^k}{k!}.$$

Comparing the coefficients of x^n on both sides, we readily find the desired result. □

We would like to note that Xiong, Tsao and Hall [12, Lemma 11] already gave a connection between general Eulerian polynomials and classical Eulerian polynomials, which states that

$$T_n(y,a,d) = \sum_{i=0}^{n} \binom{n}{i} d^i A_i(y)(ay-a)^{n-j}. \tag{2.7}$$

For $a = 1$ and $d = 2$, Theorem 2.5 and Corollary 2.6 establish some new relation between $A_n(y)$ and $B_n(y)$. Precisely, we have the following result.

Theorem 2.7. *We have*

$$\ln(B(x,y)) = x(y+1) + \sum_{n\geq 2} 2^n y A_{n-1}(y) \frac{x^n}{n!} \tag{2.8}$$

and

$$B_n(y) = \sum_{m_1+2m_2+\cdots+nm_n=n} \frac{n!(y+1)^{m_1}}{\prod_{i=1}^{n} i^{m_i} m_i!} \cdot \prod_{i=1}^{n-1} \left(\frac{yA_i(y)2^{i+1}}{i!} \right)^{m_{i+1}} \quad (n \geq 1).$$

Remark: The polynomial $B_n(y)$ enjoys a combinatorial interpretation in terms of the descent statistic of type B over signed permutations. We failed to find a bijective proof of Theorem 2.7 by mimicking that of Theorem 2.1. Such a proof would be welcome.

Problem 2.8. *Give a bijective proof of Theorem 2.7.*

3. Open problems

In Theorem 2.1 we have a series expansion of $\ln(A(x,y))$. The next question is to study the series expansion of $\exp(A(x,y))$. Since $A_0(y) = 1$, we may directly consider the series expansion of

$$E^A(x,y) = \sum_{n\geq 0} E_n^A(y) \frac{x^n}{n!} = \exp\left(\sum_{n\geq 1} A_n(y) \frac{x^n}{n!} \right).$$

Suppose that

$$E_n^A(y) = \sum_{k\geq 0} E_{n,k}^A y^k.$$

One may use the exponential formula of exponential generating functions to give a formal combinatorial interpretation of $E_n^A(y)$. For example, it is not difficult to see that $E_{n,0}^A$ is equal to the nth bell number, namely, the number of ways to partition a set of n labeled elements. In the study of Mahonian permutation statistics, Babson and Steingrímsson [1] introduced a class of generalized permutation patterns by requiring that two adjacent letters in a pattern must be adjacent in the permutation. For example, for a permutation $\pi = \pi_1\pi_2\cdots\pi_n$, a patter 12-3 is a subword $\pi_i\pi_{i+1}\pi_j$ such that $\pi_i < \pi_{i+1} < \pi_j$. Let $u_r(n)$ denote the number of permutations of length n containing exactly r occurrences of the pattern 12-3. Claesson and Mansour [5, Table 1] gave the first few of these numbers. Computer experiments show that the following conjecture is true.

Conjecture 3.1. *For any $n \geq 0$ we have $E_{n,1}^A = u_1(n+1)$.*

It is well known that $A_n(y)$ has only real zeros for any $n \geq 1$. The following conjecture also seems true.

Conjecture 3.2. *For any $n \geq 1$ the polynomial $E_n^A(y)$ has only real zeros.*

Brenti [2] also introduced type D Eulerian polynomials $D_n(y)$, whose exponential generating function is as follows:

$$\bar{D}(x,y) = 1 + xy + \sum_{n\geq 2} D_n(y)\frac{x^n}{n!} = \frac{(y-1)(1-x\exp(x(y-1)))}{y-\exp(2x(y-1))}. \tag{3.1}$$

Suppose that $\ln(\bar{D}(x,y)) = \sum_{n\geq 0} \bar{D}_n(y)\frac{x^n}{n!}$. But computer experiments show that some coefficients of $\bar{D}_n(y)$ are negative. It would be interesting to establish a result similar to Theorem 2.1 for $\ln(\bar{D}(x,y))$. Similarly, we may consider the series expansions of

$$E^B(x,y) = \sum_{n\geq 0} E_n^B(y)\frac{x^n}{n!} = \exp\left(\sum_{n\geq 1} B_n(y)\frac{x^n}{n!}\right),$$

$$E^D(x,y) = \sum_{n\geq 0} E_n^D(y)\frac{x^n}{n!} = \exp\left(\sum_{n\geq 1} D_n(y)\frac{x^n}{n!}\right).$$

The following conjectures also seem true.

Conjecture 3.3. *For any $n \geq 1$ the polynomial $E_n^B(y)$ has only real zeros.*

Conjecture 3.4. *For any $n \geq 1$ the polynomial $E_n^D(y)$ has only real zeros.*

Carlitz [3] gave an extension of the Eulerian polynomials in the algebra of the q-series. Let $\mathfrak{S}_n$ denote the set of permutations of $[n] = \{1, 2, \ldots, n\}$. In combinatorics the classical Eulerian polynomials and their q-extensions are intimately related to several permutation statistics including the number of excedances, number of descents, major index, and inversion number.

Given a permutation $\pi = \pi_1\pi_2\ldots\pi_n \in \mathfrak{S}_n$, the number i $(1 \leq i \leq n)$ is called an excedance of π if $\pi_i > i$. The number of excedances of π is denoted by $\mathrm{exc}(\pi)$. It is known that

$$A_n(y) = \sum_{\pi\in\mathfrak{S}_n} y^{\mathrm{exc}(\pi)}.$$

For his q-Eulerian polynomial Carlitz [4] obtained the following combinatorial interpretation:

$$A_n(y,q) = \sum_{\pi\in\mathfrak{S}_n} y^{\mathrm{des}(\pi)} q^{\mathrm{maj}(\pi)} \quad (n \geq 0),$$

where $\mathrm{maj}(\pi)$ is defined as the sum of the descents of π. Stanley [10] studied the following q-analogue of $A_n(y)$:

$$\bar{A}_n(y,q) = y\sum_{\pi\in\mathfrak{S}_n} y^{\mathrm{des}(\pi)} q^{\mathrm{inv}(\pi)} \quad (n \geq 1),$$

and proved that

$$\bar{A}(x,y,q) = \sum_{n\geq 0} \bar{A}_n(y,q)\frac{x^n}{(q;q)_n} = \frac{1-y}{1-ye_q((1-y)x)}, \tag{3.2}$$

where $(q;q)_n = (1-q)(1-q^2)\cdots(1-q^n)$ and $e_q(x) = \sum_{n\geq 0} x^n/(q;q)_n$. Shareshian and Wachs [9] studied the following q-analogue of $\bar{A}_n(y)$:

$$\tilde{A}_n(y,q) = y \sum_{\pi\in\mathfrak{S}_n} y^{\mathrm{exc}(\pi)} q^{\mathrm{maj}(\pi)} \quad (n\geq 0),$$

and proved that

$$\tilde{A}(x,y,q) = \sum_{n\geq 0} \tilde{A}_n(y,q)\frac{x^n}{(q;q)_n} = \frac{(1-yq)e_q(x)}{e_q(xyq)-yqe_q(x)}. \tag{3.3}$$

Given (3.2) and (3.3), it is natural to ask the following question.

Problem 3.5. *Give a q-extension of Theorem 2.1 for the above two polynomials $\bar{A}(x,y,q)$ and $\tilde{A}(x,y,q)$.*

Acknowledgment

I am indebted to an anonymous reviewer for providing specific suggestions that helped me to improve the exposition.

Statements and Declarations

The author declares that he has no known competing interests or personal relationships that could have appeared to influence the work reported in this paper.

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