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AVERAGE EDGE ORDER OF NORMAL 3-PSEUDOMANIFOLDS

BIPLAB BASAK AND RAJU KUMAR GUPTA

ABSTRACT. In their work, Feng Luo and Richard Stong introduced the concept of the average edge order, denoted as μ_0. They demonstrated that if $\mu_0(K) \leq \frac{9}{2}$ for a closed triangulated 3-manifold K, then K must be a sphere. Building upon this foundation, Makoto Tamura extended similar results to compact triangulated 3-manifolds with nonempty boundaries in [12, 13]. In our present study, we extend these findings to normal 3-pseudomanifolds. Specifically, we establish that for a normal 3-pseudomanifold K with singularities, $\mu_0(K) \geq \frac{30}{7}$. Moreover, equality holds if and only if K is a one-vertex suspension of a triangulation of $\mathbb{RP}^2$ with seven vertices. Furthermore, we establish that when $\frac{30}{7} \leq \mu_0(K) \leq \frac{9}{2}$, the 3-pseudomanifold K can be derived from some boundary complexes of 4-simplices by a sequence of possible operations, including connected sums, bistellar 1-moves, edge contractions, edge expansions, vertex folding, and edge folding.

1. INTRODUCTION

Consider a normal 3-pseudomanifold denoted by K. Its *average edge order*, denoted by $\mu_0(K)$, is defined as $\mu_0(K) = \frac{3F(K)}{E(K)}$, where $F(K)$ and $E(K)$ represent the number of faces and edges in K respectively. Essentially, $\mu_0(K)$ represents the average number of faces incident on edges within K. In their research [10], Feng Luo and Richard Stong demonstrated that a small average edge order in a closed connected triangulated 3-manifold suggests a relatively simple topology and imposes constraints on its triangulation. The implication of this finding can be summarized as:

Proposition 1.1 ([10]). *Let K be any triangulation of a closed connected 3-manifold M. Then*

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(*i*) $3 \leq \mu_0(K) < 6$, *equality holds if and only if* K *is the triangulation of the boundary of a* 4*-simplex.*
(*ii*) *For any* $e > 0$, *there are triangulations* K_1 *and* K_2 *of* M *such that* $\mu_0(K_1) < 4.5 + e$ *and* $\mu_0(K_2) > 6 - e$.
(*iii*) *If* $\mu_0(K) < 4.5$, *then* K *is a triangulation of* $\mathbb{S}^3$.
(*iv*) *If* $\mu_0(K) = 4.5$, *then* K *is a triangulation of* $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{S}^1$, *or* $\mathbb{S}^2 \tilde{\times} \mathbb{S}^1$. *Furthermore, in the last two cases the triangulations can be described.*

Building upon the previous discussion, Makoto Tamura contributed further insights in [12] by extending similar results to 3-manifolds with nonempty boundaries. In [13], Tamura revised the definition of the average edge order as follows: $\mu_0(K) = 3\frac{F_0(K)}{E_0(K)}$, where $E_0(K) = E_i(K) + \frac{E_\partial(K)}{2}$ and $F_0(K) = F_i(K) + \frac{F_\partial(K)}{2}$. Here, $E_i(K)$ (respectively, $F_i(K)$) represents the number of edges (respectively, faces) within $\operatorname{int} K = K \setminus \partial(K)$, while $E_\partial(K)$ (respectively, $F_\partial(K)$) denotes the number of edges (respectively, faces) within $\partial(K)$. With this refined definition of the average edge order, Tamura established the following theorem:

Proposition 1.2 ([13, Theorem 1.2])**.** *Let* K *be any triangulation of a compact connected* 3*-manifold* M *with nonempty boundary. Then*

(*i*) $2 \leq \mu_0(K) < 6$, *and equality holds if and only if* K *is the triangulation of one* 3*-simplex.*
(*ii*) *For any rational number* r *with* $4 < r < 6$, *there is a triangulation* K' *of* M *such that* $\mu_0(K') = r$.
(*iii*) *If* $\mu_0(K) < 4$, *then* K *is a triangulation of* $\mathbb{B}^3$. *There are an infinite number of distinct such triangulations, but for any constant* $c < 4$ *there are only finitely many triangulations* K *with* $\mu_0(K) \leq c$.
(*iv*) *If* $\mu_0(K) = 4$, *then* K *is a triangulation of* $\mathbb{B}^3, \mathbb{D}^2 \times \mathbb{S}^1$ *or* $\mathbb{D}^2 \tilde{\times} \mathbb{S}^1$. *Furthermore, in the last two cases the triangulations can be described.*

In [4, 5], similar work is done for colored triangulations of manifolds. Some classification results for closed connected smooth low–dimensional manifolds according to average edge order are given in [6]. In this article, we establish comparable conclusions for normal 3- pseudomanifolds without boundaries. In fact, we prove the following result:

Theorem 1.3. *Let* K *be a normal* 3*-pseudomanifold with singularities, and let* $\mu_0(K)$ *represent the average edge order of* K. *Then*

(*i*) $\mu_0(K) \geq \frac{30}{7}$, *and equality holds if and only if* K *is a one-vertex suspension of* $\mathbb{RP}^2$ *with seven vertices.*
(*ii*) *If* $\mu_0(K) \leq \frac{9}{2}$, *then* K *is obtained from some boundary complexes of* 4*-simplices by a sequence of possible operations, including connected sums, bistellar* 1*-moves, edge contractions, edge expansions, vertex folding, and edge folding. In particular, either* $|K|$ *is a handlebody with its boundary coned off or a suspension of* $\mathbb{RP}^2$.

(*iii*) *If K contains exactly n singular vertices, then $\mu_0(K) < 6 + n$. Additionally, a sequence of normal 3-pseudomanifolds, denoted by $\{K_m\}_{m\geq 1}$, with two singularities exists, such that $\mu_0(K_m) \to 8$ as $m \to \infty$.*

2. Preliminaries

A *d-simplex* is the convex hull of $d + 1$ affinely independent points. A 0-simplex is a point, a 1-simplex is an edge, a 2-simplex is a triangle, and so on. Let σ be a simplex. Then, *faces* of σ are defined as the convex hulls of nonempty subsets of the vertex set of σ. We denote a face τ of σ as $\tau \leq \sigma$. A *simplicial complex* K is a finite collection of simplices in $\mathbb{R}^m$ for some $m \in \mathbb{N}$ that satisfies the following conditions: if $\sigma \in K$, then $\tau \leq \sigma$ implies $\tau \in K$; for any two faces σ and τ in K, either $\sigma \cap \tau = \emptyset$ or $\sigma \cap \tau \leq \sigma$ and $\sigma \cap \tau \leq \tau$. We assume that the *empty simplex* $\emptyset$ (considered as a simplex of dimension -1) is a member of every simplicial complex. If K is a simplicial complex, then its *geometric carrier* $|K| = \cup_{\sigma \in K} \sigma$ is the union of all the simplices in K together with the subspace topology induced from $\mathbb{R}^n$ for some $n \in \mathbb{N}$. The *dimension* of K is defined as the maximum of the dimension of simplices in K. If σ and τ are two skew simplices in $\mathbb{R}^n$ for some $n \in \mathbb{N}$, then their *join* $\sigma * \tau$ is defined as $\{\lambda p + \mu q \,|\, p \in \sigma, q \in \tau; \lambda, \mu \in [0, 1] \text{ and } \lambda + \mu = 1\}$. Similarly, we define the *join* of two skew simplicial complexes K_1 and K_2 as $\{\sigma * \tau | \sigma \in K_1, \tau \in K_2\}$. The *link* of a face σ in a simplicial complex K is defined as $\{\tau \in K \,|\, \sigma \cap \tau = \emptyset, \tau * \sigma \in K\}$. The link of σ is denoted by $\mathrm{lk}(\sigma, K)$. The *star* of a face σ in a simplicial complex K is defined as $\{\beta | \beta \leq \sigma * \alpha \text{ and } \alpha \in \mathrm{lk}(\sigma, K)\}$, and is denoted by $\mathrm{st}(\sigma, K)$.

A d-dimensional simplicial complex K is called *pure* if all its facets possess the same dimension. A simplicial complex K is called a *normal d-pseudomanifold without a boundary* if it is pure, every face of dimension $d - 1$ is contained in precisely two facets, and the link of every face of dimension $\leq d-2$ is connected. If some faces of dimension $d-1$ are contained in only one facet, then K is called a *normal d-pseudomanifold with boundary*. For a normal 3-pseudomanifold K, the link of a vertex v is a triangulated closed connected surface. If $|\mathrm{lk}(v)| \cong \mathbb{S}^2$, then we call v a nonsingular vertex; otherwise, v is called a singular vertex. Let K be a simplicial complex. Let $V(K)$, $E(K)$, $F(K)$, and $T(K)$ represent the counts of vertices, edges, faces, and tetrahedra in K, respectively. In this article, we will use the abbreviations V, E, F, and T to refer to $V(K)$, $E(K)$, $F(K)$, and $T(K)$ when the context is clear that we are discussing a simplicial complex K. The numbers g_2 and g_3 for a 3-dimensional simplicial complex K are defined as $g_2(K) = E - 4V + 10$, and $g_3(K) = F - 3E + 6V - 10$.

Remark 2.1. *Let*

$$\{[1,2,4],[1,3,4],[1,2,5],[1,3,6],[1,5,6],$$
$$[2,4,6],[2,3,5],[2,3,6],[3,4,5],[4,5,6]\}$$

denote a set of facets comprising the triangulation L of $\mathbb{RP}^2$ with six vertices. Consider $K = (uv * \mathrm{lk}(6)) \cup (u * L \setminus \{6\}) \cup (v * L \setminus \{6\})$. We define K as the one-vertex suspension of the complex L. Topologically, $|K|$ represents the suspension of $|L|$. Consequently,

$$\{[1,2,4,u],\ [1,3,4,u],\ [1,2,5,u],\ [1,2,4,v],\ [1,3,4,v],\ [1,2,5,v],\ [1,3,u,v],\ [1,5,u,v],\ [2,4,u,v],\ [2,3,5,u],\ [2,3,5,v],\ [2,3,u,v],\ [3,4,5,u],\ [3,4,5,v],\ [4,5,u,v]\}$$

forms the set of facets of K.

Proposition 2.2 ([2, Lemma 2.6]). *Let K be a normal 3-pseudomanifold, and $v \in K$ be a vertex in K. Then $g_2(K) \geq g_2(\mathrm{st}(v,K)) = g_2(\mathrm{lk}(v,K))$.*

For definitions of operations like connected sum, bistellar 1-move, edge contraction, edge expansion, vertex folding, and edge folding, please see [2]. The subsequent results can be obtained by substituting $n = 1$ into [2, Theorem 4.4] and $m = 1$ into [2, Theorem 5.5].

Proposition 2.3. *Let K be a normal 3-pseudomanifold with exactly one singularity at t, such that $|\mathrm{lk}(t,K)|$ is a torus or Klein bottle. Then, $g_2(K) \leq 15$ implies that K is obtained from some boundary complexes of 4-simplices by a sequence of operations of types connected sums, bistellar 1-moves, edge contractions, edge expansions, and a vertex folding. Furthermore, $|K|$ is a handlebody with its boundary coned off.*

Proposition 2.4. *Let K be a normal 3-pseudomanifold with exactly two $\mathbb{RP}^2$ singularities. Then, $g_2(K) \leq 12$ implies K is obtained from some boundary complexes of 4-simplices by a sequence of operations consisting of connected sums, bistellar 1-moves, edge contractions, edge expansions, and an edge folding.*

Remark 2.5. *If K is a normal 3-pseudomanifold and K^{ψ}_{uv} is obtained from K by an edge folding at uv, then $g_2(K^{\psi}_{uv}) = g_2(K) + 3$. Furthermore, if K is a triangulation of a 3-sphere then K^{ψ}_{uv} is a triangulation of the suspension of $\mathbb{RP}^2$ because $|K^{\psi}_{uv}|$ is obtained from a 3-ball by the following procedure: (i) choose two connected regions in the boundary of the 3-ball having a point in common, (ii) identify those two regions in reverse orientation, and (iii) then take the resulting topological space with its boundary conned off.*

3. Average edge order of a normal 3-pseudomanifold

Consider K as a normal 3-pseudomanifold with singularities, containing n singular vertices denoted by $t_1, t_2, \ldots, t_r, p_1, p_2, \ldots, p_{r'}$, where $r + r' = n$ for some $r, r' \geq 0$. According to surface classification, each vertex $x \in K$ has a link, $\mathrm{lk}(x)$, which is either a triangulated sphere, a triangulation of connected sums of tori, or a triangulation of connected sums of projective planes. Specifically, if $|\mathrm{lk}(t_i)| \cong \#h_i\mathbb{T}^2$ and $|\mathrm{lk}(p_j)| \cong \#m_j\mathbb{RP}^2$ for $h_i \geq 0$,

$m_j \geq 0$, $1 \leq i \leq r$, and $1 \leq j \leq r'$, where $\mathbb{T}^2$ and $\mathbb{RP}^2$ represent a torus and a projective plane, respectively, then $\chi(t_i) = 2 - 2h_i$ and $\chi(p_j) = 2 - m_j$.

Let V, E, F, and T denote the count of vertices, edges, faces, and tetrahedra in K, respectively. Considering $g_2(K) = E - 4V + 10$ and $g_3(K) = F - 3E + 6V - 10$, it follows that $g_2(K) + g_3(K) = F - 2E + 2V$. Additionally, we have

$$g_2(K) + g_3(K) = \sum_{v \in K} (2 - \chi(\mathrm{lk}(v))).$$

Thus, the average edge order of a normal 3-pseudomanifold is,

$$\begin{aligned}
\mu_0(K) &= \frac{3F}{E} = 3\frac{(g_2(K) + g_3(K) + 2E - 2V)}{E} \\
&= 3\frac{(\sum_{v \in K}(2 - \chi(\mathrm{lk}(v))) + 2E - 2V)}{E} \\
&= 3\frac{(2V - \sum_{v \in K} \chi(\mathrm{lk}(v)) + 2E - 2V)}{E} \\
&= 3\frac{(-\sum_{v \in K} \chi(\mathrm{lk}(v)) + 2E)}{E} \\
&= 6 - 3\frac{\sum_{v \in K} \chi(\mathrm{lk}(v))}{E} \\
&= 6 - 3\frac{2(V - n)}{E} - 3\frac{\sum_{i=1}^{r} \chi(\mathrm{lk}(t_i))}{E} - 3\frac{\sum_{i=1}^{r'} \chi(\mathrm{lk}(p_i))}{E} \\
&= 6 - 6\frac{(V - n)}{E} - 3\frac{\sum_{i=1}^{r}(2 - 2h_i)}{E} - 3\frac{\sum_{i=1}^{r'}(2 - m_i)}{E} \\
&= 6 - 6\frac{V}{E} + 6\frac{n}{E} - 6\frac{(r + r')}{E} + 6\frac{\sum_{i=1}^{r} h_i}{E} + 3\frac{\sum_{i=1}^{r'} m_i}{E} \\
&= 6 - 6\frac{V}{E} + 6\frac{\sum_{i=1}^{r} h_i}{E} + 3\frac{\sum_{i=1}^{r'} m_i}{E}.
\end{aligned}$$

Proof of Theorem 1.3 (ii). Suppose that $\mu_0(K) \leq \frac{9}{2}$, then we obtain

$$
\begin{aligned}
& 6-6\frac{V}{E}+6\frac{\sum_{i=1}^{r} h_i}{E}+3\frac{\sum_{i=1}^{r'} m_i}{E} \le \frac{9}{2} \\
\implies & -6\frac{V}{E}+6\frac{\sum_{i=1}^{r} h_i}{E}+3\frac{\sum_{i=1}^{r'} m_i}{E} \le \frac{9}{2}-6 \\
\implies & 4\frac{V}{E}-4\frac{\sum_{i=1}^{r} h_i}{E}-2\frac{\sum_{i=1}^{r'} m_i}{E} \ge 1 \\
& \implies E-4V+10 \le 10-4\sum_{i=1}^{r} h_i-2\sum_{i=1}^{r'} m_i \\
& \implies g_2(K) \le 10-4\sum_{i=1}^{r} h_i-2\sum_{i=1}^{r'} m_i.
\end{aligned}
$$

Consider h as the maximum value in the set $\{h_1, h_2, \ldots, h_r\}$, and let m be the maximum value in $\{m_1, m_2, \ldots, m'_r\}$. Given Lemma 2.2, which states that $g_2(K) \ge g_2(\mathrm{lk}(t))$ for any $t \in K$, we derive:

$$6h \le 10-4\sum_{i=1}^{r} h_i-2\sum_{i=1}^{r'} m_i$$

and

$$3m \le 10-4\sum_{i=1}^{r} h_i-2\sum_{i=1}^{r'} m_i.$$

It is evident that the inequalities above are satisfied if and only if one of the following situations arises:

(a) $n = 1$, $h = 1$, $m = 0$
(b) $n = 1$, $h = 0$, $m = 1$
(c) $n = 1$, $h = 0$, $m = 2$
(d) $n = 2$, $h = 0$, $m = 1$
(e) $n = 3$, $h = 0$, $m = 1$

Due to the fact that the sum of Euler characteristics for the vertex links of a normal 3-pseudomanifold is always even, we can infer that cases (b) and (e) are not feasible.

Case (a) and (c): In both cases, $g_2(K) \le 6$, and K features precisely one singularity at either t_1 or p_1, where $\mathrm{lk}(t_1)$ forms a triangulated torus and $\mathrm{lk}(p_1)$ constitutes a triangulated Klein bottle. Moreover, $g_2(\mathrm{lk}(t_1)) = g_2(\mathrm{lk}(p_1)) = 6$. Additionally, according to Proposition 2.2, we establish $g_2(K) \ge 6$. Consequently, $g_2(K) = 6$. Henceforth, according to Proposition 2.3, K is obtained from some boundary complexes of 4-simplices by a sequence of operations, including connected sums, bistellar 1-moves, edge contractions, edge expansions, and vertex folding. Furthermore, $|K|$ is a handlebody with its boundary coned off.

CASE (d): In this instance, K encompasses two $\mathbb{RP}^2$ singularities at vertices p_1 and p_2. According to Proposition 2.2, we have $3 \leq g_2(K) \leq 6$. Consequently, by Proposition 2.4, we deduce that K is obtained from some boundary complexes of 4-simplices by a sequence of possible operations, including connected sums, bistellar 1-moves, edge contractions, edge expansions, and an edge folding. Moreover, K is the triangulation of suspension of $\mathbb{RP}^2$ by Remark 2.5.

Hence, by combining cases (a), (c) and (d), the proof of Theorem 1.3 (*ii*) follows. □

Proof of Theorem 1.3 (*i*). Consider K as a normal 3-pseudomanifold with singularities, where $\mu_0(K) \leq \frac{30}{7}$. Therefore, $\mu_0(K) \leq \frac{30}{7} < \frac{9}{2}$. Consequently, based on the preceding discussions, we have only three potential cases for K: (a), (c), and (d). In both cases (a) and (c), $\mu_0(K) = \frac{9}{2} > \frac{30}{7}$, rendering them implausible. Now, let's discuss the case (d). Assume K contains two $\mathbb{RP}^2$ singularities at vertices u and v. Hence, $3 \leq g_2(K) \leq 6$. Then, we obtain

$$\begin{aligned}
\mu_0(K) = 6 - \frac{6(V-1)}{E} &\leq \frac{30}{7} \\
\implies 6 - \frac{6(V-1)}{4V - 10 + g_2(K)} &\leq \frac{30}{7} \\
\implies \frac{18V + 6(g_2(K-9))}{4V - 10 + g_2(K)} &\leq \frac{30}{7} \\
\implies 126V - 378 + 42g_2(K) &\leq 120V + 30g_2(K) - 300 \\
\implies V &\leq 13 - 2g_2(K).
\end{aligned}$$

Considering that for any normal 3-pseudomanifold with singularities Δ, $V(\Delta) \geq 7$, the aforementioned inequality holds if and only if $g_2(K) = 3$ and $V(K) = 7$. Consequently, this implies that $\mu_0(K) = \frac{30}{7}$, and K represents a triangulation of the one-vertex suspension of $\mathbb{RP}^2$. Conversely, according to Remark 2.1, we obtain a triangulation K of the one-vertex suspension of $\mathbb{RP}^2$ with seven vertices, satisfying $\mu_0(K) = \frac{30}{7}$ and $g_2(K) = 3$. This concludes the proof of Theorem 1.3 (*i*). □

Proof of Theorem 1.3 (*iii*). Let h and m denote the same as previously stated. Therefore, $g_2(K) \geq 6h$, and $g_2(K) \geq 3m$. Consequently, $E(K) \geq 6h$ and $E(K) \geq 3m$. Additionally, we have

$$\mu_0(K) = 6 - 6\frac{V}{E} + 6\frac{\sum_{i=1}^{r} h_i}{E} + 3\frac{\sum_{i=1}^{r'} m_i}{E}.$$

Thus,

$$\begin{aligned}\mu_0(K) &= 6 - 6\frac{V}{E} + 6\frac{\sum_{i=1}^{r} h_i}{E} + 3\frac{\sum_{i=1}^{r'} m_i}{E} \\ &\leq 6 + 6\frac{rh}{E} + 3\frac{r'm}{E} \\ &< 6 + 6\frac{rh}{6h} + 3\frac{r'm}{3m} \\ &= 6 + r + r' \\ &= 6 + n.\end{aligned}$$

Hence, $\mu_0(K) < 6 + n$.

Consider V_m, E_m, and F_m as the number of vertices, edges, and faces, respectively, of the minimal triangulation $\mathbb{T}_m$ representing a torus with genus $m \geq 3$. According to [8], we find: $V_m = \lceil \frac{7+\sqrt{1+48m}}{2} \rceil$, $E_m = 3V_m + 6m - 6$, $F_m = 2V_m + 4m - 4$. Now, if we take the suspension of the surface $\mathbb{T}_m$, we obtain a normal 3-pseudomanifold denoted as K_m with two singularities, where: $V(K_m) = V_m + 2$, $E = 2V_m + E_m$, and $F = 2E_m + F_m$. Then,

$$\mu_0(K_m) = \frac{3F}{E} = \frac{6E_m + 3F_m}{2V_m + E_m} = \frac{24V_m + 48m - 48}{5V_m + 6m - 6}.$$

Thus, $\mu_0(K_m) \to 8$ as $m \to \infty$. This completes the proof of Theorem 1.3(iii). □

Remark 3.1. If K_1 represents a normal 3-pseudomanifold obtained from a stacked sphere K' with $g_2(K') = 0$ via a vertex folding, then the following relations hold: $E(K') = 4V(K') - 10$, $F(K') = 6V(K') - 20$, $T(K') = 3V(K') - 10$. Moreover, $V(K_1) = V(K') - 3$, $E(K_1) = E(K') - 6 = 4V(K') - 16$, $F(K_1) = F(K') - 4 = 6V(K') - 24$, $T(K_1) = T(K') - 2 = 3V(K') - 12$. Now, we find that

$$\mu_0(K_1) = \frac{3F}{E} = 3\frac{6V - 24}{4V - 16} = \frac{9}{2}\left(\frac{V - 4}{V - 4}\right) = \frac{9}{2}.$$

Corollary 3.2. *If K is a normal 3-pseudomanifold and $\mu_0(K) < \frac{30}{7}$, then K is triangulation of a 3-sphere.*

Proof. If K represents a normal 3-pseudomanifold and $\mu_0(K) < \frac{30}{7}$, according to Theorem 1.3, K does not contain any singular vertices. This indicates that K forms a triangulation of a 3-manifold, where $\mu_0(K) < \frac{30}{7}$. Consequently, the result follows from Proposition 1.1(iii). □

4. On the upper bound of average edge order

The upper bound provided in Theorem 1.3 varies according to the number of singularities. In this context, we present several examples of normal 3-pseudomanifolds with singularities that possess a higher average edge order, sourced from the simplicial complex library of GAP [7]. Denote by [SCLib,

$n]$ the simplicial complex located at the nth position within the simplicial complex library in GAP.

Example 4.1. *Let* $K_1 = [\mathrm{SCLib}, 60]$ *and* $K_2 = [\mathrm{SCLib}, 61]$*, which represent two normal* 3*-pseudomanifolds with the vector* $(V, E, F, T) = (11, 55, 154, 77)$*. Notably, both* K_1 *and* K_2 *exhibit* $\mu_0(K_1) = \mu_0(K_2) = 8.4$*. In both complexes, the link of each vertex forms a triangulation consisting of* 6 *copies of* $\mathbb{RP}^2$*. The automorphism groups for* K_1 *and* K_2 *are* D_{22} *(a dihedral group with* 22 *elements) and* C_{11} *(a cyclic group with* 11 *elements), respectively.*

Example 4.2. *Let* $K_3 = [\mathrm{SCLib}, 540]$ *be a normal* 3*-pseudomanifold with the vector* $(V, E, F, T) = (17, 136, 544, 272)$*. Consequently,* $\mu_0(K_3) = 12$*. The link of each vertex in* K_3 *forms a triangulation consisting of* 18 *copies of* $\mathbb{RP}^2$*. The automorphism group of* K_3 *is* $C_{17} : C_{16}$*, a semidirect product of cyclic groups of orders* 16 *and* 17*.*

Example 4.3. *Let* $K_4 = [\mathrm{SCLib}, 587]$ *be a normal* 3*-pseudomanifold with the vector* $(V, E, F, T) = (19, 171, 684, 342)$*. Consequently,* $\mu_0(K_4) = 12$*. The link of each vertex in* K_4 *forms a triangulation consisting of* 20 *copies of* $\mathbb{RP}^2$*. The automorphism group of* K_4 *is* $C_{19} : C_{18}$*.*

Example 4.4. *Let* $K_5 = [\mathrm{SCLib}, 541]$ *be a normal* 3*-pseudomanifold with the vector* $(V, E, F, T) = (17, 136, 680, 340)$*. Consequently,* $\mu_0(K_5) = 15$*. The link of each vertex in* K_5 *forms a triangulation consisting of* 26 *copies of* $\mathbb{RP}^2$*. The automorphism group of* K_5 *is the same as the automorphism group of* K_3*, i.e.,* $C_{17} : C_{16}$*.*

It is interesting to note that in all the above examples, each vertex in the complexes is a singular vertex. This draws attention towards the study of normal 3-pseudomanifolds with all vertices being singular. Additionally, it's worth mentioning that all of these complexes exhibit neighborly properties, with even the example of K_5 being 3-neighborly. If a normal 3-pseudomanifold K is 3-neighborly, then $\mu_0(K) = 3\binom{V}{3}/\binom{V}{2} = V - 2$. These observations motivate us to study the following questions of interest.

Question. *How many vertices are needed at minimum to construct a neighborly normal* 3*-pseudomanifold, where the link of each vertex forms a triangulated surface of genus* g *for* $g \in \mathbb{N}$*?*

Question. *Is there a* 3*-neighborly normal* 3*-pseudomanifold* K *with* $V(K) > n$ *for every positive integer* $n \in \mathbb{N}$*?*

References

1. B. Bagchi and B. Datta, *Lower bound theorem for normal pseudomanifolds*, Expo. Math. **26** (2008), 327–351.
2. B. Basak, R. K. Gupta, and S. Sarkar, *A characterization of normal 3-pseudomanifolds with at most two singularities*, Discrete Math. **346** (2023), no. 12, Paper No. 113588, 15 pp.
3. B. Basak and E. Swartz, *Three-dimensional normal pseudomanifolds with relatively few edges*, Adv. Math. **365** (2020), 107035, 1–25.

4. M. R. Casali, *The average edge order of 3-manifold coloured triangulations*, Canad. Math. Bull. **37** (1994), no. 2, 154–161.
5. ______, *Average order of colored triangulations: The general case*, Osaka J. Math. **35** (1998), 249–262.
6. A. Cavicchioli and F. Spaggiari, *On graph-theoretical invariants of combinatorial manifolds*, Electron. J. Combin. **26** (2019), no. 3, 1–15.
7. F. Effenberger and J. Spreer, *simpcomp—a gap toolkit for simplicial complexes*, Version 4.12.0 (2022), `http://www.igt.uni-stuttgart.de/LstDiffgeo/simpcomp`.
8. M. Jungerman and G. Ringel, *Minimal triangulations on orientable surfaces*, Acta Math. **145** (1980), 121–154.
9. G. Kalai, *Rigidity and the lower bound theorem i*, Invent. Math. **88** (1987), 125–151.
10. F. Luo and R. Stong, *Combinatorics of triangulations of 3-manifolds*, Trans. Amer. Math. Soc. **337** (1993), 891–906.
11. E. Swartz, *Thirty-five years and counting*, arXiv:1411.0987.
12. M. Tamura, *The average edge order of triangulations of 3-manifolds*, Osaka J. Math. **33** (1996), 761–773.
13. ______, *The average edge order of triangulations of 3-manifolds with boundary*, Trans. Amer. Math. Soc. **350** (1998), 2129–2140.

Department of Mathematics, Indian Institute of Technology Delhi, New Delhi 110016, India.
E-mail address: `biplab@iitd.ac.in`

Department of Mathematics, Indian Institute of Technology Delhi, New Delhi 110016, India.
E-mail address: `Raju.Kumar.Gupta@maths.iitd.ac.in`