Contributions to Discrete Mathematics

Volume 21, Number 1, Pages 109–122
ISSN 1715-0868

COPS AND ROBBER WITH DECOYS

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Abstract. We introduce a variation of the Cops and Robber game in which the robber side consists of a robber and a decoy which are indistinguishable to the cops except under certain conditions. The cops win when one of them moves onto the same vertex as the actual robber (i.e. not the decoy) after a finite number of turns. The robber can throw the decoy to a neighbouring vertex on any turn beyond his first; such a turn for the robber consists of throwing (or dropping) the decoy and then moving. The current decoy disappears as the next is thrown so there is only a single decoy in play at any time. We characterize decoy-copwin graphs in the case where the cop can distinguish between the robber and decoy only when he is on the same vertex as one of them. We also characterize such graphs if the cop can distinguish between the robber and decoy only when he has cornered at least one of them.

1. Introduction

The game of Cops and Robber is a pursuit game played on a reflexive graph, i.e. a graph with a loop at every vertex. There are two opposing sides, a set of $k > 0$ cops and a single robber. The cops begin the game by each choosing a vertex to occupy, and then the robber chooses a vertex. The two sides alternate turns. During a turn for the cop side, each cop makes a move where a move is to slide along an edge or along a loop (i.e. pass). A move for the robber during his turn is the same. A round of the game consists of a turn for the cops together with the robber's subsequent turn. There is perfect information, i.e. both sides can see the locations of all the players, and the cops win if any of them and the robber occupy the same vertex at the same time, after a finite number of turns. Graphs on which one cop suffices to win are called *copwin* graphs. In general, graphs on which k

Received by the editors June 4, 2024, and in revised form March 24, 2025.

2000 *Mathematics Subject Classification.* 05C75, 05C57, 05C76, 91A43.

Key words and phrases. Pursuit game, Cops and Robber, structural characterizations, graph products, subgraphs, misinformation.

N. E. Clarke acknowledges support from NSERC (Discovery Grant 2015-06258); D. DesRoches acknowledges support from an NSERC USRA.

cops can guarantee a win are called *k-copwin* graphs. The minimum number of cops that suffice to win on a graph G is the *copnumber* of G.

In the Cops and Robber literature, there are many papers that consider variations of the game in which constraints are imposed on the cops with respect to movement, information, etc. In recent years, there has been increasing interest in concentrating on the robber side, via additional constraints or additional powers; see for example [2, 5, 9]. In this paper, we introduce such a variation of the Cops and Robber game in which the robber side consists of a robber and a decoy which are indistinguishable to the cops except under certain conditions. See also [8, 10]. (These conditions will be made precise at the beginning of the corresponding sections; we consider two sets of conditions and thus two variants of the decoy version of the game.) The cops win when one of them moves onto the same vertex as the actual robber (i.e. not the decoy) after a finite number of turns. The robber can throw the decoy to a neighbouring vertex on any turn beyond his first (dropping the decoy on the robber's current vertex is allowed); such a turn for the robber consists of throwing the decoy and then moving. To the cop side, these actions appear to happen simultaneously. The previous decoy disappears as the next is thrown so there is only a single decoy in play at any time. Note that the decoys from successive robber turns appear/disappear simultaneously so that the cop side can never see the robber's actual position during the previous round. (Of course, it is sometimes possible for the cops to deduce this information.) Note that if the robber were allowed to throw the decoy on his first turn, the cops would not even be able to win on a 3-cycle. Graphs on which one cop suffices to win are called *decoy-copwin* graphs. Other graphs are *decoy-robberwin*. Our first observation: *A decoy-copwin graph is copwin.*

In the original papers, Nowakowski and Winkler [11] and Quilliot [12] completely characterize copwin graphs. In [6], Clarke and MacGillivray give a relational characterization of k-copwin graphs, for all finite k, and then use the relational characterization to obtain a vertex elimination order characterization of such graphs. The motivation for this paper lies in attempting to discover a characterization of decoy-copwin graphs. We give such characterizations for both variants; these are Theorems 2.4 and 3.6.

We now introduce our notation and basic definitions. All graphs G on which games are played are finite, connected, and reflexive. For $a, b \in V(G)$, we use $a \simeq b$ to indicate that a and b are adjacent (possibly equal). For $x \in V(G)$, $N[x] = \{y \mid y \simeq x\}$ and $N(x) = N[x] \setminus \{x\}$. We denote the *degree* of x by $\deg(x) = |N(x)|$. A *universal vertex* is one of degree $|V(G)| - 1$; a *near-universal vertex* is one of degree $|V(G)| - 2$. A (sub)graph with a universal vertex is a *universal (sub)graph*. We use P_n to denote a path with n vertices, and C_n a cycle with n vertices. If H is a subgraph of G, we write $H \subseteq G$. If $X \subseteq V(G)$, then $G[X]$ is the subgraph induced in G by the vertices of X, and $G - X = G[V(G) \setminus X]$.

The *strong product* of G and H, denoted $G\boxtimes H$, has vertex set $V(G\boxtimes H) = V(G) \times V(H)$, with $(u_1, v_1)(u_2, v_2) \in E(G \boxtimes H)$ if and only if one of the following holds:

(1) $u_1u_2 \in E(G)$ and $v_1 = v_2$,
(2) $u_1 = u_2$ and $v_1v_2 \in E(H)$, or
(3) $u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$.

We will often denote a vertex (a, b) of $G \boxtimes H$ by ab. We note that $G \boxtimes H$ can be thought of as $|V(G)|$ copies of H (joined by appropriate edges). We shall denote the copy of H corresponding to $v \in V(G)$ as $v \cdot H$. (Note that since all graphs considered are reflexive, the strong and categorical products coincide.)

A mapping $f : V(G) \to V(H)$ is a *homomorphism* if, for $x, y \in V(G)$, $f(x) \simeq f(y)$ whenever $x \sim y$. A subgraph H of a graph G is a *retract* of G if there is a homomorphism $f : V(G) \to V(H)$ such that $f(x) = x$, for all $x \in V(H)$. A class of graphs is said to be a *variety* if it is closed under finite (strong) products and retracts. A vertex u of a graph G is a *corner* or an *irreducible* vertex if there exists a vertex v in G such that $N[u] \subseteq N[v]$; we say that v *dominates* u and that a robber on u is *cornered* by a cop on v. A vertex ordering $(x_1, x_2, \ldots, x_n)$ on G is a *domination elimination ordering* [3, 4] if, for each $i \in \{1, 2, \ldots, n-1\}$, there is a $j_i > i$ such that $N_i(x_i) \subseteq N_i[x_{j_i}]$ in $G_i = G - \{x_1, x_2, \ldots, x_{i-1}\}$. If, in addition, for each i, $x_i \sim x_{j_i}$, then this domination elimination ordering is a *copwin ordering* [11]. A main result of [11, 12] is that: *a finite graph G is copwin if and only if G has a copwin ordering.* In [7], a strategy is given that can be used by the cop to guarantee a win on a copwin graph (a slight improvement is given in [5]).

2. Variant 1

In the first variant of the Decoy Cops and Robber game that we consider, *the robber and the decoy are indistinguishable to the cops unless a cop is sharing a vertex with at least one of the robber and the decoy.* Recall that the robber cannot throw a decoy on his first turn since, otherwise, even C_3 would be a decoy-robberwin graph. We will denote the set of graphs on which a single cop can guarantee a win under the rules of this variant DCW_1 (for **D**ecoy-**C**op**w**in under the **1**st variant). Of course, by guaranteeing a win, we mean that if at any time the cops must choose arbitrarily between two vertices that contain the robber and the decoy, we cannot assume that the cop side is lucky and chooses the one with the robber.

Theorem 2.1. *Let T be a finite tree. Then $T \in DCW_1$.*

Proof. Consider the cop's starting position to be the root of the tree. Intuitively, the cop always moves towards the robber and decoy and this never gives conflicting directions since, when it is time for the cop's turn, the robber and decoy are necessarily in the same subtree with respect to the cop's position. We present the cop's strategy in detail in two phases.

During the first phase, the cop proceeds according to his winning strategy in the original Cops and Robber game (i.e. the cop moves toward the robber along the unique path joining their positions). This phase continues until the cop is adjacent to at least one of the robber and the decoy at the end of a round.

Note that, to this point in the game, although the path joining the positions of the cop and the robber and the path joining the positions of the cop and the decoy at the beginning of a round may diverge (at the vertex adjacent to the positions of both the robber and the decoy), the resulting move for the cop will be unaffected. Specifically, suppose the cop is occupying a vertex v_C with the robber and decoy on vertices v_R and v_D, respectively. If one of the $v_C v_R$-path and the $v_C v_D$-path is a subpath of the other, then the cop moves toward whichever of v_R and v_D is closer (thereby moving toward both). Of course the cop cannot distinguish between the robber and decoy during this phase of the strategy. Otherwise the paths diverge at a vertex v whose children (with respect to the root) include v_R and v_D. The cop moves toward v.

The second phase begins with the cop adjacent to at least one of the robber and the decoy at the beginning of a round. Since the robber and the decoy will always be in the same subtree rooted at the cop's current vertex w, if the robber and the decoy are on the same level of the tree at the beginning of this second phase (i.e. at the same distance from the root), then they are occupying the same vertex. (Else the robber would have been on w when the cop moved there, and the game would have already ended.) Otherwise if the robber and the decoy are on different levels of the tree, the cop will either capture the robber (if the robber is the one of the robber and decoy closer to the root) or else move to the same vertex as the decoy and continue the game. Note that this can happen only a finite number of times before the cop wins or else is located on the parent of a leaf u on which both the robber and the decoy are located. Hence $T \in DCW_1$. $\square$

Since the robber cannot throw a decoy on his first move, the following result is obvious.

Observation 2.2. *Let G_U be a graph with a universal vertex. Then $G_U \in DCW_1$.*

Let $\mathcal{T}$ be the set of all finite trees, and $\mathcal{G}_U$ the set of all universal graphs. Let $\mathcal{G}_U^{\mathcal{T}}$ be the set of graphs obtained by identifying a vertex $v_i \in V(G_U)$, $G_U \in \mathcal{G}_U - \{K_1\}$, with any vertex of a finite tree $T_i \in \mathcal{T} - \{K_1\}$, for one or more i's. (If we choose to think of $\mathcal{G}_U^{\mathcal{T}}$ as the set of graphs obtained by attaching a (possibly trivial) tree to every vertex of a universal graph, then $\mathcal{T}, \mathcal{G}_U \subseteq \mathcal{G}_U^{\mathcal{T}}$.)

Corollary 2.3. *Let $G_U^{\mathcal{T}} \in \mathcal{G}_U^{\mathcal{T}}$. Then $G_U^{\mathcal{T}} \in DCW_1$.*

Proof. Suppose the cop starts on a universal vertex of $G_U \subseteq G_U^{\mathcal{T}}$. By Observation 2.2, if the robber begins on a vertex of the subgraph G_U, the cop will apprehend the robber on his next move. So, to avoid immediate capture, the robber must choose a vertex w in T_i such that w is not that vertex $v \in V(G_U) \cap V(T_i)$ where T_i is attached. The cop moves to v and follows the strategy given in the proof of Theorem 2.1. Since the robber cannot move past the cop and off the tree onto another vertex of $G_U^{\mathcal{T}}$, the cop wins on a leaf of T_i if not sooner. Thus $G_U^{\mathcal{T}} \in DCW_1$. □

Combining the results of Theorem 2.1, Observation 2.2, and Corollary 2.3, we see that if $G \in \mathcal{T} \cup \mathcal{G}_U \cup \mathcal{G}_U^{\mathcal{T}}$ then G is decoy-copwin. The next theorem shows that, in fact, we have a characterization of decoy-copwin graphs for this variant.

Theorem 2.4. $DCW_1 = \mathcal{T} \cup \mathcal{G}_U \cup \mathcal{G}_U^{\mathcal{T}}$.

Proof. We need only show that $DCW_1 \subseteq \mathcal{T} \cup \mathcal{G}_U \cup \mathcal{G}_U^{\mathcal{T}}$. Let $G \in DCW_1$ be a graph such that $G \notin \mathcal{T} \cup \mathcal{G}_U \cup \mathcal{G}_U^{\mathcal{T}}$. Let G' be the graph obtained by iteratively pruning vertices of degree 1 from G until no such vertices remain; note that G' is nonempty since $G \notin \mathcal{T}$. Conceptually, this captures the notion of removing trees which have a vertex identified with some vertex of a nontree G'. Note that any vertex v of G which is not in G' must be in a tree T_v attached to some $v \in V(G')$. Further, if there was a universal vertex in G', then $G \in \mathcal{G}_U \cup \mathcal{G}_U^{\mathcal{T}}$, contradicting the choice of G. Thus, regardless of the cop's initial move, the robber can choose to begin on a vertex in a cycle C of G' which is not adjacent to the cop's position. On subsequent turns, the robber moves only on C and throws his decoy on C, a strategy that prevents the cop from being able to guarantee a win. □

Aside. If the robber were to be permitted to throw a decoy on his first move (i.e. choose initial positions for himself and the decoy at distance at most 2 apart during the first round), then we would have $DCW_1 = \mathcal{T}$.

Given the characterization of decoy-copwin graphs under this variant presented in Theorem 2.4, Observation 2.5 is easy to see. This is because if H is a retract of a graph $G \in \mathcal{T}$, then $H \in \mathcal{T}$, and similarly for retracts of graphs in $\mathcal{G}_U$ and $\mathcal{G}_U^{\mathcal{T}}$.

Observation 2.5. *Let G be a graph and let H be a retract of G. If $G \in DCW_1$, then $H \in DCW_1$.*

We might then wonder if the graphs in DCW_1 form a variety. Consider, however, the graph $P_2 \boxtimes P_4$. Since $P_2 \boxtimes P_4$ is not decoy-copwin under variant 1, this class of graphs is not closed under finite products and thus not a variety.

Although the following result is a corollary of Theorem 2.4, we choose to highlight a classic proof technique in the Cops and Robber literature that we will use again in the proof of Theorem 3.4.

Corollary 2.6. *Let u be a leaf of a graph G. Then $G \in DCW_1$ if and only if $G - u \in DCW_1$.*

Proof. Since $G - u$ is a retract of G, necessity is given by Observation 2.5. For sufficiency, let $G - u \in DCW_1$, and let ϕ be the retraction map from G to $G - u$ with $\phi(u) = v$, say. Our argument here will be a classical one, originating all the way back to [1, 11]. The cop restricts himself to playing on $G - u$, and plays his winning strategy there against the image of the robber side's position under ϕ; i.e. if the robber side has a player on vertex u, the cop will consider this player to be on $\phi(u) = v$. In this way, the cop captures the image under ϕ of the robber (henceforth, simply the robber's image) on $G-u$. The cop side can be thought to be playing "parallel" games on G and $G - u$.

If the image of the robber is captured on a vertex of G other than u, then the robber's image and his actual position coincide and the cop has won on G. So suppose the cop has captured the image of one of the robber and the decoy on v. If that player is actually located on v, then either the cop wins or else it is the decoy that has been captured and the game continues. So suppose that player is actually located on u. The issue now is one of information: the cop does not know whose image he has captured. Since neither the robber nor the decoy is on v, it must be the case that one of them is on u and the other is on $w \simeq v$. (It is easy to see this by considering the robber side's previous move.) The cop moves to w. If it is the robber that is located there, the game is over. Otherwise the cop moves back to v (and captures the robber if the robber has moved there from u) or else captures the robber on u on the next cop turn. □

Intuitively, what's happening with this variant is that the information available to the cop side is so limited that the cop can only force a win on a leaf. The only graphs on which the cop can guarantee a win besides trees are those on which he can win before the robber is allowed to start throwing the decoy. Of course those are precisely the graphs in $\mathcal{G}_U \cup \mathcal{G}_U^{\mathcal{T}}$. Taken altogether, these are the graphs as in Theorem 2.4.

3. Variant 2

In the second variant of the Decoy Cops and Robber game, *the robber and the decoy are indistinguishable to the cops unless at least one of the robber and the decoy is cornered.* Note that once the robber is cornered, the cops can see the robber's next move. This ensures that a cornered robber cannot subsequently escape. (This advantage to the cop side does not apply if it is the decoy who is cornered; i.e. if the decoy is cornered, the cop cannot see the robber's next move.) We will denote the set of graphs on which a single cop can guarantee a win under the rules of this variant DCW_2 (for **D**ecoy-**C**op**w**in under the **2**nd variant). *An easy observation:* $DCW_1 \subseteq DCW_2$.

Example 3.1 (A decoy-robberwin graph under variant 2)**.** *Consider the graph shown in Figure 1. We give a winning strategy for the robber side. The robber begins on R, on the "opposite side" of the graph from the cop. If the cop never moves to C, the robber wins by passing forever. If the* *cop moves to C, the robber can throw the decoy to D and pass. Since the cop's position does not corner either of the robber's position R or the decoy's position D, the cop cannot distinguish between them and cannot guarantee the capture of the robber on the cop's next move. The robber can prolong the game indefinitely by continuing to move on the 3-cycle induced by vertices D, C, and R in this way. This graph is therefore decoy-robberwin.*

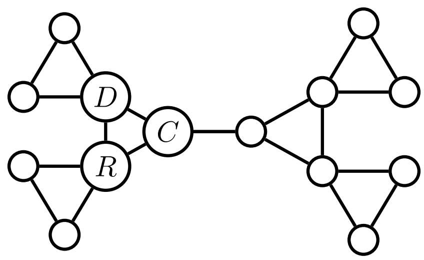

FIGURE 1. A Decoy-Robberwin Graph (Variant 2)

Our main result in the section is Theorem 3.6 which is our characterization of decoy-copwin graphs under this variant. The construction on which our characterization is based is the Cloister Construction (Theorem 3.4). We proceed with a lemma.

Lemma 3.2. *If a graph G has a near-universal vertex v and, for $u \not\sim v$, u is a corner, then $G \in DCW_2$.*

Proof. Let $v \in V(G)$ be a near-universal vertex. Let $u \not\sim v$ be a corner with corresponding dominating vertex d. The cop begins on v, and the robber must begin on u in order to avoid capture on the cop's first move. The cop moves to d and wins on his next turn. Therefore, $G \in DCW_2$. □

Lemma 3.2 can be extended to any graph G with a vertex v such that $V(G) \setminus \{v\}$ can be partitioned into $N(v)$ and a set of corners in which every corner u_i is dominated by a vertex $v_i \in N(v)$; this is Theorem 3.3. We define such a graph to be a *locally-universal* graph, with locally-universal vertex v. The class of locally-universal graphs will be denoted $\mathcal{G}_{LU}$.

Theorem 3.3. *Let $G \in \mathcal{G}_{LU}$. Then $G \in DCW_2$.*

The graphs in $\mathcal{G}_{LU}$ are a specific class to which the Cloister Construction can be applied, where $H \in \mathcal{G}_U \subseteq DCW_1$ as in the statement of the theorem below. Thus Theorem 3.3 is proven as Case 1 of the proof of Theorem 3.4.

Theorem 3.4 (Cloister Construction)**.** *Suppose the vertex set of a graph G can be partitioned into two sets ($V(H)$ and $V(K)$) on which we induce subgraphs $H \in DCW_1$ and K. Let $\phi : V(G) \to V(H)$ be a retract such that, in G, every $u_i \in V(K)$ is cornered by $\phi(u_i) \in V(H)$. Then $G \in DCW_2$.*

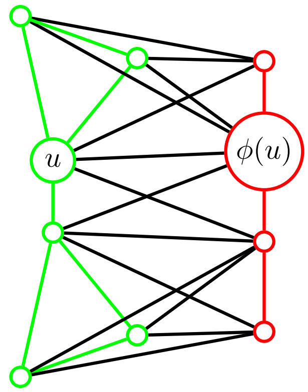

FIGURE 2. An illustration of the Cloister Construction with H indicated in red and K in green

Proof. The main idea of our proof is similar to that of the proof of Corollary 2.6, namely that the cops play two games in parallel - the second variant on G and the first variant on H. On H, they play their winning strategy for the first variant of the game against the image of the robber side under the retraction map ϕ. By doing so, the cops capture the image of the robber on H and can then capture the actual robber on G in at most one additional move. The key here of course is that, by construction, the information available to the cops on G under the second variant is sufficient to allow them to play their parallel game on H under the first variant; i.e. a cop on G can tell when he has captured the image of the robber on H. We proceed with this argument below in detail in three cases.

CASE 1: Suppose first that $H \in \mathcal{G}_U$.
Note that G is a locally-universal graph, with locally-universal vertex v say. The cop begins on v, the universal vertex of H. If the robber begins on $w \simeq v$, he will be caught on the cop's first move. So suppose the robber chooses a vertex u_i such that $u_i \not\simeq v$. We know that u_i is cornered by a vertex $v_i = \phi(u_i)$ and $v \sim v_i$. The cop will move to v_i on his first turn and corner the robber. Therefore $G \in DCW_2$.

CASE 2: Suppose $H \in \mathcal{T}$.
Our goal here is to show that the cop is able to follow his winning strategy for graphs in $\mathcal{T} \subseteq DCW_1$ on H and capture $\phi(R)$, where R is the robber's position on G. As in the proof of Theorem 2.1, we will consider the cop's starting position on H to be the root of the tree. In a tree, the cop's position can be thought of as the root of two subtrees, one of which contains the robber; call this subtree the robber's subtree. (In the original game, the cop's strategy is to move so as to decrease the robber's subtree until the robber is caught on a leaf if not sooner.) Here, we show that the cop uses his strategy for trees in DCW_1 on H to restrict the robber's image (under ϕ) to an increasingly small region of the graph G until, in a finite number

of moves, the cop captures R on H or else captures $\phi(R)$ on H, thereby capturing R on G in at most two additional moves.

We know that, on a tree, the robber is unable to move past the cop and into the other subtree without being caught. Here, we also wish to talk about the robber being unable to move past the cop. In this context, moving past the cop means that the robber moves from a vertex R such that $\phi(R)$ is on the robber's subtree of H to a vertex $R^{'}$ such that $\phi(R^{'})$ is on the cop's subtree of H. If the robber is able to bypass the cop, the cop's strategy as given in the previous paragraph will fail. Thus we need to show that there is no edge e that allows the robber to avoid being cornered by moving past the cop. Without loss of generality, we assume the robber's position R is incident with e. Let the other endpoint of e be R'.

(a) If both endpoints of e are on H, we have a cycle in H, contradicting the assumption that $H \in \mathcal{T}$.

(b) If $R \in V(H)$ and $R' \in V(K)$, then R' has a dominating vertex $\phi(R') = v \in V(H)$. Since $N[R'] \subseteq N[v]$, $v \sim R$, creating a cycle in H. (This cycle contains the unique path in H between v and the cop's position C, the unique path in H between C and R, and the edge vR. Note that the unique path in H between C and R does not contain v since v and R are in different subtrees separated by C.)

(c) If $R \in V(K)$ and $R' \in V(H)$, R has a dominating vertex $v_r \in V(H)$. Since $N[R] \subseteq N[v_r]$, $v_r \simeq R'$, creating a cycle in H. (This cycle contains the unique path in H between v_r and C, the unique path in H between C and R', and the edge v_rR'.)

(d) Otherwise, $R, R' \in V(K)$. Both R and R' have dominating vertices, v_r and v respectively, such that $v, v_r \in V(H)$. Since $N[R] \subseteq N[v_r]$, $v_r \simeq R'$. Since $N[R'] \subseteq N[v]$, $v \simeq v_r$. As illustrated in Figure 3, this creates a cycle in H. (This cycle contains the unique path in H between v and C, the unique path in H between C and v_r, and the edge vv_r.)

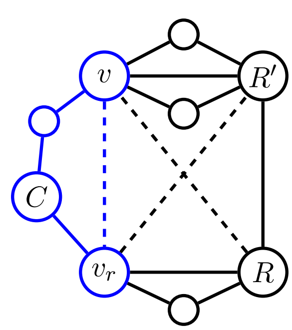

FIGURE 3. Illustration of Case 2(d) in the proof of Theorem 3.4; the problematic cycle is shown in blue

Since we have a contradiction in all cases, such an e does not exist and the robber cannot bypass the cop on G and avoid capture. Since the robber cannot bypass the cop's position, the cop will continue to play his DCW_1 tree strategy on H based on (R or) $\phi(R)$. This guarantees that the robber will be captured on H or cornered on K in a finite number of turns. Therefore, $G \in DCW_2$.

CASE 3: Finally, suppose $H \in \mathcal{G}_U^{\mathcal{T}}$.
The cop begins on the universal vertex v of the universal subgraph G_U of H. If the robber chooses a vertex adjacent to v, the cop will catch the robber on his first move. If the robber begins on $w \in V(K)$ that is mapped to one of v's neighbours under ϕ, the cop can move to $\phi(w)$ to corner the robber on his first move. Otherwise, to avoid capture on the cop's first or second move, the robber must choose a vertex that is on or is mapped to one of the finite trees attached to G_U. Without loss of generality, we designate neighbours of v as the roots of each finite tree. The cop moves to the root of the tree that corresponds to the robber's position, or the image of the robber's position under ϕ. The cop then follows the strategy outlined in Case 2 above and wins. Therefore, $G \in DCW_2$. □

We now show that the condition given by the Cloister Construction is, in fact, sufficient for graphs in DCW_2, thereby characterizing decoy-copwin graphs for this variant. We begin by defining the problematic 3-cycles from Figure 1 formally below.

A *tent* is a cycle of length 3 in a graph G in which no vertex on the cycle is dominated by another vertex on the cycle. The cycle with vertices D, C, and R in Figure 1 is a tent.

Suppose $\Delta \subseteq G$ is a tent or an induced cycle of length $n \geq 4$; we shall call such a subgraph of G a Δ-subgraph. If $v \in V(G) \setminus V(\Delta)$, then v is said to be an off-Δ vertex. If $m \geq 1$ of the n vertices of a Δ-subgraph of G are cornered by the same off-Δ vertex in G, we say that the Δ-subgraph is *externally dominated*.

Our motivation for Lemma 3.5 is to establish that, if we induce a subgraph on all the noncorners in G (as in the proof of Theorem 3.6), we will be able to show that all the vertices of all the cycles in this subgraph share a common neighbour.

Lemma 3.5. *Let $G \in DCW_2$ with at least one Δ-subgraph. Let $\mathcal{S}$ be any set of induced cycles of length at least 4 and tents in G (none of which are externally dominated). Then all the vertices of $\mathcal{S}$ have a common neighbour.*

Proof. Suppose not. There are several cases.

CASE 1: Suppose first that G has at least two Δ-subgraphs, $\Delta_i, \Delta_j \in \mathcal{S}$, whose vertices do not all share a common neighbour. (Recall that, for each of these Δ_k, $k \in \{i, j\}$, there is no vertex $v_k \in V(G)$ which corners at least one of the n_k vertices of Δ_k.) We will suppose the cop starts on an arbitrary vertex, w_j say, of Δ_j or on an off-Δ_j vertex, v_j say (which, by definition, dominates no corners of Δ_j). There are two subcases according to whether Δ_j is a tent or else an induced cycle of length at least 4.

(1) Suppose first that Δ_j is a tent.

(a) And suppose the cop starts on w_j of Δ_j. The robber can start on Δ_i, choosing u_i on Δ_i such that $u_i \not\sim w_j$. We know that such a u_i exists since otherwise w_j is a common neighbour of the vertices of Δ_i and Δ_j. Since the cop is not immediately adjacent to the robber's position, the robber is able to move/pass safely on Δ_i and also to deploy a decoy on another vertex of Δ_i not occupied by the cop.

Since no vertex of Δ_i dominates another and since Δ_i does not have a vertex cornered by some off-Δ_i vertex, the cop cannot distinguish between the robber and the decoy as long as they choose to remain on Δ_i, and may chase the robber indefinitely in this way; i.e. $G \notin DCW_2$, a contradiction.

(b) Otherwise the cop starts on v_j, an off-Δ_j vertex. By construction, there exists at least one vertex of Δ_i or Δ_j, say u_k on Δ_k, $k \in \{i, j\}$, that is not adjacent to v_j. As in (a) above, the robber may evade the cop on Δ_k indefinitely; i.e. $G \notin DCW_2$, a contradiction.

(2) Otherwise, Δ_j is an induced cycle (of length at least 4).

(a) Suppose first that the cop starts on an off-Δ_j vertex, v_j say.

(i) If v_j is adjacent to all the vertices of Δ_j then, as in Case (1)(a), the robber can choose to start on a vertex, u_i say, on Δ_i such that $u_i \not\sim v_j$ and can evade the cop indefinitely. We know that such a u_i exists since otherwise v_j is a common neighbour of the vertices of Δ_i and Δ_j.

(ii) Otherwise there exists at least one vertex, say y, of Δ_j such that $y \not\sim v_j$. Then the neighbours of y on Δ_j, x and z say, are not cornered by v_j. (Also by assumption since Δ_j is not externally dominated.) So the robber can safely start on y. If the cop remains on v_j, the robber and decoy can move among x, y, z without the cop being able to guarantee capture. If the cop moves to one of x and z, the robber can throw the decoy to y and move to the other of x and z to avoid capture. (If the cop moves to another off-Δ_j vertex, the argument is analogous. If the cop moves to a vertex of Δ_j other than x or z, the robber can remain at distance at least 2 from the cop.) Since Δ_j has no corners, as in Case (1), the robber can evade the cop in this way indefinitely.

(b) Suppose instead that the cop starts on a vertex, w_j say, of Δ_j. Since Δ_j is an induced cycle of length at least 4, there exists at least one other vertex of Δ_j that is not adjacent to w_j. The robber can choose to begin on this vertex and evade the cop indefinitely as in Case (2)(a)(i).

Thus if Δ_j is an induced cycle (of length at least 4), then in all cases the robber is able to evade the cop indefinitely, and $G \notin DCW_2$, a contradiction.

Case 2: Otherwise, for every $\Delta_i, \Delta_j \in \mathcal{S}$, all the vertices of Δ_i and Δ_j share a common neighbour. So it must be the case that there exist at least three Δ-subgraphs without a common neighbour, namely $\Delta_i, \Delta_j, \Delta_k \in \mathcal{S}$ (but all the vertices of any pair of them have a common neighbour).

Even if the cop begins on a common neighbour v of two of these subgraphs, there exists a vertex on the third, Δ_k say, which is not adjacent to the cop's position; otherwise v is a common neighbour of (all the vertices on) all three subgraphs. As in Case 1(1)(a), without an off-Δ_k vertex that corners at least one vertex of Δ_k, the robber can evade the cop indefinitely, and thus $G \notin DCW_2$, a contradiction. □

Theorem 3.6 (Cloister Characterization)**.** *For a graph G, $G \in DCW_2$ if and only if $V(G)$ can be partitioned into sets $V(H)$ and $V(K)$ on which we induce subgraphs $H \in DCW_1$ and K and, in G, every $u_i \in V(K)$ is cornered by $\phi(u_i) \in V(H)$, with $\phi : V(G) \to V(H)$ a retract.*

Proof. Sufficiency is given by Theorem 3.4. Conversely, for $G \in DCW_2$, we must satisfy two conditions:

(1) G has induced subgraphs $H \in DCW_1$ and $K = G - V(H)$, with $\{V(H), V(K)\}$ a partition of $V(G)$; and
(2) for all vertices $u_i \in V(K)$, u_i is cornered by a vertex $\phi(u_i) = v_i \in V(H)$, with $\phi : V(G) \to V(H)$ a retract.

Let $G \in DCW_2$. First, we need to establish the partition of $V(G)$ as in our first condition. To satisfy condition (2), all vertices in $V(K)$ must be corners, and so $V(H)$ must contain all the noncorners in G. We do not require, however, that every vertex in $V(H)$ be a noncorner. So we start by inducing a subgraph H' on all the noncorners of G. If G contains a subgraph isomorphic to C_3 and that subgraph is not a tent, then at least one vertex of C_3 is a corner, and thus at most two vertices of C_3 are in H'. From Lemma 3.5, we know that for any arbitrary Δ-subgraph of G, either (a) Δ is one of a group of induced cycles (of length at least 4) and tents all of whose vertices share at least one common neighbour on or off cycle (i.e. a collection of cycles satisfying the conditions to apply Lemma 3.5), or (b) at least one vertex of Δ is cornered by an off-Δ vertex, v say. In the latter case, at most $n-1$ vertices of the Δ-subgraph are noncorners. The key is that these noncorners do not themselves form a cycle in H'. (We are not concerned if they form a cycle together with some corners.) Thus, in either case, all (the vertices of all the) cycles in H' will share at least one common neighbour. We claim that H' is connected (and establish this claim at the end of the proof), and so we relabel H' as H. Define another subgraph $K = G - V(H)$ in which any $u \in V(K)$ is a corner because all noncorners of G are in H.

To satisfy our second condition, we must first guarantee that every $u_i \in V(K)$ is cornered by some $v_i \in V(H)$. Let $u_i \in V(K)$ be arbitrary; since all noncorners of G are in H, u_i must be cornered by some vertex w_{i_1}. If $w_{i_1} \in V(H)$, u_i is cornered by a vertex of H. Otherwise, w_{i_1} must itself be cornered by w_{i_2}, and since (by the definition of corner) $N[u_i] \subseteq N[w_{i_1}]$ and $N[w_{i_1}] \subseteq N[w_{i_2}]$, transitively $N[u_i] \subseteq N[w_{i_2}]$. Therefore, w_{i_2} corners u_i. If $w_{i_2} \in V(H)$, u_i is cornered by a vertex of H. Otherwise, we repeat the step above with w_{i_3} that corners w_{i_2}. We will eventually reach w_{i_m} such that w_{i_m} corners all previous vertices $u_i, w_{i_1}, \dots, w_{i_{m-1}}$. If $w_{i_m} \in V(H)$, we stop this process. Otherwise (when H is empty), we will exhaust our vertices, leaving a vertex w_{i_m} that corners all the other vertices of G, making it a universal vertex. Therefore, if $G \notin \mathcal{G}_U$ then, for arbitrary $u_i \in V(K)$, u_i is cornered by some $v_i\ (= w_{i_m}) \in V(H)$.

We now define a retraction $\phi : V(G) \to V(H)$ such that for $u_i \in V(K)$ cornered by $v_i \in V(H)$, $\phi(u_i) = v_i$. It remains to be shown that ϕ is a homomorphism. Let $u_i, u_j \in V(K)$. We know u_i is cornered by $v_i \in V(H)$. If $u_i \simeq u_j$, then $v_i \simeq u_j$. Similarly, we know u_j is cornered by $v_j \in V(H)$, and so $v_j \simeq v_i$. Thus we have $\phi(u_i) = v_i \simeq v_j = \phi(u_j)$ for $u_i \simeq u_j$. Therefore, $\phi : V(G) \to V(H)$ is a retraction map from G to H as required by condition (2).

Finally, we complete our proof by establishing our claim that $(H' =)\ H$ is connected. Since G is connected, for any pair $u, v \in V(H) \neq \emptyset$, there exists a path $P_{u,v}$ from u to v in G. Clearly, $\phi(P_{u,v})$ defines a walk from u to v in H. And so H is also connected. □

Aside. If the robber were permitted to throw a decoy on his first move, the Cloister Characterization would continue to hold but with $DCW_1 = \mathcal{T}$.

Open Problem. Perhaps we could build a hierarchy of graph classes by defining an ith variant of the Decoy Cops and Robber game in such a way that DCW_i is DCW_{i-1} plus a layer of corners as we see in our Cloister Characterization for $i = 2$.

We conclude our study of decoy-copwin graphs under variant 2 by applying Theorem 3.6 to obtain some specific results for strong products.

Corollary 3.7. *The products* $P_4 \boxtimes P_4$, $P_4 \boxtimes P_5$, $P_5 \boxtimes P_5$, $P_2 \boxtimes P_n$, $P_3 \boxtimes P_n \in DCW_2$, $n \in \mathbb{N}$.

Proof. For a strong grid $P_m \boxtimes P_n$ to be in DCW_2, the subgraph induced on the strong grid's noncorners must be in DCW_1. If the resulting subgraph is also a strong grid $P_{m'} \boxtimes P_{n'}$, we know that $m', n' \leq 3$. □

The proofs of Corollaries 3.8 and 3.9 follow similarly from Theorem 3.6 and are thus omitted.

Corollary 3.8. *For* $G_1 \in \mathcal{G}_U$ *and* $G_2 \in DCW_2$, $G_1 \boxtimes G_2 \in DCW_2$.

Corollary 3.9. *For* $G_1, G_2 \in \mathcal{G}_{LU}$, $G_1 \boxtimes G_2 \in DCW_2$.

Acknowledgements

The authors are grateful to D. A. Pike whose careful reading of the manuscript and helpful comments improved the paper, and to the anonymous referees whose suggestions, insights, and questions (such as the open problem above in Section 3) also improved the paper.

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