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A FINITE-BOUND PARTITION IDENTITY GENERALIZING A PROBLEM BY ANDREWS AND DEUTSCH

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Abstract. We introduce a finite-bound extension of a partition identity which was originally proposed as a problem by Andrews and Deutsch in 2016, and given a generalized form in 2018 by Smoot and Yang. We also give a simple bijective extension of the original proof.

1. Background

In 2016 Andrews and Deutsch proposed the following problem:

> American Mathematical Monthly Problem 11908.
>
> Let n, k be nonnegative integers. Show that the number of partitions of n having k even parts is the same as the number of partitions of n in which the largest repeated part is k (defined to be 0 if the parts are all distinct).

As an example, they list the three partitions of 7 with two even parts:

$$4+2+1, \qquad 3+2+2, \qquad 2+2+1+1+1,$$

and the three partitions of 7 in which the largest repeated part is 2:

$$3+2+2, \qquad 2+2+2+1, \qquad 2+2+1+1+1.$$

This problem was given different proofs in [2, Solutions I and II], both of which are reproduced below. The first was a manipulation of generating functions by Madhyastha [2, Solutions I and editorial comment]. This sort of manipulation is well-known and constitutes a useful set of techniques for partition theory.

On the other hand, it is also interesting to investigate how these sorts of partition identities manifest themselves explicitly, i.e., through direct bijections. For example, a delightful proof of this problem by direct construction was given in [2, Solution II, Editorial comment], [5, Theorem 23], which we

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reproduce below. Indeed, such a direct approach can easily be extended to a generalization of the original result:

Theorem 1.1 (N. Smoot, M. Yang)**.** *Let n, k, d be nonnegative integers. The number of partitions of n with k parts divisible by d is the same as the number of partitions of n in which k is the largest part to occur at least d times.*

Another more recent set of investigations in partition identities is through *finite-bound* proofs. That is, for either side of a given partition identity, we place restrictions on the size of the parts involved, usually keeping the size below an arbitrary bound m. When m becomes arbitrarily large, the finite bound result reduces to the original identity.

For example, Euler's famous identity, in which the number of partitions into odd parts equals the number of partitions into distinct parts, has been given a finite bound extension by Andrews in [1]. Of course, Euler's identity has a natural generalization to Glaisher's theorem; this can also be given a finite bound extension, as was shown in [6]. From here, various other finite-bound equinumerosity results are possible, e.g., [3], [7, Theorem 4.1].

In this paper we give a finite-bound extension of Theorem 1.1, as well as proofs which correspond to finite-bound extensions of the original proofs. In Section 2.1 we give the generating function identity which implies the original problem, together with the bijective proof of the more general Theorem 1.1. In Section 2.2, Theorem 2.3 we give our finite bound theorem. A generating function proof is given in Section 2.3, while a direct bijection is given in Section 2.4. We give examples of the bijection in Section 2.5. Finally, in Section 3 we briefly discuss possible directions for future work and highlight the advantages of these techniques in the study of partition identities.

2. A finite-bound generalization of Theorem 1.1

2.1. **Original Proofs.**

The counting functions for each class of partitions stated in the original problem have a natural generating function, and Madhyastha gave a proof of the problem through manipulation of their generating functions [2, Solution I]. Madhyastha's proof may be given a natural extension to Theorem 1.1, showing that

$$\frac{q^{dk}}{(q^d;q^d)_k}\frac{(q^d;q^d)_\infty}{(q;q)_\infty} = \frac{q^{dk}}{(q;q)_k}\frac{(q^{d(k+1)};q^d)_\infty}{(q^{k+1};q)_\infty}. \tag{2.1}$$

Here, and in the first proof of Theorem 2.3, q is a formal variable; for any indeterminate a and nonnegative integer k (we also allow $k = \infty$) the q-shifted factorial is defined as $(a;q)_k = \prod_{0\le j<k}(1-aq^j)$ (empty products being 1).

The bijective proof of the more generalized Theorem 1.1 may be given as follows

Proof of Theorem 1.1. Let λ be a partition of n with k parts divisible by d. Write

$$\lambda = \mu \cup o,$$

in which μ is the subpartition containing the k parts divisible by d, and o contains the remaining parts.

We map μ to its conjugate, which we denote μ^*. Notice that μ^* contains k as its largest part, and that indeed k must repeat dq_k times for some integer $q_k > 0$.

Next we map o, which contains no parts divisible by d, to its corresponding partition δ in which every part may only repeat up to $d-1$ times—say, using Glaisher's map [4]. Both of these actions constitute bijections. Our new partition, then, is

$$\kappa = \mu^* \cup \delta.$$

Notice that any part i of κ which is less than or equal to k will occur $dq_i + r_i$ times, in which dq_i comes from μ^* and q_i is a nonnegative integer (which is possibly 0 except when $i = k$), and r_i, with $0 \leq r_i \leq d-1$ comes from δ. Thus parts less than k may repeat any number of times. On the other hand, any part greater than k must only have a contribution from δ, and must therefore occur strictly less than d times.

This mapping is well-defined, as well as easily reversable: for any partition κ in which k is the largest part to occur at least d times, take all parts i which occur $dq_i + r_i$ times with $q_i \geq 0$ (which is positive for $i = k$), and define the subpartition μ^* by collecting i to occur dq_i times. The largest part will be k. What is left will be parts i which repeat r_i times for $0 \leq r_i < d$. Collect these terms into δ.

We map μ^* to its conjugate μ, which will contain k parts, all divisible by d. Also we map δ to o, which contains no multiples of d. Our new partition is $\lambda = \mu \cup o$. □

As an example, consider $n = 10$, $d = 3$, $k = 2$. We have five partitions of 10 with 2 parts divisible by 3, which we give here in frequency notation: $6^1 3^1 1^1$, $4^1 3^2$, $3^2 2^2$, $3^2 2^1 1^2$, $3^2 1^4$.

Consider first the subpartitions μ. Our first partition has $\mu : 6^1 3^1$. Conjugating gives us $\mu^* : 2^3 1^3$. The other partitions have $\mu : 3^2$, which conjugates to $\mu^* : 2^3$.

For the first partition, we have $o : 1^1$, which maps to itself, $\delta : 1^1$, via Glaisher's bijection. Combining μ^* and δ, our new partition is $2^3 1^4$.

For the remaining four partitions, o constitutes a partition of 4 (excepting the partition $3^1 1^1$, which contains a part divisible by 3). We map these subpartitions, $4^1, 2^2, 2^1 1^2, 1^4$ via Glaisher's map to $4^1, 2^2, 2^1 1^2, 3^1 1^1$, respectively. Combining with $\mu^* : 2^3$, we have our new partitions. In summary, we have the bijection

$$\begin{aligned} 6^1 3^1 1^1 &\longmapsto 2^3 1^4, \\ 4^1 3^2 &\longmapsto 4^1 2^3, \\ 3^2 2^2 &\longmapsto 2^5, \\ 3^2 2^1 1^2 &\longmapsto 2^4 1^2, \\ 3^2 1^4 &\longmapsto 3^1 2^3 1^1. \end{aligned}$$

2.2. **New Result.** The proof of Theorem 1.1 is especially satisfying, but it is not obvious that anything more can be done with it. However, we have found that Theorem 1.1 may be extended to a finite-bound result.

Definition 2.1. *Let $A(n,k,d,m)$ be the set of integer partitions of n in which exactly k parts are divisible by d, and all other parts are strictly less than md.*

Definition 2.2. *Let $B(n,k,d,m)$ be the set of integer partitions of n such that:*

- *For $m<k$, the largest part is kd, and all parts greater than md are divisible by d;*
- *For $m \geq k$, the part k occurs at least d times, none of the parts exceed md, and any part i such that $k<i\leq m$ occurs less than d times.*

With these definitions, we have the following:

Theorem 2.3. *For $n,k,d,m \in \mathbb{Z}_{\geq 1}$, $|A(n,k,d,m)| = |B(n,k,d,m)|$.*

Of course, for $m\to\infty$ (indeed, for $m>n$), this result reduces to Theorem 1.1. We give two different proofs of Theorem 2.3.

2.3. **Generating Function Proof.** First, we demonstrate the following identity, a q-series extension of (2.1) used in Madhyastha's proof:

$$\frac{q^{dk}}{(q^d;q^d)_k}\frac{(q^d;q^d)_m}{(q;q)_{dm}} = \begin{cases} \frac{q^{kd}}{(q^{d(m+1)};q^d)_{k-m}}\frac{1}{(q;q)_{md}}, & \text{if } m<k, \\ \frac{q^{kd}}{(q;q)_k}\frac{(q^{d(k+1)};q^d)_{m-k}}{(q^{k+1};q)_{m-k}}\frac{1}{(q^{m+1};q)_{md-m}}, & \text{if } m\geq k. \end{cases} \tag{2.2}$$

We first interpret (2.2). Examining the left-hand side,

$$\frac{q^{dk}}{(q^d;q^d)_k}\frac{(q^d;q^d)_m}{(q;q)_{dm}}, \tag{2.3}$$

we notice that the first factor of (2.3) enumerates partitions into exactly k parts divisible by d. The second factor of (2.3) enumerates partitions in which *no* parts are divisible by d, but moreover one in which all parts are bounded above by md; indeed, because the part md is specifically excluded, the parts must be *strictly less* than md. Thus the left-hand side of (2.2) enumerates partitions of $A(n,k,d,m)$.

Examining the right-hand side of (2.2) in the case that $m < k$, we note that the very last factor, $1/(q;q)_{md}$, enumerates partitions into parts of size up to md, while $1/(q^{d(m+1)};q^d)_{k-m}$ enumerates partitions of size larger than md but smaller than kd, in which the parts are divisible by d. Finally, the factor of q^{kd} confirms that kd does in fact occur. Thus, we have partitions of type $B(n,k,d,m)$ for $m < k$ enumerated by the right-hand side of (2.2).

For $m \geq k$, $1/(q;q)_k$ enumerates partitions into parts of size up to and including k with no additional restrictions. To insure that k occurs at least d times, we have the factor $q^{kd} = q^{k+k+\cdots+k}$. On the other hand, the factor $1/(q^{m+1};q)_{md-m}$ enumerates partitions into parts of size strictly greater than k and less than or equal to md, but with no other restrictions. Finally, the factor

$$\frac{(q^{d(k+1)};q^d)_{m-k}}{(q^{k+1};q)_{m-k}}$$

consists of factors of the form

$$\frac{1-q^{dj}}{1-q^j} = 1 + q^j + q^{2j} + \cdots + q^{(d-1)j},$$

for $k < j \leq m$. Thus, the parts strictly larger than k and less than or equal to m must occur fewer than d times. Thus the right-hand side of (2.2) enumerates $B(n,k,d,m)$ in both cases.

Proof. We have two cases to prove, but in both cases, we begin with (2.3). First we suppose that $m < k$. Then we have

$$(q^d;q^d)_k = (q^d;q^d)_m(q^{d(m+1)};q^d)_{k-m}.$$

Thus substituting this for $(q^d;q^d)_k$ in (2.3), we have

$$\begin{aligned}\frac{q^{dk}}{(q^d;q^d)_k}\frac{(q^d;q^d)_m}{(q;q)_{dm}} &= \frac{q^{dk}}{(q^d;q^d)_m(q^{d(m+1)};q^d)_{k-m}}\frac{(q^d;q^d)_m}{(q;q)_{dm}}\\ &= \frac{q^{dk}}{(q^{d(m+1)};q^d)_{k-m}}\frac{1}{(q;q)_{dm}}.\end{aligned}$$

Similarly, for $m \geq k$,

$$(q^d;q^d)_m = (q^d;q^d)_k(q^{d(k+1)};q^d)_{m-k}.$$

Substituting into (2.3),

$$\frac{q^{dk}}{(q^d;q^d)_k}\frac{(q^d;q^d)_m}{(q;q)_{dm}} = \frac{q^{dk}}{(q^d;q^d)_k}\frac{(q^d;q^d)_k(q^{d(k+1)};q^d)_{m-k}}{(q;q)_{dm}} \tag{2.4}$$

$$= q^{dk}\frac{(q^{d(k+1)};q^d)_{m-k}}{(q;q)_{dm}}. \tag{2.5}$$

Next, we note that

$$(q;q)_{dm} = (q;q)_k(q^{(k+1)};q)_{m-k}(q^{(m+1)};q)_{md-m}.$$

With this substitution into (2.4)–(2.5) we have

$$q^{dk}\frac{(q^{d(k+1)};q^d)_{m-k}}{(q;q)_{dm}} = q^{dk}\frac{(q^{d(k+1)};q^d)_{m-k}}{(q;q)_k(q^{(k+1)};q)_{m-k}(q^{(m+1)};q)_{md-m}}$$
$$= \frac{q^{dk}}{(q;q)_k}\frac{(q^{d(k+1)};q^d)_{m-k}}{(q^{(k+1)};q)_{m-k}}\frac{1}{(q^{(m+1)};q)_{md-m}}. \qquad \square$$

2.4. **Bijective Proof.** We give an extension of the direct constructive proof that we already gave for Theorem 1.1.

Proof. Let $\lambda \in A(n,k,d,m)$. We write

$$\lambda = \mu \cup o,$$

in which μ contains the k parts divisible by d, and o contains the remaining parts.

We begin by considering the subpartition μ. Here we map μ to its conjugate, μ^*, in which k occurs at least d times (indeed, the number of occurrences of k is a positive multiple of d). We can therefore write μ^* as

$$\mu^* := \sum_{i=1}^{k} i(dq_i),$$

in which we interpret the parts as i, and the number of occurrences as dq_i for some $q_i \in \mathbb{Z}_{\geq 0}$ (with $q_k > 0$).

We will in turn break $\mu^* = \mu_0^* \cup \epsilon$ into two smaller subpartitions and express each in frequency notation:

$$\mu_0^* : \prod_{i=1}^{\min(m,k)} i^{dq_i}, \qquad \epsilon : \prod_{m<i\leq k} (di)^{q_i}.$$

Notice that for ϵ we reinterpret the terms: the parts are now di as i ranges from 1 to m. Notice that for $m < k$, the largest part from either of these subpartitions will be dk, and it will come from ϵ. Note also that ϵ is empty for $m \geq k$.

Next, we consider the subpartition o. Notice that these parts are bounded strictly above by md, and that none of them are divisible by d. Write o first with frequency notation

$$o := \prod_{\substack{1\leq j<md,\\ d\nmid j}} j^{N_j},$$

in which j is a given part, and $N_j \geq 0$ the number of occurrences of that part. If we convert to summation convention, we have

$$o := \sum_{\substack{1\leq j<md,\\ d\nmid j}} j\cdot(N_j).$$

For each $j \le md$ there exists a unique integer $L_j \ge 0$ such that

$$m < jd^{L_j} \le md. \tag{2.6}$$

We write N_j in base d, but we collect together all terms corresponding to powers of d which match or exceed L_j:

$$N_j = \sum_{l \ge 0} a_{j,l} \cdot d^l = \sum_{0 \le l \le L_j - 1} a_{j,l} \cdot d^l + M_j \cdot d^{L_j}. \tag{2.7}$$

Here for $l \ge 0$, $0 \le l \le L_j - 1$ we have $0 \le a_{j,l} \le d-1$. Now we re-examine our original subpartition o:

$$\sum_{\substack{1 \le j < md, \\ d \nmid j}} j \cdot (N_j) = \sum_{\substack{1 \le j < md, \\ d \nmid j}} j \cdot \left(\sum_{0 \le l \le L_j - 1} a_{j,l} \cdot d^l + M_j \cdot d^{L_j} \right).$$

We thus have our new subpartition

$$\delta := \sum_{\substack{1 \le j < md, \\ d \nmid j, \\ 0 \le l \le L_j - 1}} \left(j \cdot d^l\right) (a_{j,l}) + \sum_{\substack{1 \le j < md, \\ d \nmid j}} \left(j \cdot d^{L_j}\right) (M_j). \tag{2.8}$$

In frequency convention, we have

$$\delta := \prod_{\substack{1 \le j < md, \\ d \nmid j, \\ 0 \le l \le L_j - 1}} \left(j \cdot d^l\right)^{a_{j,l}} \prod_{\substack{1 \le j < md, \\ d \nmid j}} \left(j \cdot d^{L_j}\right)^{M_j}. \tag{2.9}$$

The first sum contributes parts of the form $i = j \cdot d^l$ for $0 \le l \le L_j - 1$, and therefore $1 \le i \le m$; indeed, by unique factorization, any possible part between 1 and m must have this form. Moreover, each i occurs less than d times.

The parts which are permitted to repeat arbitrarily many times are $i = j \cdot d^{L_j}$, with $m < i \le md$. Again, by the possible range of j and through unique factorization, every possible part between m and md must have this form. These parts repeat M_j times, for any integer $M_j \ge 0$.

We will send λ to the following partition:

$$\kappa = \mu_0^* \cup \epsilon \cup \delta.$$

We need to show that $\kappa \in B(n, k, d, m)$.

Case 1: $m < k$.
In this case, ϵ is not empty, and has largest part dk. The parts of μ_0^* are all $\le m < k$, and the parts of δ are all $\le md < kd$. Thus our largest part will be kd.

A part i with $1 \le i \le m$ will have the form $i = jd^l$ for some j between 1 and m, and some nonnegative power l of d. It occurs $dq_i + a_{j,l}$ times; here

dq_i comes from μ_0^*, and $a_{j,l}$ comes from δ. Per the conditions of the division algorithm, this is an unrestricted nonnegative integer.

Any part i with $m < i \leq md$, has the form jd^{L_j} and also occurs an unrestricted number M_j of times, coming from δ. Neither μ_0^* nor ϵ contribute to these parts.

Finally, for parts greater than md, we have only terms dj from ϵ. The only restrictions is by size (the parts must be greater than md), and divisibility by d.

We thus have a well-defined partition $\kappa \in B(n, k, d, m)$ with no set restrictions beyond those of $B(n, k, d, m)$.

Case 2: $m \geq k$
In this case, ϵ is empty, and the largest part from μ_0^* is k. Indeed, any part i with $1 \leq i \leq k$ occurs dq_i times from μ_0^*. It may also occur in δ; however, because $i \leq k \leq m$, it has the form $i = jd^l$ with $d \nmid j$ and $0 \leq l \leq L_j - 1$, and can only occur $a_{j,l}$ times, i.e., no more than $d-1$ times. Thus the number of occurrences of i in κ has the form $dq_i + a_{j,l}$; it is therefore arbitrary, per the conditions of the division algorithm. Specifically for $i = k$, we have the additional condition that $q_k > 0$, so that k must occur at least d times.

Similarly, all parts $i > m$ can only come from δ, and be $\leq md$. Moreover, we have already shown that such an i may repeat M_j times, with $i = jd^{L_j}$, $d \nmid j$.

Only the parts i with $k < i \leq m$ have a restriction on their occurrences, appearing less than d times.

Finally, the largest parts in μ^* are $k \leq m \leq md$, and the largest possible parts of δ are md. Thus $\kappa \in B(n, k, d, m)$. □

2.5. **Examples.** For the sake of illustration, we give two examples. First, we take $d = 3$, $k = 7$, $m = 4$, $n = 123$. Notice that $m < k$. Moreover, we have $md = 12$ and $kd = 21$. We consider the partition

$$\lambda := 15^2 12^1 11^1 9^1 8^1 7^4 6^2 5^1 3^1 2^2 1^1.$$

The parts divisible by 3 are 15 (occurring twice), 12, 9, 6 (occurring twice), and 3. That is, there are $k = 7$ parts divisible by $d = 3$. The largest part not divisible by 3 is $11 < 12 = md$. Thus, $\lambda \in A(123, 7, 3, 4)$.

We build our bijection first by separating λ into its key subpartitions:

$$\mu := 15^2 12^1 9^1 6^2 3^1,$$
$$o := 11^1 8^1 7^4 5^1 2^2 1^1.$$

First,we map μ to its conjugate μ^*:

$$15^2 12^1 9^1 6^2 3^1 \longleftrightarrow 7^3 6^3 4^3 3^3 2^3.$$

Notice that each part i of μ^* lies between 1 and $k = 7$, and occurs a multiple of 3 times (possibly 0).

In turn we split μ^* into μ_0^* and ϵ. Because $\min(m, k) = m$, ϵ is not empty. We construct ϵ by examining the parts of μ^* which are greater than $m = 4$.

The factor of 3 in the number of occurrences is now multiplied by our part to produce a new part. What is left of the number of occurrences after dividing out 3 is our new occurrence:

$$7^3 6^3 = 7^{3\cdot 1} 6^{3\cdot 1} \longleftrightarrow 21^1 18^1 = \epsilon.$$

The remaining parts of μ^* give us μ_0^*:

$$\mu_0^* = 4^3 3^3 2^3.$$

We now turn our attention to

$$o := 11^1 8^1 7^4 5^1 2^2 1^1.$$

For $1 \leq j < 12, 3 \nmid j$, there exists a unique L_j such that $4 < jd^{L_j} \leq 12$. We may compute L_j for each part j of o:

$$\text{For } j = 11, 8, 7, 5, L_j = 0.$$
$$\text{For } j = 2, L_j = 1.$$
$$\text{For } j = 1, L_j = 2.$$

Thus for $j = 1, 2$, we can expand the number of occurrences by base 3 up to the power of 2 for $j = 1$, and 1 for $j = 2$. But the occurrences for these parts in o are 1 and 2, respectively, with the resulting maximum power being 0 in the base 3 expansions. Therefore we can simply apply Glaisher's bijection. In this case, it is a trivial mapping:

$$2^2 1^1 \longrightarrow 2^2 1^1.$$

Since $L_j = 0$ for the remaining parts of o, we will not apply Glaisher's bijection at all; we should multiply each j by 3^{L_j}. However, this factor is simply 1, and we may therefore simply reproduce the partition o:

$$\delta := 11^1 8^1 7^4 5^1 2^2 1^1.$$

Thus we have

$$\begin{aligned}\kappa &= 21^1 18^1 4^3 3^3 2^3 11^1 8^1 7^4 5^1 2^2 1^1 \\ &= 21^1 18^1 11^1 8^1 7^4 5^1 4^3 3^3 2^5 1^1.\end{aligned}$$

Our largest part is $kd = (7)(3) = 21$, and the parts greater than $md = 12$ are 21, 18, both divisible by $d = 3$. Thus $\kappa \in B(123, 7, 3, 4)$.

For our second example, we consider the $d = 3$, $k = 4$, $m = 7$, $n = 189$. Notice that $m \geq k$. Moreover, we have $md = 21$ and $kd = 12$. Let us take, say,

$$\lambda := 24^1 21^1 20^1 17^1 15^1 14^4 9^1 7^2 2^5 1^3.$$

Here, the parts divisible by 3 are 24, 21, 15, 9. Thus there are $k = 4$ parts divisible by $d = 3$. The largest part not divisible by 3 is 20, which is less than $md = 21$. Thus, $\lambda \in A(189, 4, 3, 7)$. Again, we split λ into

$$\begin{aligned}\mu &:= \; 24^1 21^1 15^1 9^1, \\ o &:= \; 20^1 17^1 14^4 7^2 2^5 1^3.\end{aligned}$$

We map μ to its conjugate μ^*:

$$24^1 21^1 15^1 9^1 \longleftrightarrow 4^9 3^6 2^6 1^3.$$

Because $m > k$, the subpartition ϵ is empty, and $\mu_0^* = \mu^*$.

Next we take

$$o := 20^1 17^1 14^4 7^2 2^5 1^3.$$

For $1 \leq j < md, d \nmid j$, we compute L_j as before, and we find that

$$\text{For } j = 20, 17, 14, L_j = 0.$$
$$\text{For } j = 7, L_j = 1.$$
$$\text{For } j = 2, 1, L_j = 2.$$

Now the mapping is nontrivial, and using (2.7)–(2.9) we get

$$\begin{aligned} &[20^1 17^1 14^4][7^2 2^{(2\cdot 3^0 + 1\cdot 3^1)} 1^{(1\cdot 3^1)}] \\ = &[20^1 17^1 14^4][7^2 6^1 2^2 3^1]. \end{aligned}$$

Putting our new subpartitions back together, we get

$$\begin{aligned} &[4^9 3^6 2^6 1^3][7^2 6^1 2^2 3^1] \\ &[20^1 17^1 14^4], \\ \kappa = &\ 20^1 17^1 14^4 7^2 6^1 4^9 3^7 2^8 1^3. \end{aligned}$$

Notice that $k = 4$ occurs at least $d = 3$ times (indeed, it occurs 9 times), and all parts are bounded above by $md = 21$. Of particular interest are the parts larger than $k = 4$ and less than or equal to $m = 7$: $7^2 6^1$. These parts occur less than 3 times. Thus, $\kappa \in B(189, 4, 3, 7)$.

3. Additional Restrictions

Of the two proofs that we have given above, both are new. However, the combinatorial bijection that we have provided is particularly interesting. One interesting advantage that it holds over the standard generating function proof is that it gives an explicit interpretation of how the given partition identity manifests itself in the parts themselves. Indeed, one can use the same finite-bound machinery (which was first developed by Andrews [1] and Nyirenda [6]) to explore other possible equivalence results between more exotic classes of partitions.

In particular, in our bijective proof we took o to include all parts j not divisible by d (and less than md, per the conditions of $A(n, k, d, m)$). We took the base d expansion of j, in which the coefficient $a_{j,l}$ of d^l is an integer between 0 and $d - 1$; we then reinterpreted $a_{j,l}$ to count occurrences of the part jd^l. Were we to impose additional restrictions on the base d expansion of the parts j, then this would in turn lead to additional restrictions on the number of occurrences of parts in our final partition.

As a simple example, recalling our definition of L_j in (2.6), we give the following definitions, in which we take $\hat{v} := (v_0, v_1, ..., v_{L_1 - 1})$ to be a vector with components in $\mathbb{Z}/d\mathbb{Z}$, and dimension L_1 (notice that $L_j \leq L_1$ for all $j \geq 1$):

Definition 3.1. *Let $A(n,k,d,m,\hat{v})$ be the set of integer partitions of n in which:*

- *Exactly k parts are divisible by d;*
- *All other parts j are strictly less than md, and have a base d expansion opening with $v_0 + v_1 d + v_2 d^2 + ... + v_{L_j-1} d^{L_j-1}$ (and possibly continuing on in some unrestricted manner).*

Definition 3.2. *Let $B(n,k,d,m,\hat{v})$ be the set of integer partitions of n such that:*

- *For $m < k$, the largest part is kd, all parts greater than md are divisible by d, and all parts $jd^l \leq m$ must occur $v_l \bmod d$ times;*
- *For $m \geq k$, the part k occurs at least d times, none of the parts exceed md, any part $i = jd^l \leq k$ occurs $v_l \bmod d$ times, and any such part $i = jd^l$ such that $k < i \leq m$ occurs exactly v_l times.*

We now have the following result:

Theorem 3.3. *For $n,k,d,m \in \mathbb{Z}_{\geq 1}$, $|A(n,k,d,m,\hat{v})| = |B(n,k,d,m,\hat{v})|$.*

We do not give this theorem as a particularly satisfying result, but only as a demonstration that much more can be done with these sorts of bijections than one may normally expect.

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