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2- AND 3-EXISTENTIALLY CLOSED TOURNAMENTS

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Abstract. A tournament has property P_k ($k \geq 1$) if for every k-subset A of its vertices and every $B \subseteq A$, there exists $x \notin A$ such that x dominates every element of B and every element of $A \setminus B$ dominates x. A tournament has property S_k if $B = \varnothing$ in the definition before. We give a characterization of those circulant tournaments of prime order having property P_2 using some results of additive number theory. Some new theoretical results are proved. It is proved that in vertex-transitive doubly regular tournaments properties S_3 and P_3 are equivalent and consequently, the Paley tournament QR_p has property P_3 for every $p \equiv 3 \bmod 4$ such that $p \geq 19$. It is also shown that the out- and in-neighborhood of every vertex of QR_p induce a circulant tournament with a special structure. As corollaries, we obtain that the out- and in-neighborhood of every vertex of QR_p has property S_3 if and only if QR_p has property S_4 and that QR_{67} has property S_4. In addition, non-vertex-transitive doubly regular tournaments of Szekeres type are considered. We show that the infinite families of Szekeres tournaments and their converses satisfy property P_3.

1. Introduction

A tournament T is *k-existentially closed* ($k \geq 1$), if for every subset $A \subseteq V(T)$ such that $|A| = k$ and every $B \subseteq A$, there exists $x \notin A$ such that x dominates every element of B and every element of $A \setminus B$ dominates x. We will say that a k-existentially closed tournament T *has property* P_k. On the other hand, a tournament T is said to *have property* S_k (defined in [4], the notation is after Schütte, see [14]), if for every subset $A \subseteq V(T)$ such that $|A| = k$ there exists $x \notin A$ such that every element of A dominates x. It is easy to see that property S_k is the special case of P_k when $B = \varnothing$ and so if T has property P_k, it also has property S_k. For $k = 1$, we have that T has property P_1 if and only if T contains neither sources nor sinks.

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The origins of this kind of adjacency problem in tournaments go back to Schütte, who posed the following question: Is there a tournament T of order n having property S_k for every k? Erdős affirmatively solved the problem in [4] and proved that if $f(k)$ is defined to be the smallest value of n for which such a tournament T exists, then $2^{k+1} - 1 \leq f(k) \leq ck^2 2^k$ for some constant $c > 0$. In fact, he verified that $f(1) = 3$ and $f(2) = 7$ (Schütte had already given these values). E. and G. Szekeres [14] improved the lower bound of the last inequality by showing that $f(k) \geq (k + 2)2^{k-1} - 1$ and confirmed that $f(3) = 19$. Later, Graham and Spencer [6] proved that if $p > k^2 2^{2k-2}$, where p is a prime such that $p \equiv 3 \bmod 4$, then the Paley tournament QR_p (formally defined in the next section) has property S_k. In [5], Erdős observed that computing the exact values of $f(k)$ is difficult and even "... an asymptotic formula for $f(k)$ seems beyond reach".

Property S_k is closely related to the domination property in tournaments and it has been studied in [13], where some interesting results on the property are obtained. Other similar adjacency properties in tournaments have been studied in [1] and [16].

Property P_k for tournaments was defined in [2] by Bonato and Cameron. They proved that if T has property P_2, then $|V(T)| \geq 7$ and QR_7 is the smallest tournament with the property P_2. Moreover, a construction (called a replication operation) is given to show that for every $k \geq 7$ with $k \neq 8$ there exists a (nonregular) tournament T with property P_2. Closing their paper, the authors posed the problem of finding examples of non-Paley k-existentially closed tournaments for $k \geq 3$. Using a random construction and computer search, Bonato, Godinowicz and Prałat in [3] show that there is a unique minimum order 3-e.c. tournament of order 19 and there are no 3-e.c. tournaments of orders 20, 21 and 22. Moreover, there are no 4-e.c. tournaments of orders 47 and 48 improving the lower bound for the minimum order of such a tournament and they verified that QR_{67} is the smallest order Paley tournament that has property P_4.

This paper is organized as follows. After some preliminaries, in Section 3, we give a characterization of those circulant tournaments of prime order having property P_2, using some results of additive number theory.

In Section 4, we prove that in vertex-transitive doubly regular tournaments properties S_3 and P_3 are equivalent and consequently, the Paley tournament QR_p has property P_3 for every $p \equiv 3 \bmod 4$ such that $p \geq 19$. It is worth mentioning that there exist vertex-transitive (regular and doubly regular) tournaments which are non-Cayley tournaments (and so, non-Paley), for instance, see [10]. It is also proved that the out- and in-neighborhood of every vertex of QR_p induce a circulant tournament with a special structure. As corollaries, we formally prove that the out- and in-neighborhood of every vertex of QR_p has property S_3 if and only if QR_p has property S_4, and that QR_{67} has property S_4.

In Section 5, non-vertex-transitive doubly regular tournaments of Szekeres type are considered. We show that the infinite families of Szekeres tournaments and their converses satisfy property P_3. We conclude with some computational results and open problems.

2. Preliminaries

Let $D = (V, A)$ be a finite digraph, where V and A denote the sets of vertices and arcs of D respectively. For $\varnothing \neq S \subseteq V(D)$ (resp. $\varnothing \neq S \subseteq A(D)$) we denote by $D[S]$ the induced (resp. arc-induced) subdigraph of D by the subset S. An arc of D is denoted by $(u, v) \in A(D)$ (or simply, $u \longrightarrow v$). We denote by $N^+_D(v)$ and $N^-_D(v)$ the out-neighborhood the in-neighborhood of a vertex $v \in V(D)$, respectively. We simply use $N^+(v)$ and $N^-(v)$ if D is clearly understood. We write $D \cong D'$ whenever the digraphs D and D' are isomorphic.

We say that a tournament T is regular if $|N^+(v)| = |N^-(v)|$ for every $v \in V(T)$; it is clear that T has an odd number of vertices. The directed triangle and the transitive subtournament of order 3 are denoted by $\overrightarrow{C}_3$ and TT_3, respectively.

Let $\mathbb{Z}_{2m+1}$ be the cyclic group of integers modulo $2m+1$ ($m \geq 1$) and J a nonempty subset of $\mathbb{Z}_{2m+1} \setminus \{0\}$ such that $|\{-j, j\} \cap J| = 1$ for every $j \in J$ (and therefore $|J| = m$). A *circulant* (or *rotational) tournament* $\overrightarrow{C}_{2m+1}(J)$ is defined by $V\left(\overrightarrow{C}_{2m+1}(J)\right) = \mathbb{Z}_{2m+1}$ and

$$A\left(\overrightarrow{C}_{2m+1}(J)\right) = \{(i, j) : i, j \in \mathbb{Z}_{2m+1} \text{ and } j - i \in J\}.$$

A circulant tournament $\overrightarrow{C}_{2m+1}(J)$ is regular and its automorphism group acts transitively on the vertex set (briefly, we say that $\overrightarrow{C}_{2m+1}(J)$ is *vertex-transitive*). Notice that $N^+(0) = J$, $N^-(0) = -J$, $N^+(i) = i + J$ and $N^-(i) = i - J$ for all $i \in \mathbb{Z}_{2m+1} \setminus \{0\}$.

Let $p \geq 3$ be a prime number. A *Paley tournament* QR_p is defined to be a circulant tournament $\overrightarrow{C}_p(J)$, where $p \equiv 3 \bmod 4$ and $J = Q_p$ is the subgroup of quadratic residues modulo p. Paley tournaments are doubly regular and *arc-transitive*, i.e. for every arc $(u, v) \in A(QR_p)$ the bijection $x \longrightarrow \frac{x-u}{v-u}$ is an automorphism of QR_p which maps (u, v) to $(0, 1)$.

From the definition of property P_k given in the introduction, we have that a tournament T has property P_k if and only if $\bigcap_{i=1}^{k} N^{\alpha_i}(v_i) \neq \varnothing$ for every k-tuple $(v_1, v_2, \ldots, v_k)$ of vertices of T and all permutations (with repetition) of $\alpha_1, \alpha_2, \ldots, \alpha_k \in \{+, -\}$. In particular, observe that T has property P_2 (resp. P_3) if for every $a, b \in V(T)$ (resp. $a, b, c \in V(T)$)

$$N^\alpha(a) \cap N^\beta(b) \neq \varnothing, \text{ (resp. } N^\alpha(a) \cap N^\beta(b) \cap N^\gamma(c) \neq \emptyset)$$

for all permutations of $\alpha, \beta \in \{+, -\}$ (resp. $\alpha, \beta, \gamma \in \{+, -\}$).

3. Circulant tournaments of prime order satisfying property P_2

We recall that given nonempty subsets $A, B \subseteq \mathbb{Z}_n$, the group of residues modulo a positive integer n, then

$$\begin{aligned} A + B &= \{a + b : a \in A, b \in B\}, \\ a + A &= \{a\} + A, \\ cA &= \{ca : a \in A\}, \\ -A &= \{-a : a \in A\}, \text{ and} \\ A - B &= A + (-B). \end{aligned}$$

We denote the set difference by $A \setminus B$.

Let $\mathbb{Z}_p$ be the set of residues modulo a prime p ($p \geq 3$). The following property is an easy consequence of the above definitions (they can be found in [17]).

Proposition 3.1. *Let A, B and C be nonempty subsets of $\mathbb{Z}_p$. Then $(A + B) \cap C = \varnothing$ if and only if $A \cap (C - B) = \varnothing$.*

We will use the famous Cauchy–Davenport theorem concerning sumsets in cyclic groups of prime order:

Theorem 3.2 ([11], Cauchy–Davenport)**.** *Let A and B be nonempty subsets of $\mathbb{Z}_p$. Then*

$$|A + B| \geq \min\{|A| + |B| - 1, p\}.$$

Let A be a nonempty subset of $\mathbb{Z}_p$. We say that A is an *arithmetic progression* if

$$A = \{a + id : i = 0, 1, \ldots, |A| - 1\},$$

where $a, d \in \mathbb{Z}_p$ and $d \neq 0$. The set A^j is an *almost arithmetic progression* if

$$A^j = \{a + id \mid 0 \leq i \leq j - 1\} \cup \{a + id \mid j + 1 \leq i \leq |A| - 1\},$$

where $a, d \in \mathbb{Z}_p$, $d \neq 0$ and $j \in \{2, \ldots, |A| - 2\}$.

In [17], A. G. Vosper classified those sets $A, B \subseteq \mathbb{Z}_p$ for which equality holds in the Cauchy–Davenport theorem. For our purposes, we only need a very special case of this characterization.

Proposition 3.3 ([17], corollary to Vosper's theorem)**.** *Let A be a nonempty subset of $\mathbb{Z}_p$ such that $A + A \neq \mathbb{Z}_p$. Then $|A + A| = 2|A| - 1$ if and only if A is an arithmetic progression.*

Proposition 3.4 ([7], corollary to Theorem 3)**.** *Let A be a nonempty subset of $\mathbb{Z}_p$ such that $|A| = \frac{p-1}{2}$. Then $|A + A| = 2|A|$ holds if and only if A is an almost arithmetic progression A^j, where $j = 2$ or $j = |A| - 2$.*

Now consider the set $\mathbb{Z}_{2n+1}$ of integers modulo $2n+1$ ($n \in \mathbb{N}$) and a nonempty subset $J \subseteq \mathbb{Z}_{2n+1} \setminus \{0\}$ such that for every $j \in \mathbb{Z}_{2n+1} \setminus \{0\}$, we have that $|\{j, -j\} \cap J| = 1$. Observe that

$$|\{j,-j\} \cap J| = 1 \Leftrightarrow \overline{J} = -J \cup \{0\} \Leftrightarrow 0 \notin J + J \tag{3.1}$$

and this implies that $|J| = |-J| = n$. In addition, if $2n+1 = p$ is a prime number, then

$$p - 2 = 2\,|J| - 1 \leq |J + J| \leq 2\,|J| = p - 1 \tag{3.2}$$

$$(p - 2 = |J| + |-J| - 1 \leq |J - J| \leq |J| + |-J| = p - 1),$$

where the first part of the inequality is an application of the Cauchy–Davenport theorem and the second one is a consequence of the above chain of equivalences. Observe that $|J+J| = p-2$ if and only if J is an arithmetic progression (Proposition 3.3).

Lemma 3.5. *There does not exist $\overrightarrow{C}_p(J)$ where $J = A^j$ is an almost arithmetic progression with $2 \leq j \leq \frac{p-3}{2}$.*

Proof. Without loss of generality, we can suppose that $J = A^j = \{1, 2, \dots, j-1, j+1, \dots, \frac{p+1}{2}\}$, where $2 \leq j \leq \frac{p-3}{2}$, otherwise there exist $a, d \in \mathbb{Z}_p$ and $d \neq 0$ such that $J' = \frac{1}{d}(J - a)$ is the required set (see [11] for more details). Then $\frac{p-1}{2} + \frac{p+1}{2} \equiv 0 \bmod p$ and so $\frac{p-1}{2} \in J$ and $-\frac{p-1}{2} \in J$, a contradiction. □

Remark: The tournament $\overrightarrow{C}_{2n+1}(1, 2, \dots, n)$ is called the *cyclic circulant tournament* for every $n \geq 1$. It is easy to check that there is only one cyclic circulant tournament up to isomorphism. We notice that in particular, $\overrightarrow{C}_p(J) \cong \overrightarrow{C}_p\left(1, 2, \dots, \frac{p-1}{2}\right)$ for $J = c\{1, 2, \dots, \frac{p-1}{2}\}$, where $c \in \mathbb{Z}_p \setminus \{0\}$. Observe that in this case, J is an arithmetic progression.

Theorem 3.7. *Let $p \geq 3$ be a prime number. Then $\overrightarrow{C}_p(J)$ has property P_2 if and only if*

(i) *J is not an arithmetic progression (equivalently,*

$$\overrightarrow{C}_p(J) \ncong \overrightarrow{C}_p\left(1, 2, \dots, \frac{p-1}{2}\right))$$

and

(ii) *$J \neq A^j$, where A^j is an almost arithmetic progression with $j = \frac{p-1}{2}$ (equivalently, $\overrightarrow{C}_p(J) \ncong \overrightarrow{C}_p\left(1, 2, \dots, \frac{p-3}{2}, \frac{p+1}{2}\right)$).*

Proof. Since $\overrightarrow{C}_p(J)$ is vertex-transitive, without loss of generality, we consider pairs of vertices $(0, a)$ with $a \in V\left(\overrightarrow{C}_p(J)\right)$. Tournament $\overrightarrow{C}_p(J)$ has property P_2 if and only if $N^{\pm}(0) \cap N^{\pm}(a) \neq \varnothing$ for every $a \in V\left(\overrightarrow{C}_p(J)\right)$.

These conditions are equivalent to

$$\begin{cases} J \cap (a+J) \neq \varnothing, \\ J \cap (a-J) \neq \varnothing, \\ -J \cap (a+J) \neq \varnothing, \quad \text{and} \\ -J \cap (a-J) \neq \varnothing. \end{cases}$$

For a contradiction, consider the following 4 cases.

CASE 1: $J \cap (a+J) = \varnothing$.

Using Proposition 3.1, this is equivalent to

$$\{a\} \cap (J-J) = \varnothing \Leftrightarrow J - J \subseteq \mathbb{Z}_p \setminus \{a\}.$$

From this, we have that $|J-J| \leq p-1$ and by (3.2), $|J-J| \geq p-2$. If $|J-J| = p-2$, then by Proposition 3.3, J is an arithmetic progression, a contradiction to condition (i) of the theorem. If $|J-J| = p-1$, then by Proposition 3.4 and Lemma 3.5, $J = A^{\frac{p-1}{2}} = c\{1, 2, 3, \ldots, \frac{p-3}{2}, \frac{p+1}{2}\}$ for some $c \in \mathbb{Z}_p$, $c \neq 0$, a contradiction to condition (ii) of the theorem.

CASE 2: $J \cap (a-J) = \varnothing$.

Using Proposition 3.1, this is equivalent to

$$\{a\} \cap (J+J) = \varnothing \Leftrightarrow J + J \subseteq \mathbb{Z}_p \setminus \{a\}.$$

From this, we have that $|J+J| \leq p-1$ and by (3.2), $|J+J| \geq p-2$. If $|J+J| = p-2$, then by Proposition 3.3, J is an arithmetic progression, a contradiction to condition (i) of the theorem. If $|J+J| = p-1$, then by Proposition 3.4 and Lemma 3.5, $J = A^{\frac{p-1}{2}} = c\{1, 2, 3, \ldots, \frac{p-3}{2}, \frac{p+1}{2}\}$ for some $c \in \mathbb{Z}_p$, $c \neq 0$, a contradiction to condition (ii) of the theorem.

CASE 3: $-J \cap (a+J) = \varnothing$.

Similar to Case 2 (use $-J$ in place of J).

CASE 3: $-J \cap (a-J) = \varnothing$.

Using Proposition 3.1, this is equivalent to $\{a\} \cap (J-J) = \varnothing$, which is Case 1.

□

4. DOUBLY REGULAR TOURNAMENTS SATISFYING PROPERTY P_3

A tournament T is *doubly regular* (or a *Hadamard tournament*, see [9]) if $T[N^+(v)]$ and $T[N^-(v)]$ are regular tournaments for every $v \in V(T)$. It is a well-known fact that doubly regular tournaments have $4n+3$ vertices ($n \geq 0$). A remarkable result proved in [12] establishes that doubly regular tournaments are equivalent to skew Hadamard matrices. It is well-known that if $u, v \in V(T)$ such that $(u, v) \in A(T)$, then

$$\begin{cases} (i) & |N^+(u) \cap N^+(v)| = n, \\ (ii) & |N^+(u) \cap N^-(v)| = n, \\ (iii) & |N^-(u) \cap N^+(v)| = n+1, \\ (iv) & |N^-(u) \cap N^-(v)| = n. \end{cases} \tag{4.1}$$

Let T be a doubly regular tournament such that $|V(T)| = 4n+3, n \geq 4$. Remember that T a doubly regular tournament, has property P_3 if for every $a, b, c \in V(T))$, $N^{\alpha}(a) \cap N^{\beta}(b) \cap N^{\gamma}(c) \neq \emptyset$ for all permutations of $\alpha, \beta, \gamma \in \{+, -\}$.

For short, in what follows we write $T[a, b, c] = T[\{a, b, c\}]$.

For every $\{a, b, c\}$, a triplet of different vertices of T, we have two posibilities:

(i) $T[a, b, c] \cong \overrightarrow{C}_3$, the oriented cycle of length three;
(ii) $T[a, b, c] \cong TT_3$, the transitive tournament of order three.

Theorem 4.1. *Let T be a doubly regular tournament such that $|V(T)| = 4n+3$, $n \geq 4$ and $\{a, b, c\} \subseteq V(T)$. Let*

$$q = \left|N^+(a) \cap N^+(b) \cap N^+(c)\right|.$$

Therefore T has property P_3 if and only if one of the following conditions hold:

(i) *If $T[a, b, c] \cong \overrightarrow{C}_3$, then $q \in [1, n-1]$.*
(ii) *If $T[a, b, c] \cong TT_3$, then $q \in [1, n-2]$.*

Proof. Let $\{a, b, c\} \subseteq V(T)$. Observe that $1 \leq q \leq n$. Define the following sets:

$$\begin{aligned} W &= N^+(a) \cap N^+(b) \cap N^+(c), \\ X &= \big(N^+(a) \cap N^+(b)\big) \setminus W, \\ Y &= \big(N^+(b) \cap N^+(c)\big) \setminus W, \text{ and} \\ Z &= \big(N^+(c) \cap N^+(a)\big) \setminus W. \end{aligned}$$

We consider two cases.

Case 1: $T[a, b, c] \cong \overrightarrow{C}_3 = (a, b, c, a)$.

Thus

$$a \in N^-(b) \cap N^+(c),\ b \in N^+(a) \cap N^-(c) \text{ and } c \in N^-(a) \cap N^+(b).$$

First, we prove that if $q = n$, then T does not have property P_3. We will use the following notation (see Figure 1):

$$\begin{aligned} A &= N^+(a) \setminus (X \cup Z \cup W \cup \{b\}), \\ B &= N^+(b) \setminus (X \cup Y \cup W \cup \{c\}), \text{ and} \\ C &= N^+(c) \setminus (Z \cup Y \cup W \cup \{a\}). \end{aligned}$$

When $|W| = n$, we have that $X = Y = Z = \varnothing$ and hence $|A| = |B| = |C| = n$. Then

$$\begin{aligned} N^-(a) &= B \cup C \cup \{c\}, \\ N^-(b) &= A \cup C \cup \{a\}, \text{ and} \\ N^-(c) &= A \cup B \cup \{b\}. \end{aligned}$$

These equalities yield $N^-(a) \cap N^-(b) \cap N^-(c) = \varnothing$.

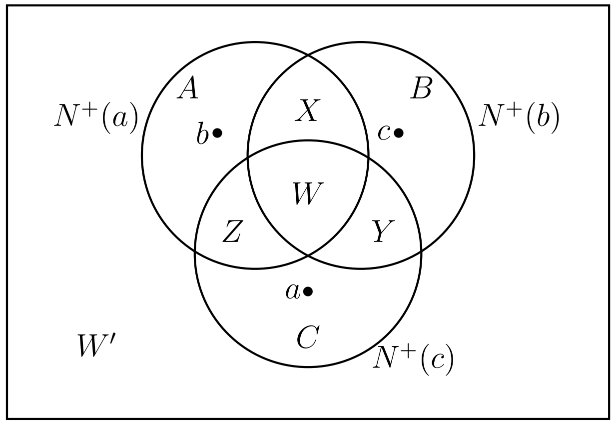

FIGURE 1. Case 1

On the other hand, suppose that $1 \leq q \leq n-1$. We show that T has property P_3. If $|W| = q$, then

$$\begin{aligned}|X| &= \left|N^+(a) \cap N^+(b) \cap N^-(c)\right| = n-q,\\ |Y| &= \left|N^+(b) \cap N^+(c) \cap N^-(a)\right| = n-q, \text{ and}\\ |Z| &= \left|N^+(a) \cap N^+(c) \cap N^-(b)\right| = n-q.\end{aligned}$$

and

$$|A| = \left|N^+(a)\right| - |X| - |Z| - |W| - 1 = 2n+1-2(n-q)-q-1 = q.$$

By an analogous argument, we get that $|B| = |C| = q$. Therefore

$$\begin{aligned}\left|N^+(a) \cup N^+(b) \cup N^+(c)\right| &= |A \cup B \cup C \cup X \cup Y \cup Z \cup W \cup \{a,b,c\}|\\ &= 3q + 3(n-q) + q + 3 = 3n + q + 3.\end{aligned}$$

If we denote $W' = N^-(a) \cap N^-(b) \cap N^-(c)$, then

$$V(T) = N^+(a) \cup N^+(b) \cup N^+(c) \cup W'$$

from which we obtain that

$$\left|W'\right| = 4n + 3 - (3n + q + 3) = n - q.$$

It implies that

$$\begin{aligned}|A| &= \left|N^+(a) \cup N^-(b) \cup N^-(c)\right| = q,\\ |B| &= \left|N^+(b) \cup N^-(a) \cup N^-(c)\right| = q, \text{ and}\\ |C| &= \left|N^+(c) \cup N^-(a) \cup N^-(b)\right| = q.\end{aligned}$$

We conclude that (i) is valid.

CASE 2: $T[a,b,c] \cong TT_3$.

Let $a \to b$, $b \to c$ and $a \to c$. Thus

$$a \in N^-(b) \cap N^-(c),\ b \in N^+(a) \cap N^-(c) \text{ and } c \in N^+(a) \cap N^+(b).$$

Notice that (see Figure 2)

$$V(T) = A \cup B \cup C \cup X \cup Y \cup Z \cup W \cup W' \cup \{a, b, c\},$$

where

$$\begin{aligned} W &= N^+(a) \cap N^+(b) \cap N^+(c), \\ X &= \big(N^+(a) \cap N^+(b)\big) \setminus W, \\ Y &= \big(N^+(b) \cap N^+(c)\big) \setminus W, \text{ and} \\ Z &= \big(N^+(c) \cap N^+(a)\big) \setminus W \end{aligned}$$

and also

$$\begin{aligned} A &= N^+(a) \setminus (X \cup Z \cup W \cup \{b, c\}), \\ B &= N^+(b) \setminus (X \cup Y \cup W \cup \{c\}), \text{ and} \\ C &= N^+(c) \setminus (Z \cup Y \cup W). \end{aligned}$$

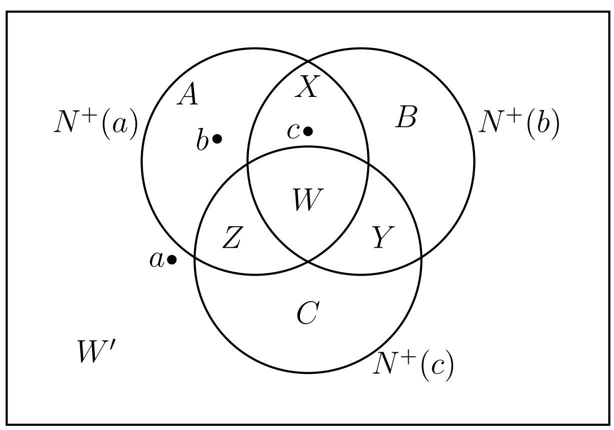

FIGURE 2. Case 2

When $|W| = n$, we obtain that $W \cup \{c\} \subseteq N^+(a) \cap N^+(b)$. But in this case $|W \cup \{c\}| = n+1$ and $|N^+(a) \cap N^+(b)| = n$, which is a contradiction and in this way, T does not have property P_3.
If $|W| = n - 1$, then $X = \varnothing$ and $|Y| = |Z| = 1$. Therefore

$$\begin{aligned} N^+(a) &= A \cup X \cup Z \cup W \cup \{b, c\}, \\ N^+(b) &= B \cup X \cup Y \cup W \cup \{c\}, \text{ and} \\ N^+(c) &= C \cup Y \cup Z \cup W. \end{aligned}$$

Consequently,

$$\begin{aligned}|A| &= \left|N^+(a)\right| - |X| - |Z| - |W| - 2\\ &= 2n+1-1-(n-1)-2 = n-1,\\ |B| &= \left|N^+(b)\right| - |X| - |Y| - |W| - 1\\ &= 2n+1-1-(n-1)-1 = n, \text{ and}\\ |C| &= \left|N^+(c)\right| - |Y| - |Z| - |W|\\ &= 2n+1-1-1-(n-1) = n.\end{aligned}$$

These equalities imply that

$$\begin{aligned}\left|W'\right| &= |V(T)| - |A\cup B\cup C| - |X\cup Y\cup Z| - |W| - |\{a,b,c\}|\\ &= 4n+3-(3n-1)-2-(n-1)-3 = 0\end{aligned}$$

and $W' = N^-(a)\cap N^-(b)\cap N^-(c) = \varnothing$. Hence T does not have property P_3.

Suppose that $1 \le q \le n-2$. We will prove that T has property P_3. Let $|W| = q$. Then

$$\begin{aligned}X &= (N^+(a)\cap N^+(b))\backslash(W\cup\{c\}),\\ Y &= (N^+(b)\cap N^+(c))\backslash W, \text{ and}\\ Z &= (N^+(a)\cap N^+(c))\backslash W\end{aligned}$$

implying that

$$|X| = n-q-1 \ge 1 \text{ and } |Y| = |Z| = n-q.$$

Therefore (see Figure 2)

$$\begin{aligned}|A| &= \left|N^+(a)\right| - |X| - |Z| - |W| - 2\\ &= 2n+1-(n-q-1)-(n-q)-q-2 = q\end{aligned}$$

and similarly, $|B| = |C| = q+1$. We have that

$$\begin{aligned}\left|W'\right| &= |V(T)| - |A\cup B\cup C| - |X\cup Y\cup Z| - |W| - |\{a,b,c\}|\\ &= 4n+3-(3q+2)-(3n-3q-1)-q-3\\ &= n-q-1 \ge 1.\end{aligned}$$

This proves (ii).

□

Corollary 4.2. *A vertex-transitive doubly regular tournament T has property P_3 if and only if it has property S_3.*

Since $f(3) = 19$ (that is, QR_{19} has property S_3) and using the already mentioned main result of [6], we have the following

Corollary 4.3. *QR_p has property P_3 for every $p \equiv 3 \bmod 4$ such that $p \ge 19$.*

Let us introduce the definition of the circulant tournament given in the preliminaries as a Cayley digraph. Let G be an abelian group such that $|G| = 2m+1$ ($m \geq 1$) whose operation is the multiplication and 1 is the identity. Consider $\varnothing \neq J \subseteq G \setminus \{1\}$ such that $\left|\{j, j^{-1}\} \cap J\right| = 1$ for every $j \in J$ and therefore $|J| = m$. The circulant tournament (in multiplicative notation) $\overrightarrow{C}^*_{2m+1}(J)$ is define by $V\left(\overrightarrow{C}^*_{2m+1}(J)\right) = G$ and

$$A\left(\overrightarrow{C}^*_{2m+1}(J)\right) = \{(i,j) : i, j \in G \text{ and } ji^{-1} \in J\}.$$

Proposition 4.4. *Let $p = 4n+3$ be a prime number. Then $QR_p[N^+(0)] \cong \overrightarrow{C}^*_{2n+1}(J)$, where $V\left(\overrightarrow{C}^*_{2n+1}(J)\right) = Q_p$ and $J = N^+_{QR_p}(0) \cap N^+_{QR_p}(1)$.*

Proof. Observe that $V\left(\overrightarrow{C}^*_{2n+1}(J)\right) = Q_p$ and $|J| = n$, since $N^+_{QR_p}(0) = Q_p$ and QR_p is doubly regular respectively. It only remains to show that for every $j \in J$, then $j^{-1} \notin J$, where $J = N^+_{QR_p}(0) \cap N^+_{QR_p}(1)$. For a contradiction, suppose that $j, j^{-1} \in J$. Since $N^+_{QR_p}(1) = 1 + N^+_{QR_p}(0)$ and $j, j^{-1} \in N^+_{QR_p}(1)$, we have that $j-1, j^{-1}-1 \in N^+_{QR_p}(0)$. Then, $j(j^{-1}-1) = 1-j \in N^+_{QR_p}(0) = Q_p$. We get a contradiction, $j-1, 1-j \in Q_p$ and $-1 \notin Q_p$ since $p \equiv 3 \bmod 4$. $\square$

Using that QR_p is vertex-transitive we get the following

Corollary 4.5. *Let $p = 4n+3$ be a prime number. Then $QR_p[N^+(i)] \cong \overrightarrow{C}^*_{2n+1}(iJ)$, where $V\left(\overrightarrow{C}^*_{2n+1}(iJ)\right) = i + Q_p$ and $iJ = N^+_{QR_p}(i) \cap N^+_{QR_p}(i+1)$ for all $i \in V(QR_p)$.*

Similarly, one can prove that:

Proposition 4.6. *Let $p = 4n+3$ be a prime number. Then $QR_p[N^-(0)] \cong \overrightarrow{C}^*_{2n+1}(J)$, where $V\left(\overrightarrow{C}^*_{2n+1}(J)\right) = -Q_p$ and $J = N^-_{QR_p}(0) \cap N^-_{QR_p}(1)$ (observe that $-Q_p$ is the set of the nonquadratic residues modulo p).*

Corollary 4.7. *Let $p = 4n+3$ be a prime number. then $QR_p[N^-(i)] \cong \overrightarrow{C}^*_{2n+1}(iJ)$, where $V\left(\overrightarrow{C}^*_{2n+1}(iJ)\right) = i - Q_p$ and $iJ = N^-_{QR_p}(i) \cap N^-_{QR_p}(i+1)$ for all $i \in V(QR_p)$.*

In [2], it was proved that if a tournament T has property P_k ($k \geq 2$), then $T[N^+(v)]$ and $T[N^-(v)]$ have property P_{k-1}. In particular, if T has property S_4, then $T[N^+(v)]$ and $T[N^-(v)]$ have property S_3. The following proposition establishes the equivalence of properties S_3 and S_4 in the case of Paley tournaments.

Proposition 4.8. *$QR_p[N^+(0)]$ (resp. $QR_p[N^-(0)]$) has property S_3 if and only if QR_p has property S_4.*

Proof. By Proposition 4.4, $QR_p[N^+(0)] \cong \overrightarrow{C}^*_{2n+1}(J)$, where $J = N^+_{QR_p}(0) \cap N^+_{QR_p}(1)$. Let $U = QR_p[N^+(0)]$ and suppose that U satisfies property S_3. We know that $V(U) = Q_p$ and

$$J = N^+_U(1) = Q_p \cap (1 + Q_p).$$

For every $i \in Q_p$,

$$N^+_U(i) = iJ = i(Q_p \cap (1 + Q_p)) = Q_p \cap (i + Q_p)\ .$$

Therefore, $N^+_U(i) = N^+_{QR_p}(0) \cap N^+_{QR_p}(i)$. Since U has property S_3, then for every $i, j \in Q_p$,

$$\begin{aligned} & N^+_U(0) \cap N^+_U(i) \cap N^+_U(j) \neq \varnothing \\ \iff & Q_p \cap (1 + Q_p) \cap (i + Q_p) \cap (j + Q_p) \neq \varnothing \\ \iff & N^+_{QR_p}(0) \cap N^+_{QR_p}(1) \cap N^+_{QR_p}(i) \cap N^+_{QR_p}(j) \neq \varnothing \end{aligned}$$

and so QR_p has property S_4, using that QR_p is arc-transitive. The proof for $QR_p[N^-(0)]$ is similar using Proposition 4.6. □

Corollary 4.9. *$QR_p[N^+(v)]$ (resp. $QR_p[N^-(v)]$) has property S_3 if and only if QR_p has property S_4 for every $v \in V(QR_p)$.*

In [6], it is stated without proof that QR_{67} has property S_4. As a consequence of Proposition 4.8 we obtain this result. Note that $Q_{67} = \{4^m \bmod 67 : 0 \leq m \leq 32\}$, that is, 4 is a generator of Q_{67} and we have that $QR_{67}[N^+(0)] \cong \overrightarrow{C}^*_{33}(J)$, where

$$J = Q_{67} \cap (1 + Q_{67}) = \{10, 15, 17, 22, 23, 24, 25, 26, 36, 37, 40, 55, 56, 60, 65\}.$$

Define the function $f : Q_{67} \to \mathbb{Z}_{67}$ by $f(4^m) = m$. It is straightforward to prove that f is bijective and consequently, $\overrightarrow{C}^*_{33}(J) \cong \overrightarrow{C}_{33}(J')$, where

$$J' = \{2, 4, 7, 8, 9, 10, 11, 13, 14, 15, 17, 21, 27, 28, 30, 32\}.$$

Since $2^{-1} = 17 \in \mathbb{Z}^*_{33}$, we have that $\overrightarrow{C}_{33}(J') \cong \overrightarrow{C}_{33}(K)$, where

$$K = \{1, 2, 4, 5, 7, 14, 15, 16, 20, 21, 22, 23, 24, 25, 27, 30\}.$$

Proposition 4.10. *$\overrightarrow{C}_{33}(K)$ has property S_3.*

Proof. For a contradiction, suppose that $N^+_U(0) \cap N^+_U(i) \cap N^+_U(j) = \varnothing$, where $U = \overrightarrow{C}_{33}(K)$. Equivalently,

$$\begin{aligned} K \cap (i + K) \cap (j + K) = \varnothing \iff & ((K \cap (i + K)) - K) \cap \{j\} = \varnothing \\ \iff & (K - K) \cap (i + (K - K)) \cap \{j\} = \varnothing, \end{aligned}$$

if we use Proposition 3.1 and the straightforward property that $(A \cap B) - C = (A - C) \cap (B - C)$ for every $A, B, C \subseteq \mathbb{Z}_n$. Since $K - K = \mathbb{Z}_{33}$, we arrive to a contradiction. □

Corollary 4.11. *QR_{67} has property S_4.*

In fact, it has computationally been checked that $\overrightarrow{C}_{33}(K)$ has property P_3 and QR_{67} has property P_4 (see [3]).

5. Tournaments of Szekeres type

Let $p \equiv 5 \bmod 8$ be a prime number with $p \geq 13$ and consider the group $\mathbb{Z}_p$. We denote by H_4 the subgroup of quartic residues modulo p. It is an easy exercise proving that 2, 4 and 8 are not quartic residues modulo p. Let us denote by $2H_4$, $4H_4$ and $8H_4$ the cosets modulo the subgroup H_4. Notice that $-H_4 = 4h_4$ and $-2H_4 = 8H_4$. Let $\mathbb{Z}_p^\circ$ be a disjoint copy of $\mathbb{Z}_p$. We define the *tournament of Szekeres type* (briefly, the *Szekeres tournament*) $\mathsf{S}_p = (V, A)$ by

$$\begin{aligned} V(\mathsf{S}_p) &= \mathbb{Z}_p \cup \mathbb{Z}_p^\circ \cup \{t\}, \\ N^+(a) &= (a + (H_4 \cup 2H_4)) \cup \{a^\circ\} \cup (a^\circ + (H_4^\circ \cup 8H_4^\circ)), \\ N^+(a^\circ) &= (a + (H_4 \cup 8H_4)) \cup \{t\} \cup (a^\circ + (4H_4^\circ \cup 8H_4^\circ)), \text{ and} \\ N^+(t) &= \mathbb{Z}_p, \end{aligned}$$

where $a \in \mathbb{Z}_p$, $a^\circ \in \mathbb{Z}_p^\circ$ and t is a new symbol (see [9]). In [15], G. Szekeres introduced this type of tournament in order to study the adjacency property defined in [14]. In that paper, it was proved that they are doubly regular. Observe that $|V(\mathsf{S}_p)| = 2p+1 = 4\frac{p-1}{2}+3$. From the definition, we have that for all $a \in \mathbb{Z}_p$ and $a^\circ \in \mathbb{Z}_p^\circ$

$$\begin{aligned} N^-(a) &= (a + (4H_4 \cup 8H_4)) \cup \{t\} \cup (a^\circ + (2H_4^\circ \cup 4H_4^\circ)), \\ N^-(a^\circ) &= (a + (2H_4 \cup 4H_4)) \cup \{a\} \cup (a^\circ + (H_4^\circ \cup 2H_4^\circ)), \text{ and} \\ N^-(t) &= \mathbb{Z}_p^\circ. \end{aligned}$$

Remark: $\mathsf{S}_p[\mathbb{Z}_p] \cong \overrightarrow{C}_p(H_4 \cup 2H_4)$ and $\mathsf{S}_p[\mathbb{Z}_p^\circ] \cong \overrightarrow{C}_p(H_4^\circ \cup 8H_4^\circ)$. Moreover $\overrightarrow{C}_p(H_4 \cup 2H_4) \cong \overrightarrow{C}_p(H_4 \cup 8H_4)$, where $\varphi : \mathbb{Z}_p \longrightarrow \mathbb{Z}_p$ defined by

$$\varphi(a) = \begin{cases} 0 & \text{if } a = 0 \\ a & \text{if } a \in H_4 \cup 4H_4 \\ 4a & \text{if } a \in 2H_4 \\ \frac{1}{4}a & \text{if } a \in 8H_4 \end{cases}$$

is an isomorphism.

Evidently, S_p is not vertex-transitive. The automorphism group $Aut(\mathsf{S}_p)$ was completely determined in [8]. In particular, we emphasize the following automorphism that will be useful in the forthcoming proof. Let $\tau \in \mathbb{Z}_p$, then $\tau^* \in Aut(\mathsf{S}_p)$, where

$$\tau^*(a) = \begin{cases} a + \tau & \text{if } a \in \mathbb{Z}_p \\ t & \text{if } a = t \\ a^\circ + \tau^\circ & \text{if } a^\circ \in \mathbb{Z}_p^\circ \end{cases} \tag{5.1}$$

for every $a \in \mathbb{Z}_p$.

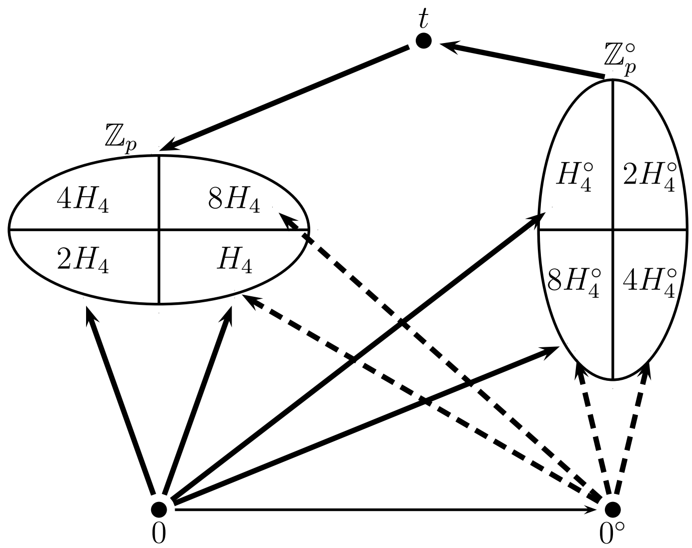

FIGURE 3. Szekeres tournament

Let us denote by T and T° the subtournaments $\mathsf{S}_p[\mathbb{Z}_p]$ and $\mathsf{S}_p[\mathbb{Z}_p^\circ]$, respectively. Recall a well-known fact of the theory of finite fields. The group of units $\mathbb{Z}_p^* = \mathbb{Z}_p \setminus \{0\}$ is cyclic. Let H be a subgroup of $\mathbb{Z}_p^*$ and A a nonempty union of cosets modulo H. Therefore, we have that

$$\sum_{a\in A} a \equiv 0 \bmod p. \tag{5.2}$$

Lemma 5.2. *For all $a, b \in V(\mathsf{S}_p) \setminus \{t\}$ we have that $N_T^\alpha(a) \cap N_T^\beta(b) \neq \varnothing$ and $N_{T^\circ}^\alpha(a) \cap N_{T^\circ}^\beta(b) \neq \varnothing$ for all permutations of $\alpha, \beta \in \{+, -\}$.*

Proof. We consider three cases:

CASE 1: $a, b \in \mathbb{Z}_p$.

Using the automorphism of S_p defined in (5.1), we will verify the properties for pairs $\{0, x\}$ $(x \neq 0)$. Without loss of generality, we can assume that $0 \longrightarrow x$. Observe that

$$\begin{aligned} N_{\mathsf{S}_p}^+(x) &= (x + (H_4 \cup 2H_4)) \cup \{x^\circ\} \cup (x^\circ + (H_4^\circ \cup 8H_4^\circ)) \\ &= N_T^+(x) \cup \{x^\circ\} \cup N_{T^\circ}^+(x) \text{ and} \\ N_{\mathsf{S}_p}^-(x) &= (x + (4H_4 \cup 8H_4)) \cup \{t\} \cup (x^\circ + (2H_4^\circ \cup 4H_4^\circ)) \\ &= N_T^-(x) \cup \{t\} \cup N_{T^\circ}^-(x). \end{aligned}$$

(i) Suppose that $N_T^-(0) \cap N_T^+(x) = \varnothing$. Since S_p is doubly regular, therefore

$$\left| N_{\mathsf{S}_p}^-(0) \cap N_{\mathsf{S}_p}^+(x) \right| = \frac{p+1}{2} \tag{5.3}$$

(see (4.1)(iii)) and

$$\begin{aligned} N^-_{\mathsf{S}_p}(0) \cap N^+_{\mathsf{S}_p}(x) &\subseteq (2H_4^\circ \cup 4H_4^\circ) \cap (x^\circ + (H_4^\circ \cup 8H_4^\circ)) \\ &\subseteq 2H_4^\circ \cup 4H_4^\circ. \end{aligned}$$

From this, $\left|N^-_{\mathsf{S}_p}(0) \cap N^+_{\mathsf{S}_p}(x)\right| \leq \frac{p-1}{2}$, a contradiction to (5.3).

(ii) Suppose that $N^\alpha_T(0) \cap N^\beta_T(x) = \varnothing$ for some permutations of $\alpha, \beta \in \{+,-\}$, where $(\alpha,\beta) \neq (-,+)$. Since S_p is doubly regular, therefore

$$\left|N^\alpha_{\mathsf{S}_p}(0) \cap N^\beta_{\mathsf{S}_p}(x)\right| = \frac{p-1}{2}$$

(see (4.1)(i),(ii),(iv)) and

$$N^\alpha_{\mathsf{S}_p}(0) \cap N^\beta_{\mathsf{S}_p}(x) \subseteq (iH_4^\circ \cup jH_4^\circ) \cap (x^\circ + (kH_4^\circ \cup lH_4^\circ))$$

for some $i,j,k,l \in \{1,2,4,8\}$. The last two relations imply that $N^\alpha_{\mathsf{S}_p}(0) \cap N^\beta_{\mathsf{S}_p}(x) = iH_4^\circ \cup jH_4^\circ$. Analogously, $N^\alpha_{\mathsf{S}_p}(0) \cap N^\beta_{\mathsf{S}_p}(x) = x^\circ + (kH_4^\circ \cup lH_4^\circ)$. Therefore, $iH_4^\circ \cup jH_4^\circ = x^\circ + (kH_4^\circ \cup lH_4^\circ)$. From this equality and using (5.2), we obtain that

$$\sum_{s \in iH_4^\circ \cup jH_4^\circ} s \equiv \sum_{s \in kH_4^\circ \cup lH_4^\circ} s \equiv 0 \bmod p \text{ and}$$

$$\sum_{s \in kH_4^\circ \cup lH_4^\circ} (x+s) \equiv \frac{p-1}{2} x \equiv 0 \bmod p,$$

which means that $\frac{p-1}{2} \equiv 0 \bmod p$ or $x \equiv 0 \bmod p$. Both congruences are impossible and we get a contradiction, so $N^\alpha_T(0) \cap N^\beta_T(x) \neq \varnothing$ for all permutations of $\alpha, \beta \in \{+,-\}$, where $(\alpha,\beta) \neq (-,+)$.

(iii) Suppose that $N^-_{T^\circ}(0) \cap N^+_{T^\circ}(x) = \varnothing$. Since S_p is doubly regular, therefore (5.3) holds and

$$\begin{aligned} N^-_{\mathsf{S}_p}(0) \cap N^+_{\mathsf{S}_p}(x) &\subseteq (4H_4 \cup 8H_4) \cap (x + (H_4 \cup 2H_4)) \\ &\subseteq 4H_4 \cup 8H_4. \end{aligned}$$

From this relation, we get a contradiction (see (i)).

(iv) Suppose that $N^\alpha_{T^\circ}(0) \cap N^\beta_{T^\circ}(x) = \varnothing$ for some permutations of $\alpha, \beta \in \{+,-\}$ where $(\alpha,\beta) \neq (-,+)$. Since S_p is doubly regular, we have that

$$\left|N^\alpha_{\mathsf{S}_p}(0) \cap N^\beta_{\mathsf{S}_p}(x)\right| = \frac{p-1}{2}$$

(see (4.1)(i),(ii),(iv)) and

$$N^\alpha_{\mathsf{S}_p}(0) \cap N^\beta_{\mathsf{S}_p}(x) \subseteq (iH_4 \cup jH_4) \cap (x + (kH_4 \cup lH_4))$$

for some $i, j, k, l \in \{1, 2, 4, 8\}$. The last two relations imply that $N^{\alpha}_{\mathsf{S}_p}(0) \cap N^{\beta}_{\mathsf{S}_p}(x) = iH_4 \cup jH_4$. Analogously, $N^{\alpha}_{\mathsf{S}_p}(0) \cap N^{\beta}_{\mathsf{S}_p}(x) = x + (kH_4 \cup lH_4)$. We proceed as in (ii) (for H_4, $2H_4$, $4H_4$ and $8H_4$) to obtain a similar contradiction. So $N^{\alpha}_{T^\circ}(0) \cap N^{\beta}_{T^\circ}(x) \neq \varnothing$ for all permutations of $\alpha, \beta \in \{+, -\}$, where $(\alpha, \beta) \neq (-, +)$.

CASE 2: $a, b \in \mathbb{Z}_p^\circ$.

Using the automorphism of S_p defined in (5.1), we will verify the properties for pairs $\{0^\circ, x^\circ\}$ $(x^\circ \neq 0^\circ)$. Without loss of generality, we can assume that $0^\circ \longrightarrow x^\circ$. In this case, observe that

$$\begin{aligned} N^+(x^\circ) &= (x + (H_4 \cup 8H_4)) \cup \{t\} \cup (x^\circ + (4H_4^\circ \cup 8H_4^\circ)) \\ &= N^+_T(x^\circ) \cup \{t\} \cup N^+_{T^\circ}(x^\circ) \text{ and} \\ N^-(x^\circ) &= (x + (2H_4 \cup 4H_4)) \cup \{x\} \cup (x^\circ + (H_4^\circ \cup 2H_4^\circ)) \\ &= N^-_T(x^\circ) \cup \{x\} \cup N^-_{T^\circ}(x^\circ). \end{aligned}$$

A completely analogous subcase analysis as carried on in Case 1 yields to the desired properties.

CASE 3: $a \in \mathbb{Z}_p$ and $b \in \mathbb{Z}_p^\circ$.

Using the automorphism of S_p defined in (5.1), we will verify the properties for pairs $\{0, x^\circ\}$. Again, we can assume that $0 \longrightarrow x^\circ$ (otherwise, using (5.1) we have that $0^\circ \longrightarrow x$ and we proceed similarly). Once more the procedure of Case 1 applies to obtain the stated properties.

□

Theorem 5.3. *Let $p \equiv 5 \bmod 8$ be a prime number with $p \geq 13$. Then S_p has property P_3.*

Proof. From the definition of the Szekeres tournament, we have to check property P_3 for subsets of 3 vertices of the following types subsets containing t, i.e.,

$$\{t, a, b\},\ \{t, a, b^\circ\} \text{ and } \{t, a^\circ, b^\circ\},$$

and subsets not containing t, i.e.,

$$\{a, b, c\},\ \{a, b, c^\circ\},\ \{a, b^\circ, c^\circ\} \text{ and } \{a^\circ, b^\circ, c^\circ\},$$

where $a, b, c \in \mathbb{Z}_p$ and $a^\circ, b^\circ, c^\circ \in \mathbb{Z}_p^\circ$. Using the automorphism of S_p defined in (5.1), we will verify property P_3 for the triplets $\{t, 0, x\}$, $\{t, 0, x^\circ\}$, $\{t, 0^\circ, x^\circ\}$, $\{0, x, y\}$, $\{0, x, y^\circ\}$, $\{0, x^\circ, y^\circ\}$ and $\{0^\circ, x^\circ, y^\circ\}$, where $x, y \in \mathbb{Z}_p$ and $x^\circ, y^\circ \in \mathbb{Z}_p^\circ$.

CASE 1: Triplets containing t.

By Lemma 5.2, $N^{\alpha}_T(0) \cap N^{\beta}_T(x) \neq \varnothing$, $N^{\alpha}_T(0) \cap N^{\beta}_T(x^\circ) \neq \varnothing$ and $N^{\alpha}_T(0^\circ) \cap N^{\beta}_T(x^\circ) \neq \varnothing$ for all permutations of $\alpha, \beta \in \{+, -\}$. Therefore,

$$N^{\gamma}_{\mathsf{S}_p}(t) \cap N^{\alpha}_{\mathsf{S}_p}(0) \cap N^{\beta}_{\mathsf{S}_p}(x) \neq \varnothing$$

for all permutations of $\alpha, \beta, \gamma \in \{+, -\}$.

Case 2: Triplets not containing t.

By Lemma 5.2,

$$\begin{aligned} N^+_{\mathsf{S}_p}(0)\cap N^+_{\mathsf{S}_p}(x) &\subseteq (H_4\cup 2H_4)\cup(H_4^\circ\cup 8H_4^\circ) \text{ and}\\ N^+_{\mathsf{S}_p}(0)\cap N^+_{\mathsf{S}_p}(y) &\subseteq (H_4\cup 2H_4)\cup(H_4^\circ\cup 8H_4^\circ), \end{aligned}$$

hence

$$\begin{aligned} &\left(N^+_{\mathsf{S}_p}(0)\cap N^+_{\mathsf{S}_p}(x)\right)\cup\left(N^+_{\mathsf{S}_p}(0)\cap N^+_{\mathsf{S}_p}(y)\right)\\ &= N^+_{\mathsf{S}_p}(0)\cap\left(N^+_{\mathsf{S}_p}(x)\cup N^+_{\mathsf{S}_p}(y)\right)\subseteq (H_4\cup 2H_4)\cup(H_4^\circ\cup 8H_4^\circ) \end{aligned}$$

and so

$$\left|N^+_{\mathsf{S}_p}(0)\cap\left(N^+_{\mathsf{S}_p}(x)\cup N^+_{\mathsf{S}_p}(y)\right)\right|\le p-1.$$

Let us suppose that $N^+_{\mathsf{S}_p}(0)\cap N^+_{\mathsf{S}_p}(x)\cap N^+_{\mathsf{S}_p}(y)=\varnothing$. Therefore using (4.1)(i),

$$\left|N^+_{\mathsf{S}_p}(0)\cap\left(N^+_{\mathsf{S}_p}(x)\cup N^+_{\mathsf{S}_p}(y)\right)\right| = p-1 \text{ and}$$

$$\begin{aligned} N^+_{\mathsf{S}_p}(0)\cap\left(N^+_{\mathsf{S}_p}(x)\cup N^+_{\mathsf{S}_p}(y)\right) &= (H_4\cup 2H_4)\cup(H_4^\circ\cup 8H_4^\circ)\\ &= N^+_{\mathsf{S}_p}(0)\setminus\{0^\circ\}. \end{aligned}$$

We arrive to a contradiction, $N^+_{\mathsf{S}_p}(0)\cap N^+_{\mathsf{S}_p}(x)\cap N^+_{\mathsf{S}_p}(y)\neq\varnothing$. By Theorem 4.1, we get the result.

□

Let D be a digraph. We define the *converse (opposite) digraph* $\overleftarrow{D}=(V,\overleftarrow{A})$, where $(u,v)\in A$ if and only if $(v,u)\in\overleftarrow{A}$. Thus $\overleftarrow{\mathsf{S}}_p$ is define by $V(\overleftarrow{\mathsf{S}}_p)=\mathbb{Z}_p\cup\mathbb{Z}_p^\circ\cup\{t\}$ and

$$N^+_{\overleftarrow{\mathsf{S}}_p}(a)=N^-_{\mathsf{S}_p}(a),\ N^+_{\overleftarrow{\mathsf{S}}_p}(a^\circ)=N^-_{\mathsf{S}_p}(a^\circ) \text{ and } N^+_{\overleftarrow{\mathsf{S}}_p}(t)=\mathbb{Z}_p^\circ,$$

where $a\in\mathbb{Z}_p$ and $a^\circ\in\mathbb{Z}_p^\circ$. In [9], it is proved that $\overleftarrow{\mathsf{S}}_p$ is also a doubly regular tournament and $Aut(\mathsf{S}_p)=Aut(\overleftarrow{\mathsf{S}}_p)$, but

Theorem 5.4 ([9])**.** $\mathsf{S}_p\ncong\overleftarrow{\mathsf{S}}_p$.

We can now prove another infinite family of tournaments to have property P_3.

Theorem 5.5. *Let $p\equiv 5 \bmod 8$ be a prime number with $p\ge 13$. Then $\overleftarrow{\mathsf{S}}_p$ has property P_3.*

Proof. Analogous to Theorem 5.3. □

We conclude with the following problems.

Problem 5.6. *Is it true that a vertex-transitive doubly regular tournament T has property P_k if and only if it has property S_k?*

Problem 5.7. *Characterize circulant tournaments $\overrightarrow{C}_{2n+1}(J)$ $(n \geq 1)$ having properties P_2 and P_3.*

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