

CONGRUENCE PROPERTIES OF 8- AND 9-COLORED
GENERALIZED FROBENIUS PARTITIONS MODULO 5

DAZHAO TANG

ABSTRACT. In his 1984 AMS Memoir, Andrews introduced the family of functions $c\phi_k(n)$, which denotes the number of k -colored generalized Frobenius partitions of n . In this paper, by employing some q -series identities and elementary generating function manipulations, we prove a characterization of $c\phi_8(5n+2)$ modulo 5. Moreover, we derive a characterization of $c\phi_9(5n+1)$ modulo 5. These two characterizations can lead to the corresponding infinite sets of Ramanujan-type congruences modulo 5 satisfied by $c\phi_8(n)$ and $c\phi_9(n)$.

1. INTRODUCTION

The purpose of this paper is to establish congruence properties for 8- and 9-colored generalized Frobenius partitions modulo 5 by utilizing some q -series manipulations. As an immediate consequence, we obtain some infinite sets of Ramanujan-type congruences modulo 5 satisfied by $c\phi_8(n)$ and $c\phi_9(n)$.

In his 1984 Memoir of the American Mathematical Society, Andrews [1] introduced the notion of a generalized Frobenius partition of n , which is a two-rowed array of nonnegative integers of the form:

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix},$$

wherein each row, which is of the same length, is arranged in weakly decreasing order with $n = r + \sum_{i=1}^r (a_i + b_i)$. Further, Andrews investigated two general classes of generalized Frobenius partitions. One of them is generalized Frobenius partitions whose parts are taken from k copies of the nonnegative integers, which is called k -colored generalized Frobenius partitions. For any

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positive integer k , let $c\phi_k(n)$ denote the number of k -colored generalized Frobenius partitions of n . Among many other things, Andrews [1, Corollary 10.1] proved that for any $n \geq 0$,

$$(1.1) \quad c\phi_2(5n+3) \equiv 0 \pmod{5}.$$

In 1994, Sellers [31] conjectured that there is an infinite family of congruences modulo any powers of 5 enjoyed by $c\phi_2(n)$, which contain (1.1) as the first special case. Eichhorn and Sellers [16] proved several special cases of the Sellers conjecture by modular forms and a computational method. This conjecture was eventually proved by Paule and Radu [30] in 2011.

By employing the theory of modular forms, Ono [29], Lovejoy [27], and Xiong [38] investigated congruence properties modulo small powers of 5 satisfied by $c\phi_3(n)$. Wang [35] proved several infinite families of congruences modulo powers of 3 for $c\phi_3(n)$. In 2011, Baruah and Sarmah [3] derived another expression of the generating function of $c\phi_4(n)$ and thus established several congruences modulo small powers of 4 satisfied by $c\phi_4(n)$. Sellers [32], Xia [37], and Hirschhorn and Sellers [22] considered congruence properties modulo 5 for $c\phi_4(n)$. Congruences modulo 7 for $c\phi_4(n)$ were considered by Lin [26], and Zhang and Wang [39]. Cui, Gu and Huang [13] proved some infinite families modulo small powers of 2 enjoyed by $c\phi_4(n)$. In 2015, Baruah and Sarmah [4] established another expression of the generating function of $c\phi_6(n)$ and proved congruences modulo 4 and 9 for $c\phi_6(n)$. Congruence properties modulo powers of 3 for $c\phi_6(n)$ were successively considered by Xia [36], Hirschhorn [20], Gu, Wang and Xia [19], and the author [33]. Cui and Gu [12] established some infinite families of congruences modulo 8 for $c\phi_6(n)$. There are other congruence properties for $c\phi_k(n)$; see, for example, [14, 15, 23, 24, 28].

In 2017, Wang and Zhang [34] established the following characterization of $c\phi_4(5n+1)$ modulo 5, which contains the results of Sellers [32], Xia [37], and Hirschhorn and Sellers [22] as special cases.

Theorem 1.1 (Wang–Zhang). *For any $n \geq 0$,*

$$(1.2) \quad c\phi_4(5n+1) \equiv \begin{cases} k+1 & \pmod{5} & \text{if } n = 6k^2 + 2k, \ k \in \mathbb{Z}, \\ 0 & \pmod{5} & \text{otherwise.} \end{cases}$$

In 2019, with the help of the theory of modular forms, Chan, Wang and Yang [7] studied $C\Phi_k(q)$ systematically, where $C\Phi_k(q)$ denotes the generating function of $c\phi_k(n)$. In particular, they [7, Theorem 5.3] obtained the following general congruence relation for $c\phi_k(n)$ by induction.

Theorem 1.2 (Chan–Wang–Yang). *Let p be a prime and let N be a positive integer such that $\gcd(p, N) = 1$. For any $\alpha \geq 1$ and $n \geq 0$,*

$$(1.3) \quad c\phi_{p^\alpha N}(n) \equiv c\phi_{p^{\alpha-1}N}(n/p) \pmod{p^{2\alpha}}.$$

The congruence relation (1.3) reveals that there is an inseparable relation on congruence properties between $c\phi_2(n)$ and $c\phi_4(n)$ modulo powers of 2. A

similar congruence relation holds between $c\phi_4(n)$ and $c\phi_8(n)$. Interestingly, (1.1) and (1.2) reveal that the congruence properties modulo 5 of $c\phi_2(n)$ and $c\phi_4(n)$ are still closely related. Therefore, a natural question is whether there exists a similar result modulo 5 for $c\phi_8(n)$. By employing Ramanujan's theta functions and some q -series manipulations, we establish the following characterization of $c\phi_8(5n+2)$ modulo 5.

Theorem 1.3. *For any $n \geq 0$,*

$$(1.4) \quad c\phi_8(5n+2) \equiv \begin{cases} 2k+2 & (\text{mod } 5) & \text{if } n = 12k^2 + 4k, k \in \mathbb{Z}, \\ 3-k & (\text{mod } 5) & \text{if } n = 12k^2 + 8k + 1, k \in \mathbb{Z}, \\ 0 & (\text{mod } 5) & \text{otherwise.} \end{cases}$$

As an immediate consequence, we obtain the following corollary which implies infinitely many Ramanujan-type congruences modulo 5 satisfied by $c\phi_8(n)$.

Corollary 1.4. *Let $p \geq 5$ be a prime and let r be an integer, $1 \leq r \leq p-1$, such that $3r+1$ is a quadratic nonresidue modulo p . Then for any $n \geq 0$,*

$$(1.5) \quad c\phi_8(5pn+5r+2) \equiv 0 \pmod{5}.$$

Utilizing the theory of modular forms, Ono [29, Main Theorem 2] obtained a characterization of $c\phi_3(5n+2)$ modulo 5 (a typo in Ono's original theorem has been corrected).

Theorem 1.5 (Ono). *For any $n \geq 0$,*

$$(1.6) \quad c\phi_3(5n+2) \equiv \begin{cases} (-1)^{k+1}(k+3) & (\text{mod } 5) & \text{if } n = 3k(k+1), k \in \mathbb{N}, \\ 0 & (\text{mod } 5) & \text{otherwise.} \end{cases}$$

According to (1.3), one readily sees that there is a close relation on congruence properties between $c\phi_3(n)$ and $c\phi_9(n)$. Motivated by (1.3) and (1.6), it is natural to ask whether there exists a result similar to (1.6) for $c\phi_9(n)$. With the help of an identity of Kolitsch (see (3.34) below) and the congruence relation (1.6), we establish the following characterization of $c\phi_9(5n+1)$ modulo 5.

Theorem 1.6. *For any $n \geq 0$,*

$$(1.7) \quad c\phi_9(5n+1) \equiv \begin{cases} 2k+1 & (\text{mod } 5) & \text{if } n = 18k^2 + 3k, k \in \mathbb{N}, \\ k-1 & (\text{mod } 5) & \text{if } n = 18k^2 + 9k + 1, k \in \mathbb{N}, \\ 2k & (\text{mod } 5) & \text{if } n = 18k^2 + 15k + 3, k \in \mathbb{N}, \\ -(2k+2) & (\text{mod } 5) & \text{if } n = 18k^2 + 21k + 6, k \in \mathbb{N}, \\ -(k+2) & (\text{mod } 5) & \text{if } n = 18k^2 + 27k + 10, k \in \mathbb{N}, \\ -(2k+1) & (\text{mod } 5) & \text{if } n = 18k^2 + 33k + 15, k \in \mathbb{N}, \\ 0 & (\text{mod } 5) & \text{otherwise.} \end{cases}$$

Similarly, we obtain the following corollary which implies infinitely many congruences modulo 5 satisfied by $c\phi_9(n)$.

Corollary 1.7. *Let $p \geq 3$ be a prime and let r be an integer, $1 \leq r \leq p-1$, such that $8r+1$ is a quadratic nonresidue modulo p . Then for any $n \geq 0$,*

$$(1.8) \quad c\phi_9(5pn + 5r + 1) \equiv 0 \pmod{5}.$$

2. PRELIMINARIES

In this section, we collect some necessary notation and lemmas which will be utilized to prove Theorems 1.3 and 1.6.

Throughout the rest of this paper, we always assume that q is a complex number such that $|q| < 1$ and adopt the following customary notation:

$$(a; q)_\infty := \prod_{k=0}^{\infty} (1 - aq^k).$$

For notational convenience, we denote

$$E(q) := (q; q)_\infty.$$

First, Ramanujan's theta functions $\varphi(q)$ and $\psi(q)$ are defined by

$$(2.1) \quad \varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} \quad \text{and} \quad \psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2}.$$

According to the Jacobi triple product identity [21, Eq. (1.1.1)], one readily obtains the following well-known product representations of $\varphi(q)$ and $\psi(q)$:

$$(2.2) \quad \varphi(q) = \frac{E(q^2)^5}{E(q)^2 E(q^4)^2},$$

$$(2.3) \quad \psi(q) = \frac{E(q^2)^2}{E(q)},$$

Replacing q by $-q$ in (2.3) gives

$$(2.4) \quad \psi(-q) = \frac{E(q)E(q^4)}{E(q^2)}.$$

Next, we need the following identities.

Lemma 2.1.

$$(2.5) \quad \frac{1}{E(q)^4} = \frac{E(q^4)^{14}}{E(q^2)^{14} E(q^8)^4} + 4q \frac{E(q^4)^2 E(q^8)^4}{E(q^2)^{10}},$$

$$(2.6) \quad \frac{1}{E(q)^4} = \frac{E(q^{10})^2}{E(q^2)^2 E(q^5)^4} + 4q \frac{E(q^{10})^5}{E(q)^3 E(q^2) E(q^5)^5}.$$

Proof. The identity (2.5) follows from [21, Eq. (1.10.1)] and (2.2) and (2.3). The identity (2.6) follows from [10, Eq. (2.4)]; see also [2, Eq. (2.6)]. $\square$

Lemma 2.2.

$$(2.7) \quad \frac{E(q^2)^5}{E(q^4)^2} = \sum_{n=-\infty}^{\infty} (6n+1)q^{3n^2+n},$$

$$(2.8) \quad \frac{E(q)^2 E(q^4)^2}{E(q^2)} = \sum_{n=-\infty}^{\infty} (3n+1)q^{3n^2+2n}.$$

The identities (2.7) and (2.8) come from [21, Eqs. (10.7.3) and (10.7.6)]; see also [11, Eqs. (0.47) and (0.48)], which can be derived easily by using the quintuple product identity [21, Eq. (10.1.2)].

Lemma 2.3 ([11, Eq. (0.49)]).

$$(2.9) \quad E(q)^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)/2}.$$

Moreover, we also require the following 5-dissection for the Euler product $E(q)$.

Lemma 2.4 ([21, Eq. (8.1.1)]). *We have*

$$(2.10) \quad E(q) = E(q^{25}) \left(\frac{1}{R(q^5)} - q - q^2 R(q^5) \right),$$

where

$$R(q) = \frac{(q; q^5)_{\infty} (q^4; q^5)_{\infty}}{(q^2; q^5)_{\infty} (q^3; q^5)_{\infty}}.$$

Finally, we will frequently make use of the following congruence, which can be derived immediately from the binomial theorem.

$$E(q^n)^5 \equiv E(q^{5n}) \pmod{5}, \quad n \in \mathbb{N}_+.$$

3. PROOFS OF THE MAIN RESULTS

The proof of Theorem 1.3 relies on the following two lemmas.

Lemma 3.1. *We have*

$$(3.1) \quad \sum_{n=0}^{\infty} c\phi_8(2n)q^n \\ = \frac{1}{E(q)^{20}} \left(\frac{E(q^4)^{55}}{E(q^2)^{14} E(q^8)^{22}} + 892q \frac{E(q^4)^{43}}{E(q^2)^{10} E(q^8)^{14}} \right. \\ \left. + 21408q^2 \frac{E(q^4)^{31}}{E(q^2)^6 E(q^8)^6} + 115584q^3 \frac{E(q^4)^{19} E(q^8)^2}{E(q^2)^2} \right. \\ \left. + 177408q^4 E(q^2)^2 E(q^4)^7 E(q^8)^{10} + 35840q^5 \frac{E(q^2)^6 E(q^8)^{18}}{E(q^4)^5} \right)$$

and

$$\begin{aligned}
 (3.2) \quad & \sum_{n=0}^{\infty} c\phi_8(2n+1)q^n \\
 &= \frac{1}{E(q)^{20}} \left(64 \frac{E(q^4)^{49}}{E(q^2)^{12}E(q^8)^{18}} + 5632q \frac{E(q^4)^{37}}{E(q^2)^8E(q^8)^{10}} \right. \\
 &\quad + 57344q^2 \frac{E(q^4)^{25}}{E(q^2)^4E(q^8)^2} + 172032q^3 E(q^4)^{13}E(q^8)^6 \\
 &\quad \left. + 114688q^4 E(q^2)^4E(q^4)E(q^8)^{14} \right).
 \end{aligned}$$

Proof. According to [7, Theorem 6.3],

$$\begin{aligned}
 (3.3) \quad & \sum_{n=0}^{\infty} c\phi_8(n)q^n = \frac{1}{E(q)^8} \left(\frac{E(q^8)^{35}}{E(q^4)^{14}E(q^{16})^{14}} + 56q \frac{E(q^8)^{29}}{E(q^4)^{12}E(q^{16})^{10}} \right. \\
 &\quad + 420q^2 \frac{E(q^8)^{23}}{E(q^4)^{10}E(q^{16})^6} + 896q^3 \frac{E(q^8)^{17}}{E(q^4)^8E(q^{16})^2} \\
 &\quad + 2352q^4 \frac{E(q^8)^{11}E(q^{16})^2}{E(q^4)^6} + 2688q^5 \frac{E(q^8)^5E(q^{16})^6}{E(q^4)^4} \\
 &\quad \left. + 2240q^6 \frac{E(q^{16})^{10}}{E(q^4)^2E(q^8)} \right).
 \end{aligned}$$

From (2.5), one sees that

$$(3.4) \quad \frac{1}{E(q)^8} = \frac{E(q^4)^{28}}{E(q^2)^{28}E(q^8)^8} + 8q \frac{E(q^4)^{16}}{E(q^2)^{24}} + 16q^2 \frac{E(q^4)^4E(q^8)^8}{E(q^2)^{20}}.$$

Substituting (3.4) into (3.3), we obtain that

$$\begin{aligned}
 (3.5) \quad & \sum_{n=0}^{\infty} c\phi_8(2n)q^n \\
 &= \frac{1}{E(q)^{20}} \left(\frac{E(q^2)^{14}E(q^4)^{27}}{E(q)^8E(q^8)^{14}} + 16q \frac{E(q^4)^{43}}{E(q^2)^{10}E(q^8)^{14}} \right. \\
 &\quad + 448q \frac{E(q^2)^4E(q^4)^{29}}{E(q)^4E(q^8)^{10}} + 420q \frac{E(q^2)^{18}E(q^4)^{15}}{E(q)^8E(q^8)^6} \\
 &\quad + 6720q^2 \frac{E(q^4)^{31}}{E(q^2)^6E(q^8)^6} + 7168q^2 \frac{E(q^2)^8E(q^4)^{17}}{E(q)^4E(q^8)^2} \\
 &\quad + 2352q^2 \frac{E(q^2)^{22}E(q^4)^3E(q^8)^2}{E(q)^8} + 37632q^3 \frac{E(q^4)^{19}E(q^8)^2}{E(q^2)^2} \\
 &\quad + 21504q^3 \frac{E(q^2)^{12}E(q^4)^5E(q^8)^6}{E(q)^4} + 2240q^3 \frac{E(q^2)^{26}E(q^8)^{10}}{E(q)^8E(q^4)^9} \\
 &\quad \left. + 35840q^4 E(q^2)^2E(q^4)^7E(q^8)^{10} \right)
 \end{aligned}$$

and

$$\begin{aligned}
 (3.6) \quad & \sum_{n=0}^{\infty} c\phi_8(2n+1)q^n \\
 &= \frac{1}{E(q)^{20}} \left(8 \frac{E(q^2)^2 E(q^4)^{35}}{E(q)^4 E(q^8)^{14}} + 56 \frac{E(q^2)^{16} E(q^4)^{21}}{E(q)^8 E(q^8)^{10}} \right. \\
 &\quad + 896q \frac{E(q^4)^{37}}{E(q^2)^8 E(q^8)^{10}} + 3360q \frac{E(q^2)^6 E(q^4)^{23}}{E(q)^4 E(q^8)^6} \\
 &\quad + 896q \frac{E(q^2)^{20} E(q^4)^9}{E(q)^8 E(q^8)^2} + 14336q^2 \frac{E(q^4)^{25}}{E(q^2)^4 E(q^8)^2} \\
 &\quad + 18816q^2 \frac{E(q^2)^{10} E(q^4)^{11} E(q^8)^2}{E(q)^4} + 2688q^2 \frac{E(q^2)^{24} E(q^8)^6}{E(q)^8 E(q^4)^3} \\
 &\quad \left. + 43008q^3 E(q^4)^{13} E(q^8)^6 + 17920q^3 \frac{E(q^2)^{14} E(q^8)^{10}}{E(q)^4 E(q^4)} \right).
 \end{aligned}$$

Substituting (2.5) and (3.4) into the parentheses portion of (3.5) and (3.6), after simplification, we obtain (3.1) and (3.2). $\square$

Next, we also need to establish the following auxiliary identities. Before stating them, we first introduce the Atkin U_m -operator, defined by

$$U_m \left(\sum_{n=n_0}^{\infty} a(n)q^n \right) = \sum_{n=\lceil n_0/m \rceil}^{\infty} a(mn)q^n.$$

Lemma 3.2.

(3.7)

$$U_5(q^{-1}E(q^2)E(q^8)^3) = -5q^5 E(q^{10})E(q^{40})^3,$$

(3.8)

$$U_5(E(q^4)^3 E(q^8)) = \frac{E(q^4)^7}{E(q^8)^3} + 5q^4 \frac{E(q^4)^4 E(q^{40})^3}{E(q^8)^2 E(q^{20})} + 20q^8 \frac{E(q^4)E(q^{40})^6}{E(q^8)E(q^{20})^2},$$

(3.9)

$$\begin{aligned}
 U_5 \left(q \frac{E(q^4)}{E(q^2)E(q^8)} \right) &= q \frac{E(q^4)^2 E(q^{10})E(q^{40})}{E(q^2)^2 E(q^8)^2 E(q^{20})} + 5q^3 \frac{E(q^4)^4 E(q^{10})^3 E(q^{40})^3}{E(q^2)^4 E(q^8)^4 E(q^{20})^3} \\
 &\quad + 5q^5 \frac{E(q^4)^6 E(q^{10})^5 E(q^{40})^5}{E(q^2)^6 E(q^8)^6 E(q^{20})^5},
 \end{aligned}$$

(3.10)

$$U_5 \left(q^2 \frac{E(q^2)^3 E(q^8)^2}{E(q^4)} \right) = 5q^4 \frac{E(q^{10})^3 E(q^{40})^2}{E(q^{20})},$$

(3.11)

$$U_5(q^3 E(q^2)^2 E(q^4)^2) = -2q \frac{E(q^2)^7 E(q^{20})}{E(q^4)^3 E(q^{10})} - 15q^3 \frac{E(q^2)^4 E(q^{20})^4}{E(q^4)^2 E(q^{10})^2}$$

$$- 20q^5 \frac{E(q^2)E(q^{20})^7}{E(q^4)E(q^{10})^3}.$$

Proof. We prove (3.7)–(3.11) one by one.

Proof of (3.7). By (2.10), we find that

$$\begin{aligned} U_5(q^{-1}E(q^2)E(q^8)^3) &= U_5\left(q^{-1}E(q^{50})E(q^{200})^3\left(\frac{1}{R(q^{10})} - q^2 - q^4R(q^{10})\right)\right. \\ &\quad \left.\times \left(\frac{1}{R(q^{40})} - q^8 - q^{16}R(q^{40})\right)^3\right) \\ &= E(q^{10})E(q^{40})^3U_5(-5q^{25}) \\ &= -5q^5E(q^{10})E(q^{40})^3. \end{aligned}$$

Proof of (3.8). It follows from (2.10) that

$$\begin{aligned} U_5(E(q^4)^3E(q^8)) &= U_5\left(E(q^{100})^3E(q^{200})\left(\frac{1}{R(q^{20})} - q^4 - q^8R(q^{20})\right)^3\right. \\ &\quad \left.\times \left(\frac{1}{R(q^{40})} - q^8 - q^{16}R(q^{40})\right)\right) \\ &= E(q^{20})^3E(q^{40})\left(\left(\frac{1}{R(q^4)^3R(q^8)} + q^8R(q^4)^3R(q^8)\right)\right. \\ &\quad \left.- 5q^4 + 3q^4\left(\frac{R(q^8)}{R(q^4)^2} - \frac{R(q^4)^2}{R(q^8)}\right)\right). \end{aligned}$$

Thanks to [2, Eqs. (1.20) and (1.21)] (or [8, Theorem 1.1]),

$$(3.12) \quad \frac{R(q^2)}{R(q)^2} - \frac{R(q)^2}{R(q^2)} = 4q \frac{E(q)E(q^{10})^5}{E(q^2)E(q^5)^5},$$

$$(3.13) \quad \frac{1}{R(q)^3R(q^2)} + q^2R(q)^3R(q^2) = \frac{E(q^2)E(q^5)^5}{E(q)E(q^{10})^5} + 2q + 4q^2 \frac{E(q)E(q^{10})^5}{E(q^2)E(q^5)^5}.$$

By means of (3.12) and (3.13),

$$\begin{aligned} (3.14) \quad U_5(E(q^4)^3E(q^8)) &= E(q^{20})^3E(q^{40}) \\ &\quad \times \left(\frac{E(q^8)E(q^{20})^5}{E(q^4)E(q^{40})^5} - 3q^4 + 16q^8 \frac{E(q^4)E(q^{40})^5}{E(q^8)E(q^{20})^5}\right) \\ &= E(q^{20})^3E(q^{40})\left(\frac{E(q^8)E(q^4)^7E(q^{20})^5}{E(q^{40})^5} \cdot \left(\frac{1}{E(q^4)^4}\right)^2\right. \end{aligned}$$

$$-3q^4 E(q^4)^4 \cdot \frac{1}{E(q^4)^4} + 16q^8 \frac{E(q^4)E(q^{40})^5}{E(q^8)E(q^{20})^5} \Bigg).$$

Replacing q by q^4 in (2.6) and plugging it into (3.14), we obtain that

$$U_5(E(q^4)^3 E(q^8)) = \frac{E(q^4)^7}{E(q^8)^3} + 5q^4 \frac{E(q^4)^4 E(q^{40})^3}{E(q^8)^2 E(q^{20})} + 20q^8 \frac{E(q^4)E(q^{40})^6}{E(q^8)E(q^{20})^2}.$$

Proof of (3.9). Fang, Xue and Yao [17, Eq. (1.3)] proved that

$$(3.15) \quad U_5\left(q^{-2} \frac{1}{\psi(-q)}\right) = \frac{E(q^2)^2 E(q^5)E(q^{20})}{E(q)^2 E(q^4)^2 E(q^{10})} + 5q \frac{E(q^2)^4 E(q^5)^3 E(q^{20})^3}{E(q)^4 E(q^4)^4 E(q^{10})^3} \\ + 5q^2 \frac{E(q^2)^6 E(q^5)^5 E(q^{20})^5}{E(q)^6 E(q^4)^6 E(q^{10})^5}.$$

The identity (3.9) follows from (2.4) and (3.15) immediately.

Proof of (3.10). From Corollary (ii) on page 49 in [5],

$$(3.16) \quad \psi(q) = A(q^5) + qB(q^5) + q^3\psi(q^{25}),$$

where

$$A(q) = (-q^2, -q^3, q^5; q^5)_\infty \quad \text{and} \quad B(q) = (-q, -q^4, q^5; q^5)_\infty.$$

Combining (2.10) and (3.16), we deduce that

$$U_5\left(q^2 \frac{E(q^2)^3 E(q^8)^2}{E(q^4)}\right) = U_5\left(q^2 E(q^{50})^3 \left(\frac{1}{R(q^{10})} - q^2 - q^4 R(q^{10})\right)^3 \right. \\ \left. \times (A(q^{20}) + q^4 B(q^{20}) + q^{12} \psi(q^{100}))\right) \\ = 5q^4 \frac{E(q^{10})^3 E(q^{40})^2}{E(q^{20})}.$$

Proof of (3.11). By virtue of (2.10),

$$U_5(q^3 E(q^2)^2 E(q^4)^2) \\ = U_5\left(q^3 E(q^{50})^2 E(q^{100})^2 \left(\frac{1}{R(q^{10})} - q^2 - q^4 R(q^{10})\right)^2 \right. \\ \left. \times \left(\frac{1}{R(q^{20})} - q^4 - q^8 R(q^{20})\right)^2\right) \\ = E(q^{10})^2 E(q^{20})^2 \left(-2q \left(\frac{1}{R(q^2)R(q^4)^2} - q^4 R(q^2)R(q^4)^2\right) \right. \\ \left. + q^3 + 2q^3 \left(\frac{R(q^4)}{R(q^2)^2} - \frac{R(q^2)^2}{R(q^4)}\right)\right).$$

Baruah and Begum [2, Eq. (1.19)] derived that

$$(3.17) \quad \frac{1}{R(q)R(q^2)^2} - q^2 R(q)R(q^2)^2 = \frac{E(q^2)E(q^5)^5}{E(q)E(q^{10})^5}.$$

With the help of (3.12) and (3.17), we further have

$$(3.18) \quad \begin{aligned} & U_5(q^3 E(q^2)^2 E(q^4)^2) \\ &= E(q^{10})^2 E(q^{20})^2 \\ & \quad \times \left(-2q \frac{E(q^4)E(q^{10})^5}{E(q^2)E(q^{20})^5} + q^3 + 8q^5 \frac{E(q^2)E(q^{20})^5}{E(q^4)E(q^{10})^5} \right) \\ &= E(q^{10})^2 E(q^{20})^2 \left(-2q \frac{E(q^2)^7 E(q^4)E(q^{10})^5}{E(q^{20})^5} \cdot \left(\frac{1}{E(q^2)^4} \right)^2 \right. \\ & \quad \left. + q^3 E(q^2)^4 \cdot \frac{1}{E(q^2)^4} + 8q^5 \frac{E(q^2)E(q^{20})^5}{E(q^4)E(q^{10})^5} \right). \end{aligned}$$

Replacing q by q^2 in (2.6) and substituting it into (3.18), after simplification, we obtain the desired identity. $\square$

Now, it is time to prove Theorem 1.3.

In what follows, all congruences are modulo 5 unless otherwise specified.

Proof of Theorem 1.3. According to (3.1) and (3.7)–(3.11), we deduce that

$$(3.19) \quad \begin{aligned} & \sum_{n=0}^{\infty} c\phi_8(10n+2)q^n \\ &= U_5 \left(\sum_{n=0}^{\infty} c\phi_8(2n)q^{n-1} \right) \\ &\equiv \frac{E(q^4)^{11}}{E(q)^4 E(q^2)^3 E(q^8)^5} U_5(q^{-1} E(q^2) E(q^8)^3) \\ & \quad + \frac{2E(q^4)^8}{E(q)^4 E(q^2)^2 E(q^8)^3} U_5(E(q^4)^3 E(q^8)) \\ & \quad - \frac{2E(q^4)^6}{E(q)^4 E(q^2) E(q^8)} U_5 \left(q \frac{E(q^4)}{E(q^2) E(q^8)} \right) \\ & \quad - \frac{E(q^4)^4}{E(q)^4 E(q^2)} U_5 \left(q^2 \frac{E(q^2)^3 E(q^8)^2}{E(q^4)} \right) \\ & \quad - \frac{2E(q^4)E(q^8)^2}{E(q)^4} U_5(q^3 E(q^2)^2 E(q^4)^2) \\ &\equiv \frac{2E(q^4)^8}{E(q)^4 E(q^2)^2 E(q^8)^3} \cdot \frac{E(q^4)^7}{E(q^8)^3} \\ & \quad - \frac{2E(q^4)^6}{E(q)^4 E(q^2) E(q^8)} \cdot q \frac{E(q^4)^2 E(q^{10}) E(q^{40})}{E(q^2)^2 E(q^8)^2 E(q^{20})} \end{aligned}$$

$$\begin{aligned}
& - \frac{2E(q^4)E(q^8)^2}{E(q)^4} \left(-2q \frac{E(q^2)^7 E(q^{20})}{E(q^4)^3 E(q^{10})} \right) \\
& = \frac{2E(q^2)^{12} E(q^4)}{E(q)^4 E(q^8)^2} \left(\frac{E(q^4)^{14}}{E(q^2)^{14} E(q^8)^4} + q \frac{E(q^4)^2 E(q^8)^4}{E(q^2)^{10}} \right) \\
& \equiv \frac{2E(q^2)^{12} E(q^4)}{E(q)^4 E(q^8)^2} \left(\frac{E(q^4)^{14}}{E(q^2)^{14} E(q^8)^4} - 4q \frac{E(q^4)^2 E(q^8)^4}{E(q^2)^{10}} \right).
\end{aligned}$$

Replacing q by $-q$ in (2.5) yields

$$(3.20) \quad \frac{E(q^4)^{14}}{E(q^2)^{14} E(q^8)^4} - 4q \frac{E(q^4)^2 E(q^8)^4}{E(q^2)^{10}} = \frac{E(q)^4 E(q^4)^4}{E(q^2)^{12}}.$$

Combining (3.19) and (3.20),

$$(3.21) \quad \sum_{n=0}^{\infty} c\phi_8(10n+2)q^n \equiv \frac{2E(q^4)^5}{E(q^8)^2}.$$

Thanks to (2.7), we find that

$$\sum_{n=0}^{\infty} c\phi_8(10n+2)q^n \equiv 2 \sum_{m=-\infty}^{\infty} (6m+1)q^{6m^2+2m},$$

from which we obtain that

$$(3.22) \quad c\phi_8(10n+2) \equiv \begin{cases} 2k+2 & \text{if } n = 2k(3k+1) \text{ for some integer } k, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, it follows from (3.2) that

$$\begin{aligned}
(3.23) \quad & \sum_{n=0}^{\infty} c\phi_8(10n+7)q^n \\
& = U_5 \left(\sum_{n=0}^{\infty} c\phi_8(2n+1)q^{n-3} \right) \\
& \equiv - \frac{E(q^4)^{10}}{E(q)^4 E(q^2)^3 E(q^8)^4} U_5 \left(q^{-3} \frac{E(q^2)^3 E(q^8)^2}{E(q^4)} \right) \\
& \quad + \frac{2E(q^4)^7}{E(q)^4 E(q^2)^2 E(q^8)^2} U_5 (q^{-2} E(q^2)^2 E(q^4)^2) \\
& \quad - \frac{E(q^4)^5}{E(q)^4 E(q^2) E(q^8)} U_5 (q^{-1} E(q^2) E(q^8)^3) \\
& \quad + \frac{2E(q^4)^2 E(q^8)}{E(q)^4} U_5 (E(q^4)^3 E(q^8)) \\
& \quad - \frac{2E(q^2) E(q^8)^3}{E(q)^4} U_5 \left(q \frac{E(q^4)}{E(q^2) E(q^8)} \right).
\end{aligned}$$

From Lemma 3.2, we find that

$$(3.24) \quad U_5\left(q^{-3}\frac{E(q^2)^3E(q^8)^2}{E(q^4)}\right) = q^{-1} \cdot U_5\left(q^2\frac{E(q^2)^3E(q^8)^2}{E(q^4)}\right) \equiv 0,$$

$$(3.25) \quad U_5(q^{-2}E(q^2)^2E(q^4)^2) = q^{-1} \cdot U_5(q^3E(q^2)^2E(q^4)^2) \\ \equiv -2\frac{E(q^2)^7E(q^{20})}{E(q^4)^3E(q^{10})},$$

$$(3.26) \quad U_5(q^{-1}E(q^2)E(q^8)^3) \equiv 0,$$

$$(3.27) \quad U_5(E(q^4)^3E(q^8)) \equiv \frac{E(q^4)^7}{E(q^8)^3},$$

$$(3.28) \quad U_5\left(q\frac{E(q^4)}{E(q^2)E(q^8)}\right) \equiv q\frac{E(q^4)^2E(q^{10})E(q^{40})}{E(q^2)^2E(q^8)^2E(q^{20})}.$$

Substituting (3.24)–(3.28) into (3.23), one sees that

$$(3.29) \quad \sum_{n=0}^{\infty} c\phi_8(10n+7)q^n \\ \equiv \frac{-2E(q^2)^{14}E(q^8)^2}{E(q)^4E(q^4)^5} \left(\frac{E(q^4)^{14}}{E(q^2)^{14}E(q^8)^4} + q\frac{E(q^4)^2E(q^8)^4}{E(q^2)^{10}} \right) \\ \equiv \frac{-2E(q^2)^{14}E(q^8)^2}{E(q)^4E(q^4)^5} \left(\frac{E(q^4)^{14}}{E(q^2)^{14}E(q^8)^4} - 4q\frac{E(q^4)^2E(q^8)^4}{E(q^2)^{10}} \right).$$

Combining (3.20) and (3.29) yields

$$(3.30) \quad \sum_{n=0}^{\infty} c\phi_8(10n+7) \equiv \frac{-2E(q^2)^2E(q^8)^2}{E(q^4)}.$$

Thanks to (2.8),

$$\sum_{n=0}^{\infty} c\phi_8(10n+7)q^n \equiv -2 \sum_{m=-\infty}^{\infty} (3m+1)q^{6m^2+4m},$$

from which we obtain

$$(3.31) \quad c\phi_8(10n+7) \equiv \begin{cases} 3-k & \text{if } n = 2k(3k+2) \text{ for some integer } k, \\ 0 & \text{otherwise.} \end{cases}$$

The congruence (1.4) follows by combining (3.22) and (3.31). $\square$

With the help of Theorem 1.3, we can prove Corollary 1.4 easily.

Proof of Corollary 1.4. Let p and r be chosen as in the statement of the corollary. With Theorem 1.3 in mind, we would ask whether there exist integers k_1 and k_2 such that

$$pn + r = 12k_1^2 + 4k_1 \quad \text{or} \quad pn + r = 12k_2^2 + 8k_2 + 1.$$

Notice that

$$3(pn + r) + 1 = (6k_1 + 1)^2 \quad \text{or} \quad 3(pn + r) + 1 = (6k_2 + 2)^2.$$

This implies that $3r+1 \equiv (6k_1+1)^2 \pmod{p}$ or $3r+1 \equiv (6k_2+2)^2 \pmod{p}$. Therefore, if $3r+1$ cannot be congruent to a square modulo p , then $pn+r$ can be represented as neither $12k^2+4k$ nor $12k^2+8k+1$ for any integer k . Now, (1.5) follows from (1.4) immediately. $\square$

Clearly, for any prime $p \geq 5$, the congruence (1.5) gives $(p-1)/2$ different congruences modulo 5 satisfied by $c\phi_8(n)$. Therefore, one actually obtains infinitely many congruences modulo 5 for $c\phi_8(n)$.

Remark: Two remarks on Theorem 1.3 are in order. First, it follows immediately from (3.21) and (3.30) that

$$(3.32) \quad c\phi_8(20n+12) \equiv 0,$$

$$(3.33) \quad c\phi_8(20n+17) \equiv 0.$$

Of course, (3.32) and (3.33) can be derived by (1.4) immediately. Second, there is a congruence modulo 5 for $c\phi_8(n)$ beyond the congruences (1.5), i.e.,

$$c\phi_8(25n+17) \equiv 0,$$

which can be derived by (1.4) along with a little calculation.

The main ingredients in the proof of Theorem 1.6 are Ono's result (1.6) and the following identity due to Kolitsch [25, Theorem 2]:

$$(3.34) \quad \overline{c\phi}_9(n) = 3\overline{c\phi}_3(3n-1),$$

where $n \geq 1$ and

$$(3.35) \quad \overline{c\phi}_m(n) = \sum_{d|\gcd(m,n)} \mu(d) c\phi_{m/d}\left(\frac{n}{d}\right),$$

and where $\mu(x)$ is the Möbius function. It is worthwhile to mention that Kolitsch proved (3.34) by utilizing some combinatorial arguments. Recently, Wang [35, pp. 3378–3379] provided a different proof by employing some q -series techniques. With the help of (3.35), (3.34) is equivalent to

$$(3.36) \quad \sum_{n=0}^{\infty} c\phi_9(n)q^n = 3 \sum_{n=0}^{\infty} c\phi_3(3n-1)q^n + \sum_{n=0}^{\infty} c\phi_3\left(\frac{n}{3}\right)q^n,$$

where we agree that $c\phi_3(-1) = 0$.

Now, we are in a position to prove Theorem 1.6.

Proof of Theorem 1.6. According to (3.36), one sees that

$$\begin{aligned} (3.37) \quad & \sum_{n=0}^{\infty} c\phi_9(5n+1)q^n \\ &= 3 \sum_{n=0}^{\infty} c\phi_3(15n+2)q^n + \sum_{n=0}^{\infty} c\phi_3\left(\frac{5n+1}{3}\right)q^n \\ &= 3 \sum_{n=0}^{\infty} c\phi_3(15n+2)q^n + \sum_{n=0}^{\infty} c\phi_3(5n+2)q^{3n+1}, \end{aligned}$$

where the last step in (3.37) follows by the fact that $c\phi_3(m) = 0$ if m is not a positive integer. Thanks to (1.6) and (2.9), we find that

$$(3.38) \quad \sum_{n=0}^{\infty} c\phi_3(5n+2)q^n \equiv 2E(q^3)^3.$$

Combining (3.37) and (3.38), we conclude that

$$\begin{aligned} & \sum_{n=0}^{\infty} c\phi_9(5n+1)q^n \\ & \equiv E(q)^3 + 2qE(q^9)^3 \\ & = \sum_{n=0}^{\infty} (-1)^n (2n+1)q^{n(n+1)/2} + 2 \sum_{n=0}^{\infty} (-1)^n (2n+1)q^{(3n+1)(3n+2)/2} \\ & = \sum_{j=0}^5 \sum_{n=0}^{\infty} (-1)^{6n+j} (12n+2j+1)q^{(6n+j)(6n+j+1)/2} \\ & \quad + 2 \sum_{n=0}^{\infty} (4n+1)q^{(3n+1)(6n+1)} - 2 \sum_{n=0}^{\infty} (4n+3)q^{(3n+2)(6n+5)}, \end{aligned}$$

from which we obtain (1.7). $\square$

The proof of Corollary 1.7 is similar to that of Corollary 1.4, thus we omit the details here.

Remark: Two remarks are necessary. First, there are also two congruences modulo 5 for $c\phi_9(n)$ outside of the congruences (1.8), namely,

$$\begin{aligned} c\phi_9(15n+11) & \equiv 0, \\ c\phi_9(25n+16) & \equiv 0, \end{aligned}$$

which can be derived by (1.7) together with some calculations. Moreover, (3.36) reveals that there is an inseparable relation on congruence properties modulo powers of 5 between $c\phi_9(n)$ and $c\phi_3(n)$. For example, one can obtain that for any $n \geq 0$,

$$\begin{aligned} c\phi_9(15n+8) & \equiv 0 \pmod{625}, \\ c\phi_9(15n+14) & \equiv 0 \pmod{625}, \end{aligned}$$

which follows immediately from (1.7) and the following two congruences due to Xiong [38, Theorem 1.1]:

$$\begin{aligned} c\phi_3(45n+23) & \equiv 0 \pmod{625}, \\ c\phi_3(45n+41) & \equiv 0 \pmod{625}. \end{aligned}$$

4. FINAL REMARKS

We conclude this paper with several remarks to motivate further investigation.

First, there are some additional congruences modulo small powers of 5 for $c\phi_8(n)$ and $c\phi_9(n)$. The numerical evidence suggests the following congruences:

$$(4.1) \quad c\phi_8(20n + 8) \equiv 0 \pmod{5},$$

$$(4.2) \quad c\phi_8(20n + 16) \equiv 0 \pmod{5},$$

$$(4.3) \quad c\phi_9(75n + 41) \equiv 0 \pmod{25}.$$

Following a similar strategy to that developed in [6], (4.1)–(4.3) could be proved by constructing appropriate modular forms and using Sturm's theorem. It would be of interest to find an elementary proof of (1.6) and (4.1)–(4.3).

Second, the congruences (1.2), (1.4), (1.6) and (1.7) imply that

$$(4.4) \quad \lim_{X \rightarrow \infty} \frac{\#\{0 \leq n < X : c\phi_k(5n + 2) \equiv 0 \pmod{5}\}}{X} = 1.$$

where $k \in \{3, 4, 8, 9\}$. Notice that

$$(4.5) \quad c\phi_1(5n + 4) \equiv 0 \pmod{5} \quad \text{and} \quad c\phi_2(5n + 3) \equiv 0 \pmod{5}.$$

Moreover, Garvan and Sellers [18] proved that if p is prime and $1 \leq r \leq p-1$, then

$$(4.6) \quad c\phi_k(pn + r) \equiv 0 \pmod{p} \implies c\phi_{pN+k}(pn + r) \equiv 0 \pmod{p},$$

where N is a positive integer. Based on (1.3), (4.4) and (4.5), a natural question is whether the following identity holds:

$$(4.7) \quad \lim_{X \rightarrow \infty} \frac{\#\{0 \leq n < X : c\phi_{p^\alpha}(5n + t) \equiv 0 \pmod{5}\}}{X} = 1,$$

where $p \in \{2, 3\}$, p^α is congruent to 3 or 4 modulo 5, $p^\alpha + t \equiv 0 \pmod{5}$ and $t \in \{1, 2\}$. However, preliminary numerical evidence suggests that no results analogous to (1.4) and (1.7) exist for $c\phi_{13}(n)$ and $c\phi_{14}(n)$.

Finally, Choi [9, p. 497] derived that for all but finitely many primes p and any $j, r \geq 1$,

$$\#\{0 \leq n < X : c\phi_k(n) \equiv r \pmod{p^j}\} \gg \sqrt{X}/\log X.$$

It follows immediately from (1.3) that

$$\lim_{X \rightarrow \infty} \frac{\#\{0 \leq n < X : c\phi_{5N}(n) \equiv 0 \pmod{5}\}}{X} \geq \frac{4}{5}.$$

Based on (4.4)–(4.6), it is natural to ask whether the following inequality holds:

$$(4.8) \quad \lim_{X \rightarrow \infty} \frac{\#\{0 \leq n < X : c\phi_k(n) \equiv 0 \pmod{5}\}}{X} \geq \frac{1}{5},$$

where k is congruent to 3 or 4 modulo 5.

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SCHOOL OF MATHEMATICAL SCIENCES

CHONGQING NORMAL UNIVERSITY

CHONGQING 401331, P. R. CHINA

E-mail address: dazhaotang@sina.com; dazhaotang@cqnu.edu.cn