



GENERALIZED COMBINATORIAL IDENTITIES FOR SPLIT $(n + t)$ -COLOR PARTITIONS

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ABSTRACT. This paper studies three generalized q -series combinatorially using split $(n + t)$ -color partitions as a combinatorial tool. This work provides a generalized approach to unify the several combinatorial identities found in the literature. In this process, we obtain several new Rogers–Ramanujan–MacMahon type partition identities.

1. INTRODUCTION

Partitions originally gained traction in 1669 when Leibnitz first brought up this concept in his letter to John Bernoulli asking if he had investigated the number of ways in which a given number can be expressed as a sum of two or more integers. After this, Euler, the true pioneer of the movement, published his book *Introductio in Analysin Infinitorum* in 1748, in which he discussed the partition function and its important properties. The combinatorial interpretations of the following famous Rogers–Ramanujan identities, formulated by MacMahon [18] in terms of restricted partitions, have sparked significant interest among researchers in further exploring these types of q -series identities and have shaped the development of partition theory throughout the twentieth century.

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} = \prod_{n=1}^{\infty} (1 - q^{5n-1})^{-1} (1 - q^{5n-4})^{-1}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q; q)_n} = \prod_{n=1}^{\infty} (1 - q^{5n-2})^{-1} (1 - q^{5n-3})^{-1}.$$

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A q -series (or basic series or Eulerian series) is a series involving coefficients of the form $(a; q)_n = (1 - a)(1 - aq) \dots (1 - aq^{n-1})$, where $(a; q)_0 = 1$ and $|q| < 1$. Various q -series identities have been interpreted combinatorially in the past with the help of many different combinatorial tools such as ordinary partitions, colored partitions, Frobenius partitions, lattice paths, associated lattice paths, etc. To extend the work of MacMahon, authors in [6], [12], [13], [17], and [23] used ordinary partitions to provide combinatorial partition identities associated with Rogers–Ramanujan type identities. After this, Agarwal and Andrews [7] introduced and studied generalized partitions called $(n + t)$ -color partitions. These color partitions have been used to provide combinatorial meaning to several unexplored basic series identities (see, for instance, [2]-[5], [7], [8], [16]). Recently Sonik and Goyal [20] studied the following generalized combinatorial identities using $(n + t)$ -color partitions and weighted lattice paths.

$$(1.1) \quad \sum_{\pi=0}^{\infty} \frac{q^{m\pi^2+t\pi}}{(q^k; q^k)_{\pi}(q^p; q^{2p})_{\pi}}, \quad t \geq 0,$$

$$(1.2) \quad \sum_{\pi=0}^{\infty} \frac{q^{m\pi^2+t\pi}}{(q^k; q^k)_{\pi}(q^p; q^{2p})_{\pi+1}}, \quad t > 0,$$

where $m, k, p \in \mathbb{Z}^+$. Also, Sonik et. al. [21] studied a special case of (1.1) with even k using restricted n -color partitions and weighted lattice paths. This work motivates us to explore a unified approach in the study of Rogers–Ramanujan type identities.

In 2014, Agarwal and Sood [10] introduced split $(n + t)$ -color partitions which is the further generalization of $(n + t)$ -color partitions. Using this new set of partitions, they gave combinatorial meaning to the following two basic functions of Gordon–McIntosh [14] found in 2000.

$$V_0(q) = 1 + 2 \sum_{n=1}^{\infty} \frac{q^{n^2}(-q; q^2)_n}{(q; q^2)_n},$$

$$V_1(q) = \sum_{n=1}^{\infty} \frac{q^{n^2}(-q; q^2)_{n-1}}{(q; q^2)_n}.$$

First, we recall some definitions before stating our main results:

Definition 1.1. *A partition of a positive integer n is a finite nonincreasing sequence of positive integers whose sum is n .*

Example 1.2. *The partitions of 2 are 2, 1 + 1.*

Definition 1.3 (Agarwal and Andrews [7]). *An $(n + t)$ -color partition, $t \geq 0$, is a partition in which a part of size n , $n \geq 0$, can appear in $(n + t)$ different colors represented as $n_1, n_2, n_3, \dots, n_{(n+t)}$.*

Example 1.4. The $(n + 1)$ -color partitions of 2 are

$2_1, 2_1 + 0_1, 2_2, 2_2 + 0_1, 2_3, 2_3 + 0_1, 1_1 + 1_1, 1_1 + 1_1 + 0_1, 1_2 + 1_1, 1_2 + 1_1 + 0_1, 1_2 + 1_2, 1_2 + 1_2 + 0_1.$

Remark 1.5: Note that zeros are allowed only when t is greater than zero. Also, zeros are not allowed to be repeated in any partition. In fact, only one copy of 0 namely 0_t is allowed to appear. For $t = 0$, these partitions are called n -color partitions.

Definition 1.6 (Agarwal and Andrews [7]). The weighted difference of two parts u_a, v_b ($u \geq v$) is defined by $u - v - a - b$ and is denoted by $((u_a - v_b))$.

Definition 1.7 (Agarwal and Sood [10]). Let m_p be a part in an $(n + t)$ -color partition of a nonnegative integer π . We split the color 'p' into two parts- 'the green part' and 'the red part' and denote them by 'g' and 'r' respectively, such that $1 \leq g \leq p, 0 \leq r \leq p - 1$ and $p = g + r$. An $(n + t)$ -color partition in which each part is split in this manner is called a split $(n + t)$ -color partition.

Example 1.8. The split $(n + 1)$ -color partitions of 2 are

$2_3, 2_3 + 0_1, 2_{2+1}, 2_{2+1} + 0_1, 2_{1+2}, 2_{1+2} + 0_1, 2_2, 2_2 + 0_1, 2_{1+1}, 2_{1+1} + 0_1, 2_1, 2_1 + 0_1, 1_2 + 1_2, 1_2 + 1_2 + 0_1, 1_2 + 1_{1+1}, 1_2 + 1_{1+1} + 0_1, 1_2 + 1_1, 1_2 + 1_1 + 0_1, 1_{1+1} + 1_{1+1}, 1_{1+1} + 1_{1+1} + 0_1, 1_{1+1} + 1_1, 1_{1+1} + 1_1 + 0_1, 1_1 + 1_1, 1_1 + 1_1 + 0_1.$

Remark 1.9: If the red part is 0 then it is not written separately. Thus, for example, 2_{3+0} is written as 2_3 .

Definition 1.10. We define the following order on the set of all parts of split $(n + t)$ -color partitions: if $u < v$ then $u_{g'+r'} < v_{g+r}$ irrespective of the subscripts. And, if $u = v$ then $u_{g'+r'} < u_{g+r}$ if and only if $g' + r' < g + r$. Further, if $u = v$ and $g' + r' = g + r$, then $u_{g'+r'} < u_{g+r}$ if and only if $g' < g$. Thus, the parts satisfy the order:

$$1_1 < 1_{1+1} < 1_2 < 1_{1+2} < 1_{2+1} < 1_3 < \cdots < 2_1 < 2_{1+1} < \cdots .$$

Also, [9], [15], [22] explored the potential of split $(n + t)$ -color partitions in interpreting some more q -series which could not be interpreted combinatorially with the help of ordinary or $(n + t)$ -color partitions. Adiga et. al. [1] generalized some of the results due to Agarwal and Sachdeva [9] to interpret the following two q -series using split $(n + t)$ -color partitions, R -weighted lattice paths and modified lattice paths.

$$(1.3) \quad \sum_{\pi=0}^{\infty} \frac{q^{m\pi^2} (-q^l; q^{2l})_{\pi}}{(q^k; q^k)_{\pi} (q^p; q^{2p})_{\pi}},$$

$$(1.4) \quad \sum_{\pi=0}^{\infty} \frac{q^{m\pi^2 + t\pi} (-q^l; q^{2l})_{\pi}}{(q^k; q^k)_{\pi} (q^p; q^{2p})_{\pi+1}},$$

where m, t, l and p be any positive integers with even $k \geq 2$.

Recently Marwah and Goyal [19] also studied these identities partition theoretically and graphically. Literature survey of [1] and [19] has led us to

the question, is it possible to extend interpretations of these series for odd k too, which may lead us to more generalized results? To answer this question and provide generalizations to identities (1.1)-(1.4), we have considered the following three generalized q -series and interpreted them combinatorially with the help of split $(n+t)$ -color partitions in this paper.

Definition 1.11. Let m, l, k, p be any positive integers. For $|q| < 1, 1 \leq i \leq 3$, we define $f_i(q)$ by

$$(1.5) \quad f_1(q) = \sum_{\pi=0}^{\infty} \frac{q^{m\pi^2+t\pi}(-q^l; q^{2l})_{\pi}}{(q^k; q^k)_{\pi}(q^p; q^{2p})_{\pi}},$$

$$(1.6) \quad f_2(q) = \sum_{\pi=0}^{\infty} \frac{q^{m\pi^2-t\pi}(-q^l; q^{2l})_{\pi}}{(q^k; q^k)_{\pi}(q^p; q^{2p})_{\pi}}, \quad m > t,$$

$$(1.7) \quad f_3(q) = \sum_{\pi=0}^{\infty} \frac{q^{m\pi^2+t\pi}(-q^l; q^{2l})_{\pi}}{(q^k; q^k)_{\pi}(q^p; q^{2p})_{\pi+1}},$$

where $t \in \mathbb{Z}^+ \cup \{0\}$ for $f_1(q)$ and $t \in \mathbb{Z}^+$ for $f_2(q)$ and $f_3(q)$.

As we can see by taking $t = 0$ and k any even number in (1.5) and even k in (1.7), we get series (1.3) and (1.4) respectively studied in [1] and [19], which makes (1.3) and (1.4) just a particular case of our generalizations. In Section 3, we have provided the proofs for the combinatorial interpretations of (1.5)–(1.7). In Section 4, we have discussed some particular cases of these generalized series that lead us to Rogers–Ramanujan type identities found in Chu and Zhang compendium [11].

2. Main Results

Theorem 2.1. Let m, l, k, p be any positive integers and $t \in \mathbb{Z}^+ \cup \{0\}$ and let $A_{\mu}(m, t, l, k, p)$ denote the number of split n -color partitions of μ such that

- (i) if x_y is the smallest or the only part in the partition, then $x \equiv y + t \pmod{k}$,
- (ii) the weighted difference of any two consecutive parts is nonnegative and $\equiv 0 \pmod{k}$,
- (iii) the red part is either 0 or l ,
- (iv) the green part is greater than or equal to m and congruent to $m \pmod{p}$.

Then,

$$\sum_{\mu=0}^{\infty} A_{\mu}(m, t, l, k, p)q^{\mu} = f_1(q).$$

Theorem 2.2. Let m, t, l, k, p be any positive integers and $m > t$ and let $B_{\mu}(m, t, l, k, p)$ denote the number of split n -color partitions of μ such that

- (i) if x_y is the smallest or the only part in the partition, then $x \equiv y \pmod{k}$,
- (ii) the weighted difference of any two consecutive parts is greater than or equal to $2t$ and congruent to $2t \pmod{k}$,
- (iii) the red part is either 0 or l ,
- (iv) the green part is greater than or equal to $m - t$ and congruent to $m - t \pmod{p}$.

Then,

$$\sum_{\mu=0}^{\infty} B_{\mu}(m, t, l, k, p)q^{\mu} = f_2(q).$$

Theorem 2.3. Let m, t, l, k, p be any positive integers and let $C_{\mu}(m, t, l, k, p)$ denote the number of split $(n + t)$ -color partitions of μ such that

- (i) the smallest part is of the form $i_{(i+t)}$ with red part equal to 0, where $i = \lambda p$; λ is a nonnegative integer,
- (ii) for all other parts, the green part is greater than or equal to m and congruent to $m \pmod{p}$ and the red part is either 0 or l ,
- (iii) the weighted difference of any two consecutive parts is nonnegative and congruent to 0 \pmod{k} .

Then,

$$\sum_{\mu=0}^{\infty} C_{\mu}(m, t, l, k, p)q^{\mu} = f_3(q).$$

In the proofs we shall write $A(j, \mu)$, $B(j, \mu)$ and $C(j, \mu)$ to denote the number of partitions of μ enumerated by $A_{\mu}(m, t, l, k, p)$, $B_{\mu}(m, t, l, k, p)$ and $C_{\mu}(m, t, l, k, p)$ respectively with the added restriction that there are exactly j parts.

3. PROOFS

3.1. Proof of Theorem 2.1. We split the partitions enumerated by $A(j, \mu)$ into the following four mutually exclusive and exhaustive classes:

- (a) those that do not contain $(i + t)_i$ or $(i + t)_{(i-l)+l}$ as a part, where $i \in \mathbb{Z}^+$,
- (b) those that contain $(m + t)_m$ as a part,
- (c) those that contain $(m + t + l)_{m+l}$ as a part,
- (d) those that contain $(i + t)_i (i > m)$ or $(i + t)_{(i-l)+l} (i > m + l)$ as a part, where $i \in \mathbb{Z}^+$.

We transform the partitions in class (a) by subtracting k from each part and ignoring the subscripts. The transformed partition will be of the type enumerated by $A(j, \mu - kj)$. Next, we transform the partitions in class (b) by deleting the part $(m + t)_m$ and then subtracting $2m$ from each of the remaining parts ignoring the subscripts. The transformed partition will be of the type enumerated by $A(j - 1, \mu - (j - 1)2m - (m + t))$. In class

(c), we transform the partitions by deleting the part $(m+t+l)_{m+l}$ and then subtracting $2(m+l)$ from each of the remaining parts ignoring the subscripts. The transformed partition will be of the type enumerated by $A(j-1, \mu - (j-1)2(m+l) - (m+t+l))$. Finally in class (d), we transform the partitions by replacing $(i+t)_i$ by $(i+t-p)_{(i-p)}$ or $(i+t)_{(i-l)+l}$ by $(i+t-p)_{(i-p-l)+l}$ as the case may be and then subtracting $2p$ from each of the remaining parts ignoring the subscripts. This will produce a partition of $\mu - (j-1)2p - p$ into j parts. Now, here by this transformation, we get only those partitions of $\mu - (j-1)2p - p$ which contain a part of the form $(i+t)_i$ or $(i+t)_{(i-l)+l}$. So, the actual number of partitions in class (d) is $A(j, \mu - (j-1)2p - p) - A(j, \mu - (j-1)2p - p - kj)$, where $A(j, \mu - (j-1)2p - p - kj)$ is the number of partitions of $\mu - (j-1)2p - p$ into j parts that do not contain a part of the form $(i+t)_i$ or $(i+t)_{(i-l)+l}$. The above transformations are reversible and so they establish a bijection between the partitions enumerated by $A(j, \mu)$ and those enumerated by $A(j, \mu - kj) + A(j-1, \mu - (j-1)2m - (m+t)) + A(j-1, \mu - (j-1)2(m+l) - (m+t+l)) + A(j, \mu - (j-1)2p - p) - A(j, \mu - (j-1)2p - p - kj)$.

This leads to the following recurrence relation:

$$(3.1) \quad A(j, \mu) = A(j, \mu - kj) + A(j-1, \mu - (j-1)2m - (m+t)) \\ + A(j-1, \mu - (j-1)2(m+l) - (m+t+l)) \\ + A(j, \mu - (j-1)2p - p) - A(j, \mu - (j-1)2p - p - kj).$$

$$(3.2) \quad h_1(z; q) = \sum_{\mu, j=0}^{\infty} A(j, \mu) z^j q^\mu, \quad \text{where } |q| < 1, |z| < |q|^{-1}.$$

Using (3.1) in (3.2) and then simplifying, we get,

$$(3.3) \quad h_1(z; q) = h_1(zq^k; q) + zq^{m+t}h_1(zq^{2m}; q) + zq^{m+t+l}h_1(zq^{2(m+l)}; q) \\ + q^{-p}h_1(zq^{2p}; q) - q^{-p}h_1(zq^{2p}q^k; q).$$

Now, $h_1(z; q)$ being uniformly convergent for $|q| < 1, |z| < |q|^{-1}$, can be represented as a power series. Let

$$(3.4) \quad h_1(z; q) = \sum_{\pi=0}^{\infty} \alpha_\pi(q) z^\pi, \quad \alpha_0(q) = 1.$$

Using (3.4) in (3.3), we get,

$$\begin{aligned} \sum_{\pi=0}^{\infty} \alpha_{\pi}(q) z^{\pi} &= \sum_{\pi=0}^{\infty} \alpha_{\pi}(q) (zq^k)^{\pi} + zq^{m+t} \sum_{\pi=0}^{\infty} \alpha_{\pi}(q) (zq^{2m})^{\pi} \\ &\quad + zq^{m+t+l} \sum_{\pi=0}^{\infty} \alpha_{\pi}(q) (zq^{2(m+l)})^{\pi} + q^{-p} \sum_{\pi=0}^{\infty} \alpha_{\pi}(q) (zq^{2p})^{\pi} \\ &\quad - q^{-p} \sum_{\pi=0}^{\infty} \alpha_{\pi}(q) (zq^{2p}q^k)^{\pi}. \end{aligned}$$

Comparing coefficients of z^{π} on both sides, we get,

$$(3.5) \quad \alpha_{\pi}(q) = \frac{q^t q^{m(2\pi-1)} (1 + q^{l(2\pi-1)})}{[1 - q^{k\pi} - q^{p(2\pi-1)} (1 - q^{k\pi})]} \alpha_{\pi-1}(q).$$

Iterating (3.5) π times and observing that $\alpha_0(q) = 1$, we find that

$$(3.6) \quad \alpha_{\pi}(q) = \frac{q^{m\pi^2+t\pi} (-q^l; q^{2l})_{\pi}}{(q^k; q^k)_{\pi} (q^p; q^{2p})_{\pi}}.$$

Thus

$$(3.7) \quad h_1(z; q) = \sum_{\pi=0}^{\infty} \frac{q^{m\pi^2+t\pi} (-q^l; q^{2l})_{\pi}}{(q^k; q^k)_{\pi} (q^p; q^{2p})_{\pi}} z^{\pi}.$$

Now

$$\begin{aligned} \sum_{\mu=0}^{\infty} A_{\mu}(m, t, l, k, p) q^{\mu} &= \sum_{\mu=0}^{\infty} \sum_{j=0}^{\infty} A(j, \mu) q^{\mu} = h_1(1; q) = f_1(q) \\ \implies \sum_{\mu=0}^{\infty} A_{\mu}(m, t, l, k, p) q^{\mu} &= f_1(q) = \sum_{\pi=0}^{\infty} \frac{q^{m\pi^2+t\pi} (-q^l; q^{2l})_{\pi}}{(q^k; q^k)_{\pi} (q^p; q^{2p})_{\pi}}. \end{aligned}$$

This proves Theorem 2.1.

Remark 3.1: Note that the combinatorial interpretation of series (1.1) can be obtained by fixing the red part ‘ l ’ equal to 0 in the condition (iii) of Theorem 2.1. So, there are only three classes in the proof, as class (b) and class (c) represent the same set of partitions. The recurrence relation for this case is

$$\begin{aligned} A(j, \mu) &= A(j, \mu - kj) + A(j-1, \mu - (j-1)2m - (m+t)) \\ &\quad + A(j, \mu - (j-1)2p - p) - A(j, \mu - (j-1)2p - p - kj). \end{aligned}$$

3.2. Outline of the proof of Theorem 2.2. We split the partitions enumerated by $B(j, \mu)$ into the following four mutually exclusive and exhaustive classes:

- (a) those that do not contain i_i or $i_{(i-l)+l}$ as a part, where $i \in \mathbb{Z}^+$,
- (b) those that contain $(m-t)_{(m-t)}$ as a part,
- (c) those that contain $(m-t+l)_{(m-t)+l}$ as a part,

- (d) those that contain $i_i(i > m - t)$ or $i_{(i-l)+l}(i > m - t + l)$ as a part, where $i \in \mathbb{Z}^+$.

By the similar arguments as we have given for the classes in the proof of Theorem 2.1, we are directed to the following recurrence relation:

$$\begin{aligned} B(j, \mu) &= B(j, \mu - kj) + B(j - 1, \mu - (j - 1)2m - (m - t)) \\ &\quad + B(j - 1, \mu - (j - 1)2(m + l) - (m - t + l)) \\ &\quad + B(j, \mu - (j - 1)2p - p) - B(j, \mu - (j - 1)2p - p - kj). \end{aligned}$$

Hence, we get the following result,

$$\sum_{\mu=0}^{\infty} B_{\mu}(m, t, l, k, p)q^{\mu} = f_2(q) = \sum_{\pi=0}^{\infty} \frac{q^{m\pi^2 - t\pi}(-q^l; q^{2l})_{\pi}}{(q^k; q^k)_{\pi}(q^p; q^{2p})_{\pi}}.$$

This proves Theorem 2.2.

3.3. Proof of Theorem 2.3. We split the partitions enumerated by $C(j + 1, \mu)$ into those classes that contain $\lambda p_{(\lambda p+t)}$ as the smallest part, where $\lambda \in \mathbb{Z}^+ \cup \{0\}$. These are infinite classes as λ varies from 0 to ∞ . We transform the partitions in these classes by deleting the part $\lambda p_{(\lambda p+t)}$ and then subtracting $2\lambda p$ from each of the remaining parts ignoring the subscripts for each λ . The transformed partition will be of the type enumerated by $A(j, \mu - 2\lambda p j - \lambda p)$ given in Theorem 2.1 for the corresponding value of λ . The above transformations are reversible and so they establish a bijection between the partitions enumerated by $C(j + 1, \mu)$ and those enumerated by $\sum_{\lambda=0}^{\infty} A(j, \mu - 2\lambda p j - \lambda p)$. This leads to the following relation

$$(3.8) \quad C(j + 1, \mu) = \sum_{\lambda=0}^{\infty} A(j, \mu - 2\lambda p j - \lambda p).$$

Let

$$(3.9) \quad h_2(z; \mu) = \sum_{\mu, j=0}^{\infty} C(j + 1, \mu) z^j q^{\mu}, \quad \text{where } |q| < 1, |z| < |q|^{-1}.$$

Now using (3.2),

$$(3.10) \quad \sum_{\mu, j=0}^{\infty} A(j, \mu - 2\lambda p j - \lambda p) z^j q^{\mu} = q^{\lambda p} h_1(z q^{2\lambda p}; q).$$

Using (3.9) and (3.10) in (3.8), we get,

$$(3.11) \quad h_2(z; q) = \sum_{\lambda=0}^{\infty} q^{\lambda p} h_1(z q^{2\lambda p}; q).$$

Now, $h_2(z; q)$ being uniformly convergent for $|q| < 1, |z| < |q|^{-1}$, can be represented as a power series. Let

$$(3.12) \quad h_2(z; q) = \sum_{\pi=0}^{\infty} \beta_{\pi}(q) z^{\pi}; \quad \beta_0(q) = 1.$$

Using (3.12), (3.4) in (3.11), we get,

$$\sum_{\pi=0}^{\infty} \beta_{\pi}(q) z^{\pi} = \sum_{\lambda=0}^{\infty} q^{\lambda p} \sum_{\pi=0}^{\infty} \alpha_{\pi}(q) (z q^{2\lambda p})^{\pi}.$$

Equating coefficients of z^{π} on both sides, we get,

$$\begin{aligned} \beta_{\pi}(q) &= \sum_{\lambda=0}^{\infty} q^{\lambda p(2\pi+1)} \alpha_{\pi}(q) \\ &= \alpha_{\pi}(q) \left[1 + q^{p(2\pi+1)} + q^{2p(2\pi+1)} + q^{3p(2\pi+1)} + \dots \right] \\ &= \frac{\alpha_{\pi}(q)}{(1 - q^{p(2\pi+1)})}. \end{aligned}$$

Using (3.6), we get,

$$\beta_{\pi}(q) = \frac{q^{m\pi^2+t\pi}(-q^l; q^{2l})_{\pi}}{(q^k; q^k)_{\pi}(q^p; q^{2p})_{\pi+1}}.$$

Thus

$$h_2(z; q) = \sum_{\pi=0}^{\infty} \frac{q^{m\pi^2+t\pi}(-q^l; q^{2l})_{\pi}}{(q^k; q^k)_{\pi}(q^p; q^{2p})_{\pi+1}} z^{\pi}.$$

Now

$$\begin{aligned} \sum_{\mu=0}^{\infty} C_{\mu}(m, t, l, k, p) q^{\mu} &= \sum_{\mu=0}^{\infty} \sum_{j=0}^{\infty} C(j+1, \mu) q^{\mu} = h_2(1; q) = f_3(q) \\ \implies \sum_{\mu=0}^{\infty} C_{\mu}(m, t, l, k, p) q^{\mu} &= f_3(q) = \sum_{\pi=0}^{\infty} \frac{q^{m\pi^2+t\pi}(-q^l; q^{2l})_{\pi}}{(q^k; q^k)_{\pi}(q^p; q^{2p})_{\pi+1}}. \end{aligned}$$

This proves Theorem 2.3.

Remark 3.2: Note that the combinatorial interpretation of series (1.2) can be obtained by fixing the red part 'l' equal to 0 in the condition (ii) of Theorem 2.3.

4. ROGERS–RAMANUJAN TYPE IDENTITIES

For some particular values of m, t, l, k , and p , Theorems 2.1, 2.2 and 2.3 enable us to provide new combinatorial meanings to the following six

identities found in Chu and Zhang's compendium [11].

$$(4.1) \quad \sum_{\mu=0}^{\infty} \frac{q^{3\mu^2}(-q; q^2)_{\mu}}{(q^4; q^4)_{\mu}(q^2; q^4)_{\mu}} = \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} [q^{10}, q, q^9; q^{10}]_{\infty} [q^{12}, q^8; q^{20}]_{\infty},$$

$$(4.2) \quad \sum_{\mu=0}^{\infty} \frac{q^{\mu^2+2\mu}(-q; q^2)_{\mu}}{(q^4; q^4)_{\mu}(q^2; q^4)_{\mu}} = \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} [q^{14}, q, q^{13}; q^{14}]_{\infty} [q^{16}, q^{12}; q^{28}]_{\infty},$$

$$(4.3) \quad \sum_{\mu=0}^{\infty} \frac{q^{3\mu^2-2\mu}(-q; q^2)_{\mu}}{(q^4; q^4)_{\mu}(q^2; q^4)_{\mu}} = \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} [q^{10}, q^3, q^7; q^{10}]_{\infty} [q^{16}, q^4; q^{20}]_{\infty},$$

$$(4.4) \quad \sum_{\mu=0}^{\infty} \frac{q^{\mu^2+2\mu}(-q; q^2)_{\mu}}{(q^4; q^4)_{\mu}(q; q^2)_{\mu+1}} = \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} [q^{20}, q^5, q^{15}; q^{20}]_{\infty},$$

$$(4.5) \quad \sum_{\mu=0}^{\infty} \frac{q^{\mu^2+2\mu}(-q; q^2)_{\mu}}{(q^2; q^2)_{\mu}(q^2; q^4)_{\mu+1}} = \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} [q^6, -q^6, -q^6; q^6]_{\infty},$$

$$(4.6) \quad \sum_{\mu=0}^{\infty} \frac{q^{\mu^2}(-q; q^2)_{\mu}}{(q; q)_{\mu}(q^2; q^4)_{\mu}} = \frac{[q^{14}, q^6, q^8; q^{14}]_{\infty}}{(q; q)_{\infty}}.$$

The right-hand side of (4.1)–(4.6) can be easily interpreted as the generating functions for certain restricted ordinary partitions. In comparison, the left-hand side is interpreted according to the generalized results provided in this paper by giving particular values to m, t, l, k , and p . The combinatorial interpretations of (4.1)–(4.6) are stated in the form of the following theorems:

Theorem 4.1. *Let $D_1(\mu) = \sum_{r=0}^{\mu} E_1(\mu-r)F_1(r)$, where $E_1(\mu)$ is the number of ordinary partitions of μ into parts congruent to $\pm 4 \pmod{10}$ and $F_1(\mu)$ denotes the number of ordinary partitions of μ into distinct parts congruent to $\pm 3, 5 \pmod{10}$. Then*

$$A_{\mu}(3, 0, 1, 4, 2) = D_1(\mu).$$

Example 4.2. $A_{12}(3, 0, 1, 4, 2) = 4 = D_1(12) = \sum_{r=0}^{12} Y_1(\mu-r)Z_1(r)$.

Table 1 shows the relevant partitions enumerated by $E_1(r)$, $F_1(r)$ and $A_{\mu}(3, 0, 1, 4, 2)$ for $0 \leq r, \mu \leq 12$.

Theorem 4.3. *Let $D_2(\mu) = \sum_{r=0}^{\mu} E_2(\mu-r)F_2(r)$, where $E_2(\mu)$ is the number of ordinary partitions of μ into parts congruent to $\pm 4, \pm 6 \pmod{14}$ and $F_2(\mu)$ denotes the number of ordinary partitions of μ into distinct parts congruent to $\pm 3, \pm 5, 7 \pmod{14}$. Then*

$$A_{\mu}(1, 2, 1, 4, 2) = D_2(\mu).$$

Theorem 4.4. *Let $D_3(\mu) = \sum_{r=0}^{\mu} E_3(\mu-r)F_3(r)$, where $E_3(\mu)$ is the number of ordinary partitions of μ into parts congruent to $\pm 2 \pmod{10}$ and*

TABLE 1. Partitions enumerated by $E_1(r)$, $F_1(r)$ and $A_\mu(3, 0, 1, 4, 2)$

r	Partitions enumerated by $E_1(r)$	$E_1(r)$	Partitions enumerated by $F_1(r)$	$F_1(r)$
0	Empty Partition	1	Empty Partition	1
1	-	0	-	0
2	-	0	-	0
3	-	0	3	1
4	4	1	-	0
5	-	0	5	1
6	6	1	-	0
7	-	0	7	1
8	4 + 4	1	5 + 3	1
9	-	0	-	0
10	6 + 4	1	7 + 3	1
11	-	0	-	0
12	6 + 6, 4 + 4 + 4	2	7 + 5	1

μ	$D_1(\mu)$	Partitions enumerated by $A_\mu(3, 0, 1, 4, 2)$	$A_\mu(3, 0, 1, 4, 2)$
0	1	Empty Partition	1
1	0	-	0
2	0	-	0
3	1	3_3	1
4	1	4_{3+1}	1
5	1	5_5	1
6	1	6_{5+1}	1
7	2	$7_7, 7_3$	2
8	2	$8_{7+1}, 8_{3+1}$	2
9	2	$9_9, 9_5$	2
10	2	$10_{9+1}, 10_{5+1}$	2
11	3	$11_{11}, 11_7, 11_3$	3
12	4	$12_{11+1}, 12_{7+1}, 12_{3+1}, 9_3 + 3_3$	4

$F_3(\mu)$ denotes the number of ordinary partitions of μ into distinct parts congruent to $\pm 1, 5 \pmod{10}$. Then

$$B_\mu(3, 2, 1, 4, 2) = D_3(\mu).$$

Theorem 4.5. Let $D_4(\mu) = \sum_{r=0}^{\mu} E_4(\mu-r)F_4(r)$, where $E_4(\mu)$ is the number of ordinary partitions of μ into parts congruent to $\pm 2, \pm 4 \pmod{10}$ and $F_4(\mu)$ denotes the number of ordinary partitions of μ into distinct parts congruent to $\pm 1, \pm 3 \pmod{10}$. Then

$$C_\mu(1, 2, 1, 4, 1) = D_4(\mu).$$

Theorem 4.6. Let $D_5(\mu) = \sum_{r=0}^{\mu} E_5(\mu-r)F_5(r)$, where $E_5(\mu)$ is the number of ordinary partitions of μ into parts congruent to $\pm 2 \pmod{6}$ and $F_5(\mu)$ denotes the number of ordinary partitions of μ into distinct parts congruent to $0, 1, 3, 5 \pmod{6}$, where parts congruent to $0 \pmod{6}$ are counted twice. Then

$$C_\mu(1, 2, 1, 2, 2) = D_5(\mu).$$

Theorem 4.7. *Let $D_6(\mu)$ denotes the number of ordinary partitions of μ into parts congruent to $\pm 1, \pm 2, \pm 3, \pm 4, \pm 5, 7 \pmod{14}$. Then*

$$A_\mu(1, 0, 1, 1, 2) = D_6(\mu).$$

5. CONCLUSION

In the present paper, three generalized q -series identities have been studied and interpreted combinatorially with the help of split $(n + t)$ -color partitions which, as their particular cases, give identities studied in [1] and [19]. Also, the main results in this paper give combinatorial meaning to six Rogers–Ramanujan type identities from Chu and Zhang’s compendium [11] as particular cases and explore the utility of split $(n + t)$ -color partitions. These results give an insight into the possibility of future research work to answer questions like:

- Is there any possible combinatorial interpretation of the q -series of the type

$$\sum_{\pi=0}^{\infty} \frac{q^{m\pi^2+t\pi}(-q^{l_1}; q^{2l_1})_{\pi}(-q^{l_2}; q^{2l_2})_{\pi}}{(q^k; q^k)_{\pi}(q^{p_1}; q^{2p_1})_{\pi}(q^{p_2}; q^{2p_2})_{\pi}},$$

where m, t, l_1, l_2, k, p_1 and p_2 be any positive integers?

- Are split $(n + t)$ -color partitions helpful in interpreting the above q -series, or do we need to develop and study some extension of split $(n + t)$ -color partitions?

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