MINIMAL PRIME AGES, WORDS AND PERMUTATION GRAPHS

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ABSTRACT. This paper is a contribution to the study of hereditary classes of finite graphs. We classify these classes according to the number of prime structures they contain. We consider such classes that are *minimal prime*: classes that contain infinitely many primes but every proper hereditary subclass contains only finitely many primes. We give a complete description of such classes. In fact, each one of these classes is a well-quasi-ordered age and there are uncountably many of them. Eleven of these ages are almost multichainable; they remain well-quasi-ordered when labels from a well-quasi-ordering are added, hence have finitely many bounds. Five ages among them are exhaustible. Among the remaining ones, only countably many remain well-quasi-ordered when one label is added, and these have finitely many bounds (except for the age of the infinite path and its complement). The others have infinitely many bounds.

Except for six examples, members of these ages we characterize are permutation graphs. In fact, every age which is not among the eleven ones is the age of a graph associated to a uniformly recurrent word on the integers.

A description of minimal prime classes of posets and bichains is also provided.

Our results hint towards the truth of three conjectures. One stating that if a hereditary class of finite graphs does not remain well-quasi-ordered when adding labels in a well-quasi ordered set to these graphs, then it is not well-quasi-ordered when adding just two constants to each of these graphs.

1. Introduction and presentation of the results

This paper is a contribution to the study of hereditary classes of finite graphs. We classify these classes according to their proper subclasses. With this idea, our simplest classes are those that contain finitely many proper subclasses, hence these classes are finite. At the next level, there are the classes that contain infinitely many proper subclasses, but every proper subclass contains only finitely many. That is such classes are infinite but proper subclasses are finite. It is a simple exercise based on Ramsey's theorem that there are only two such classes: the class of finite cliques and the class of their complements (the class of finite independent sets). Pursuing this idea further, we would like to attach a rank to each class, preferably an ordinal. If we do this, it turns out that a class has a rank if and only if the set of its proper subclasses ordered by set inclusion is well founded. This latter condition amounts to the class being well-quasi-ordered (this follows from Higman's characterization of well-quasi-orders [28]). This puts forward the importance of well-quasi-ordered hereditary classes.

A basic construction of well-quasi-ordered hereditary classes of finite graphs and more generally of finite structures goes as follows: chose a finite hereditary class of finite binary structures and take its closure under lexicographical sums over elements of the class. The fact that this latter class is well-quasi-ordered is a consequence of a theorem of Higman [28] . An important property of such a class is that it contains only finitely many prime structures (see Definition 4.1). A concrete example of such a class is the class of finite cographs (prime structures in this class have cardinality at most two). A natural question then arises: under what conditions a class that contains infinitely many primes is well-quasi-ordered?

Among hereditary classes which contain infinitely many prime members, we show that there are minimal ones with respect to set inclusion (Theorem 4.7). Furthermore, we show that the minimal ones are well-quasi-ordered ages (Theorem 4.11). We obtain some general results that we are able to refine in some special cases like graphs, ordered sets, and bichains. We give a complete description of minimal prime ages of graphs (Theorem 5.22), of posets, and bichains (Corollary 5.24). It turns out that there are $2^{\aleph_0}$ such ages (Corollary 5.14). Eleven of these ages are almost multichainable; they remain well-quasi-ordered when labels in a well-quasi-ordering are added, five being exhaustible. Among the remaining ones, countably many remain well-quasi-ordered when one label is added and these have finitely many bounds (except for the age of the infinite path and its complement). The others have infinitely many bounds (Theorem 5.25).

Except for six examples, members of these ages are permutation graphs. In fact, every age which is not among the eleven ones is the age of a graph associated to a uniformly recurrent word on the integers (this is a consequence of Theorems 5.10, 5.11 and 5.13).

Our classification result hints towards the truthfulness of the following three conjectures.

Conjectures. (1) *If a hereditary class of finite graphs does not remain well-quasi-ordered when adding labels in a well-quasi ordered set to these graphs, then it is not well-quasi-ordered when we add just two constants to each of these graphs.*

(2) *If a hereditary class $\mathcal{C}$ of finite graphs remains well-quasi-ordered when adding labels to these graphs in any well-quasi ordered set, then the set of countable graphs whose age is included in $\mathcal{C}$ is well-quasi-ordered.*

(3) *Every well-quasi-ordered hereditary class of finite graphs is better-quasi ordered in the sense of Nash–Williams [46].*

Our description of minimal prime classes uses a description of minimal prime graphs [60] and previous work by Sobrani [64, 65] and the authors [47, 51] on properties of uniformly recurrent words and the associated graphs. The completeness of our description is based on classification results of Chudnovsky, Kim, Oum and Seymour [19] and by Malliaris and Terry [42].

2. Organisation of the paper

In Section 3 we present some prerequisites on graphs, posets and words. In Section 4 we consider binary relational structures with a finite signature, we give the definition of a minimal prime hereditary class of binary structures and prove their existence. Section 4 contains also the proof of Theorem 4.7 (see Subsection 4.2) and a proof of Theorem 4.11 (see Subsection 4.2.1). In Section 5 we start with the classification results of Chudnovsky, Kim, Oum and Seymour [19] and Malliaris and Terry [42]. Then, we present our main results on minimal prime ages. In Section 5.5 we look at the number of bounds of our minimal prime ages. In Section 6 we provide a proof of Theorem 5.10 and a characterization of order types of realizers of transitive orientations of 0-1 graphs. In Section 7 we characterize the modules of a 0-1 graph. We prove among other things, that if, for a 0-1 word μ, the graph G_μ is not prime, then μ contains large factors of 0's or 1's. Section 8 is devoted to the study of the relation between embeddings of 0-1 words and their corresponding graphs. Results obtained in this section will be used in the proof of Theorem 5.11. In Section 9 we give a proof of Theorem 5.11. Theorem 5.13 is proved in Section 10. We investigate bounds of 0-1 graphs in Section 11, and give a proof of Theorem 5.25.

3. Prerequisites

3.1. **Graphs, posets and relations.** This paper is mostly about graphs and posets. Sometimes, we will need to consider binary relational structures, that is ordered pairs $R := (V, (\rho_i)_{i \in I})$ where each ρ_i is a binary relation or a unary relation on V. The set V, sometimes denoted by $V(R)$, is the *domain* or *base* of R. The sequence $s := (n_i)_{i \in I}$ of arity n_i of ρ_i is the *signature* of R (this terminology is justified since we may identify a unary relation on V,

that is a subset U of V, with the binary relation made of pairs (u, u) such that $u \in U$). We denote by Ω_s the collection of finite structures of signature s. In the sequel we will suppose the signature finite, i.e. I finite. For example, we will consider *bichains*, i.e., relational structures $R := (V, (\leq', \leq''))$ made of a set V and two linear orders $\leq'$ and $\leq''$ on V.

The framework of our study is the theory of relations as developed by Fraïssé and subsequent investigators. At the core is the notion of embeddability, a quasi-order between relational structures. We recall that a relational structure R is *embeddable* in a relational structure R', and we set $R \leq R'$, if R is isomorphic to an induced substructure of R'. Several important notions in the study of these structures, like hereditary classes, ages, bounds, derive from this quasi-order. For example, a class $\mathcal{C}$ of relational structures of signature s is *hereditary* if it contains every relational structure that embeds into a member of $\mathcal{C}$. The *age* of a relational structure R is the class $\text{Age}(R)$ of all finite relational structures, considered up to isomorphy, which embed into R. This is an *ideal* of Ω_s that is a nonempty, hereditary and *up-directed* class $\mathcal{C}$ (any pair of members of $\mathcal{C}$ are embeddable in some element of $\mathcal{C}$). A characterization of ages was given by Fraïssé (see chapter 10 of [25]). Namely, a class $\mathcal{C}$ of finite relational structures is the age of some relational structure if and only if $\mathcal{C}$ is an ideal of Ω_s. We recall that a *bound* of a hereditary class $\mathcal{C}$ of finite relational structures (e.g. graphs, ordered sets), also called *minimal forbidden structure* or *obstruction*, is any relational structure $R \notin \mathcal{C}$ such that every proper induced substructure of R belongs to $\mathcal{C}$. For a wealth of information on these notions see [25].

3.1.1. *Graphs.* Unless otherwise stated, the graphs we consider are undirected, simple and have no loops. That is, a *graph* is a pair $G := (V, E)$, where E is a subset of $[V]^2$, the set of 2-element subsets of V. Elements of V are the *vertices* of G and elements of E its *edges*. The *complement* of G is the graph $\overline{G}$ whose vertex set is V and edge set $\overline{E} := [V]^2 \setminus E$. If A is a subset of V, then the pair $G_{\restriction A} := (A, E \cap [A]^2)$ is the *graph induced by G on A*, also called the graph induced by A in G. A *path* is a graph P such that there exists a one-to-one map f from the set $V(\mathrm{P})$ of its vertices into an interval I of the chain $\mathbb{Z}$ of integers in such a way that $\{u, v\}$ belongs to $E(\mathrm{P})$, the set of edges of P, if and only if $|f(u) - f(v)| = 1$ for every $u, v \in V(\mathrm{P})$. If $I = \{1, \dots, n\}$, then we denote that path by P_n; its *length* is $n - 1$ (so, if $n = 2$, P_2 is made of a single edge, whereas if $n = 1$, P_1 is a single vertex).

3.1.2. *Posets.* Throughout, $P := (V, \leq)$ denotes an ordered set (poset), that is a set V equipped with a binary relation $\leq$ on V which is reflexive, antisymmetric and transitive. We say that two elements $x, y \in V$ are *comparable* if $x \leq y$ or $y \leq x$, otherwise, we say they are *incomparable*. The *dual* of P denoted P^* is the order defined on V as follows: if $x, y \in V$, then $x \leq y$ in P^* if and only if $y \leq x$ in P.

According to Szpilrajn [66], every order $\leq$ on a set V has a *linear extension*, that is a linear (or total) order $\preceq$ on the set V such that $x \preceq y$ whenever $x \leq y$, for all $x, y \in V$. Let $P := (V, \leq)$ be a poset. A *realizer* of P is a family $\mathcal{L}$ of linear extensions of the order of P whose intersection is the order of P. Observe that the set of all linear extensions of P is a realizer of P. The *dimension* of P, denoted $dim(P)$, is the least cardinal d for which there exists a realizer of cardinality d [20]. It follows from the Compactness Theorem of First Order Logic that an order is the intersection of at most n linear orders ($n \in \mathbb{N}$) if and only if every finite restriction of the order has this property. Hence, the class of posets with dimension at most n is determined by a set of finite obstructions, each obstruction is a poset Q of dimension $n+1$ such that the deletion of any vertex of Q leaves a poset of dimension n; such a poset is said to be *critical*. For $n \geq 2$, there are infinitely many critical posets of dimension $n + 1$. For $n = 2$, critical posets of dimension three were characterized by Kelly [32]. For $n \geq 3$, the task of characterizing critical posets of dimension $n + 1$ is considered as hopeless.

3.1.3. *Comparability and incomparability graphs.* The *comparability graph*, respectively the *incomparability graph*, of a poset $P := (V, \leq)$ is the graph, denoted by $\mathrm{Comp}(P)$, respectively $\mathrm{Inc}(P)$, with vertex set V and edges the pairs $\{u, v\}$ of comparable distinct vertices (that is, either $u < v$ or $v < u$), respectively incomparable vertices. A graph $G := (V, E)$ is a *comparability graph* if the edge set is the set of comparabilities of some order on V. From the Compactness Theorem of First Order Logic, it follows that a graph is a comparability graph if and only if every finite induced subgraph is a comparability graph. Hence, the class of comparability graphs is determined by a set of finite obstructions. The complete list of minimal obstructions was determined by Gallai [26] (see [27] for an English translation). The list can also be found in [68] Figures 4(a) and 4(b).

3.1.4. *Permutation graphs.* A graph $G := (V, E)$ is a *permutation graph* if there is a linear order $\leq$ on V and a permutation σ of V such that the edges of G are the pairs $\{x, y\} \in [V]^2$ which are reversed by σ.

Denoting by $\leq_\sigma$ the set of ordered pairs (x, y) such that $\sigma(x) \leq \sigma(y)$, the permutation graph G is the comparability graph of the poset whose order is the intersection of $\leq$ and the dual of $\leq_\sigma$. Hence, a permutation graph is the comparability graph of an order, which is the intersection of two linear orders, that is the comparability graph of an order of dimension at most two [20]. The converse holds if the graph is finite. As it is well known, a finite graph G is a permutation graph if and only if G and $\overline{G}$ are comparability graphs [20]; in particular, a finite graph is a permutation graph if and only if its complement is a permutation graph.

The comparability graph of an infinite order, which is the intersection of two linear orders, is not necessarily a permutation graph. A one way infinite path is a permutation graph, but the complement of this infinite path is not a permutation graph. There are examples of infinite posets which are

the intersection of two linear orders and whose comparability and incomparability graphs are not permutation graphs. For an example see Figure 8. However, via the Compactness Theorem of First Order Logic, an infinite graph is the comparability graph of a poset, which is the intersection of two linear orders, if and only if each finite induced subgraph is a permutation graph (sometimes these graphs are called permutation graphs, while there is no possible permutation involved). For more about permutation graphs, see [35], [69].

3.1.5. *Initial segment and ideal.* An *initial segment* of a poset $P := (V, \leq)$ is any subset I of V such that for every $x \in V$, if $y \in I$ and $x \leq y$, then $x \in I$. An *ideal* is any nonempty initial segment J of P which is up-directed (that is, $x, y \in J$ implies that $x, y \leq z$ for some $z \in J$). If X is a subset of V, the set $\downarrow X := \{y \in V : y \leq x \text{ for some } x \in X\}$ is the least initial segment containing X, we say that it is *generated* by X. If X is a singleton, say $X = \{x\}$, we denote by $\downarrow x$, instead of $\downarrow X$, this initial segment and say that it is *principal*. We denote by $\mathbf{I}(P)$, resp. $\mathbf{Id}(P)$, the set of initial segments, respectively ideals, of P, ordered by set inclusion.

3.2. **Well-quasi-order.** We recall the notions of well-quasi-order and of better-quasi-order; for further details see, e.g., [45]. A poset is *well-founded* if every nonempty subset has some minimal element. Such a poset has a *level decomposition* $(P_\alpha)_{\alpha < h(P)}$ indexed by ordinal numbers. Level P_α is the set of minimal elements of $P \setminus \bigcup\{P_\beta : \beta < \alpha\}$ and $h(P)$, the *height* of P, is the least ordinal α such that $P_\alpha = \emptyset$. The poset is *level-finite* if each level P_α is finite. A quasi-ordered-set (quoset) Q is *well-quasi-ordered* (w.q.o.) if every infinite sequence of elements of Q contains an infinite increasing subsequence. If Q is an ordered set, this amounts to say that every nonempty subset of Q contains finitely many minimal elements (this number being nonzero). Equivalently, Q is w.q.o. if and only if it contains no infinite descending chain and no infinite antichain.

3.2.1. *Labelled classes.* Among classes of structures which are w.q.o. under the embeddability quasi-order some remain w.q.o. when the structures are labelled by the elements of a quasi-order. Precisely, let $\mathcal{C}$ be a class of relational structures, e.g., graphs, posets, etc., and Q be a quasi-ordered set or a poset. If $R \in \mathcal{C}$, a *labelling of R by Q* is any map f from the domain of R into the domain of Q. Let $\mathcal{C} \cdot Q$ denote the collection of (R, f) where $R \in \mathcal{C}$ and $f : V(R) \to V(Q)$ is a labelling. This class is quasi-ordered by the order defined as follow: for every $(R, f), (R', f') \in \mathcal{C} \cdot Q$, $(R, f) \leq (R', f')$ if there exists an embedding $h : V(R) \to V(R')$ such that $f(x) \leq (f' \circ h)(x)$ for all $x \in V(R)$. We say that $\mathcal{C}$ is *very well-quasi-ordered* (vw.q.o. for short) if for every finite Q, the class $\mathcal{C} \cdot Q$ is w.q.o. The class $\mathcal{C}$ is *hereditary w.q.o.* if $\mathcal{C} \cdot Q$ is w.q.o. for every w.q.o. Q. The class $\mathcal{C}$ is *n-w.q.o.* if for every n-element poset Q, the poset $\mathcal{C} \cdot Q$ is w.q.o. The class $\mathcal{C}$ is *n^--w.q.o.* if the class $\mathcal{C}_{n^-}$ of $(R, a_1, \ldots, a_n)$ where $R \in \mathcal{C}$ and $a_1, \ldots, a_n \in V(R)$ is w.q.o.

We do not know if these four notions are different. In the case of posets covered by two chains (that is of width at most two) we proved that they are identical [62].

We will use the notion of hereditary well-quasi-ordering in Theorems 4.3 and 4.13 and the notion of 1^--well-quasi-ordering in Lemma 4.15. We recall the following result (Proposition 2.2 of [50]).

Theorem 3.1. *Provided that the signature s is bounded, the cardinality of bounds of every hereditary and hereditary w.q.o. subclass of Ω_s is bounded.*

3.2.2. *Better-quasi-order.* Prove that some classes of countable structures are w.q.o. under embeddability may require a strengthening of that notion, e.g; the notion of *better-quasi-order* (chapter) (see Subsection 4.1). The operational definition of b.q.o. is not intuitive. Since we are not going to prove properties of b.q.o.'s, we use the following definition, based on the idea of labelling considered above. Let Q be a quasi-ordered set and $Q^{<\omega_1}$, the set of maps $f : \alpha \to Q$, where α is any countable ordinal. If f and g are two such maps, we set $f \leq g$ if there is a one-to-one preserving map h from the domain α into the domain β of g such that $f(\gamma) \leq g(h((\gamma))$ for all $\gamma < \alpha$. This relation is a quasi-order; the quasi-ordered set Q is a *better-quasi-order* if $Q^{<\omega_1}$ is w.q.o. We just recall that b.q.o.'s are w.q.o.'s. As for w.q.o.'s, finite sets and well-ordered sets are b.q.o.'s, finite unions, finite products, subsets and images of b.q.o.s by order preserving maps are b.q.o.'s. But, contrarily to w.q.o.'s, if Q is b.q.o., then the set $\mathbf{I}(Q)$ of initial segments of Q is b.q.o. (see [25] for more). Nash–Williams 1965 [46] p. 700, asserted that "one is inclined to conjecture that most w.q.o. sets which arise in a reasonably 'natural' manner are likely to be b.q.o." It is not known if the answer is positive for hereditary classes of finite graphs. The first classes to consider are probably those which are minimal prime. As we will see later, due to the description, the minimal prime classes of finite graphs are b.q.o.

Instead of labelling ordinals, we may label chains and compare them as above. We will need the following result of Laver [38] (Theorem, page 90).

Theorem 3.2. *The class of countable chains labelled by a b.q.o. is b.q.o.*

3.2.3. *Jónsson posets.*

Definition 3.3. *A poset P is a* Jónsson *poset if it is infinite and every proper initial segment has a strictly smaller cardinality than P.*

Jónsson posets were introduced by Oman and Kearnes [31]. Countable Jónsson posets were studied and described in [51, 57, 6]. We recall (see Proposition 3.1 [6]):

Theorem 3.4. *Let P be a countable poset. The following statements are equivalent.*

(i) P is Jónsson;
(ii) P is well-quasi-ordered and each ideal distinct from P is finite;

(iii) P is level-finite, has height ω, and for each $n < \omega$, there is $m < \omega$ such that each element of height at most n is below every element of height at least m.

Lemma 3.5. *Every infinite well-founded poset P which is level finite contains an initial segment which is Jónsson.*

Proof. We apply Zorn's Lemma to the set $\mathcal{J}$ of infinite initial segments of P included in the first ω-levels. For that, we prove that $\mathcal{J}$ is closed under intersections of nonempty chains. Indeed, let $\mathcal{C}$ be a nonempty chain (with respect to set inclusion) of members of $\mathcal{J}$. Set $J := \cap \mathcal{C}$. Let $n < \omega$, let P_n be the n-th level of P and $\mathcal{C}_n := \{C \cap P_n : C \in \mathcal{C}\}$. The members of $\mathcal{C}_n$ are finite, nonempty and linearly ordered by set inclusion. Hence, $J_n := \cap \mathcal{C}_n$ is nonempty. Since $J = \cup\{J_n : n \in \mathbb{N}\}$, $J \in \mathcal{J}$. $\qquad\square$

Jónsson posets are behind the study of minimal prime hereditary classes (See Theorem 4.6 in Section 4).

3.3. Words. Let Σ be a finite set. A Σ-sequence is any map u from an interval I of the set $\mathbb{Z}$ of integers in Σ. The set I is the *domain* of u. Two Σ-sequences u and u' are *isomorphic* if there is a translation t on $\mathbb{Z}$ mapping the domain I of u onto the domain I' of u' so that $u(i) = u'(t(i))$ for all $i \in I$. If the domain of a Σ-sequence u is $\{0, \ldots, n-1\}$, $\mathbb{N}$, $\mathbb{N}^* := \{0, -1, \ldots, -n \ldots\}$ or $\mathbb{Z}$, the sequence is a *word*. Words appear as representatives of equivalence classes of sequences. Except when their domain is $\mathbb{Z}$, the representatives are unique. The elements of Σ are called *letters* and Σ is the *alphabet*. When the alphabet is $\{0, 1\}$, we use the terminology 0-1 sequences or 0-1 word. If u is a 0-1 sequence with domain I and if I' is a subset of I, the *restriction* of u to I' is denoted by $u_{\restriction I'}$. If I is finite, the sequence u is finite and the *length* of u, denoted $|u|$, is the number of elements of I. We denote by $\square$ the empty sequence. If u is a finite word and v is a word, finite or infinite with domain $\mathbb{N}$, the *concatenation* of u and v is the word uv obtained by writing v after u. If v has domain $\mathbb{N}^*$, the word vu is defined similarly. A word v is a *factor* of u if $u = u_1 v u_2$. This defines an order on the collection Σ^* of finite words, the *factor ordering*.

3.3.1. *Hereditary classes of words.* A subset $\mathcal{C}$ of Σ^* is *hereditary* if it contains every factor of every member of $\mathcal{C}$. In other words, this is an initial segment of Σ^* ordered with the factor ordering. The *age* of a word u is the set $\mathrm{Fac}(u)$ of all its finite factors endowed with the factor ordering. This is a hereditary subset of Σ^*. In fact, if the alphabet is at most countable, a set $\mathcal{C}$ of finite words is the age of a word u if and only if $\mathcal{C}$ is an *ideal* for the factor ordering. Note that the domain of u is not necessarily $\mathbb{N}$.

A nonempty subset $\mathcal{C}$ of Σ^* is *inexhaustible* if it is not reduced to the empty word and if for every $v \in \mathcal{C}$ there is some w such that $vwv \in \mathcal{C}$.

Lemma 3.6. *A hereditary class $\mathcal{C}$ of finite words is inexhaustible if and only if $\mathcal{C}$ is a union of inexhaustible ages.*

Proof. To prove the necessary condition we use the following claims.

Claim 3.7. *If $\mathcal{D}$ is an inexhaustible subset of Σ^*, then $\downarrow \mathcal{D}$ is inexhaustible.*

Proof of Claim 3.7. Let $u \in\downarrow \mathcal{D}$. There exists $u' \in D$ such that u is a factor of u'. We write $u' := u'_1 u u'_2$. Since $\mathcal{D}$ is inexhaustible there exists w' such that $u'w'u' \in D$. Let $w := u'_2 w' u'_1$. The word uwu is a factor of $u'w'u'$ hence is in $\downarrow \mathcal{D}$. $\qquad\square$

Claim 3.8. *If $\mathcal{D}$ is an inexhaustible subset of Σ^*, then for all $u \in \mathcal{D}$ there is a sequence $u_0, \ldots u_n \ldots$ satisfying $u_0 = u$, $u_{n+1} := u_n v_n u_n$, $u_n \in \mathcal{D}$ and $u_{n+1} \in \mathcal{D}$. By construction, the set $E := \{u_n : n \in \mathbb{N}\}$ is an inexhaustible set.*

Claim 3.9. *If $\mathcal{C}$ is inexhaustible and hereditary, then for all $u \in \mathcal{C}$ there exists an inexhaustible age $\mathcal{A}$ such that $u \in \mathcal{A} \subseteq \mathcal{C}$.*

Proof of Claim 3.9. We apply Claim 3.8 with $\mathcal{D} := \mathcal{C}$. The set E defined in Claim 3.8 is inexhaustible. The set $\downarrow E$ is the age of the word u_∞ having the words u_n as prefixes for $n \geq 0$. $\qquad\square$

The sufficient condition is obvious. $\qquad\square$

A word u is *recurrent* if every finite factor occurs infinitely often. This amounts to the fact that $\mathrm{Fac}(u)$ is inexhaustible. In fact:

Theorem 3.10. *Let μ be a nonempty 0-1 sequence on an interval of $\mathbb{Z}$. The following are equivalent.*
- *(i)* $\mathrm{Fac}(\mu)$ *is inexhaustible;*
- *(ii)* μ *is recurrent;*
- *(iii)* *There exists a word ν on $\mathbb{Z}$ so that $\mathrm{Fac}(\mu) = \mathrm{Fac}(\nu) = \mathrm{Fac}(\nu_{\upharpoonright \mathbb{N}}) = \mathrm{Fac}(\nu_{\upharpoonright \mathbb{N}^*})$.*

This is a variation of Proposition 2 in section II-2.3 page 40 of [51]. For results along these lines, see [8].

Let u be a word and p be a nonnegative integer. The word u is *periodic* with *period p* if $u(i) = u(i + p)$ whenever i and $i + p$ belong to the domain of u. The word u is *ultimately periodic* if there is an integer $l > 0$ such that $u(i) = u(i + p)$ for every $i \geq l$ whenever i and $i + p$ belong to the domain of u, that is $u = vw$ where v and w are two nonempty words with w periodic.

A word u is *uniformly recurrent* if for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that each factor $u(p) \ldots u(p+n)$ of length n occurs as a factor of every factor of length m.

The fact that a word is uniformly recurrent can be expressed in terms of properties of its age ordered by the factor ordering.

Theorem 3.11. *Let u be a word with domain $\mathbb{N}$ over a finite alphabet. The following properties are equivalent:*
- *(i)* u *is uniformly recurrent;*
- *(ii)* $\mathrm{Fac}(u)$ *is inexhaustible and well-quasi-ordered;*

(iii) Fac(u) *is a countable Jónsson poset.*

The only nontrivial implication is (ii) $\Rightarrow$ (iii) (see Lemma II-2.5 in [51] p. 47). See Theorem 5 in [61].

Let u and v be two words. The word v is a *prefix* of u if $u = vu'$. This is a *suffix* of u if $u = u''v$. These relations define two orders on the collection Σ^* of finite words, the *prefix* and the *suffix* orders. The prefix order, as well as the suffix order are (ordered) trees (for each $u \in \Sigma^*$, the set of elements below is a chain). A first consequence is that every ideal is a chain. A more significant consequence, is that if $\mathcal{C}$ is an initial segment of Σ^* for one of these orders, then $\mathcal{C}$ is a w.q.o. if and only if it is an union of finitely many chains (indeed, if $\mathcal{C}$ is a w.q.o., then as every w.q.o. this is a finite union of ideals (see [45]). From this we deduce:

Theorem 3.12. *Let u be a word with domain $\mathbb{N}$ over a finite alphabet. Then* Fac(u) *is well-quasi-ordered by the prefix order, respectively, the suffix order, if and only if u is ultimately periodic, respectively, periodic.*

Proof. Suppose that Fac(u) is w.q.o. by the prefix order. Then Fac(u) is a finite union of ideals. For each integer $n \in \mathbb{N}$, the set $\mathrm{pref}_n(u) := \{u_{\restriction[n,m[} : n \leq m\}$ is an ideal. This ideal been included in a finite union of initial segments is included in one of them, and in fact equal. Thus there are only finitely many sets of the form $\mathrm{pref}_n(u)$. If for $n < n'$, $\mathrm{pref}_n(u) = \mathrm{pref}_{n'}(u)$, then the sequences $\mathrm{pref}_n(u)$ and $\mathrm{pref}_{n'}(u)$ give the same word. It follows that u is ultimately periodic. If Fac(u) is w.q.o. by the suffix order, we observe first that u is recurrent. Then we apply Theorem 3.10: there is a word w on $\mathbb{Z}$ such that Fac(w) = Fac(u). With the same argument as above, for $n \in \mathbb{Z}$ the set $\mathrm{suff}_n(w) := \{w_{\restriction[m,n[} : m \leq n\}$ is finite. This ensures that w is periodic. It follows that u is periodic too. The converses of these implications are obvious. $\qquad\square$

Note that if Fac(u) is covered by finitely many chains, then it is b.q.o.

3.3.2. *Bounds of hereditary classes of words.* This notion was defined before in case of hereditary class of finite relational structures.

Let $\mathcal{C}$ be a hereditary class of finite words. A *bound* of $\mathcal{C}$ is any finite word $v \notin \mathcal{C}$ such that every proper factor of v belongs to $\mathcal{C}$. Equivalently, if $v := v_0 \ldots v_{n-1}$, then v is a bound of $\mathcal{C}$ if and only if $v \notin \mathcal{C}$ and the words $v_0 \ldots v_{n-2}$ and $v_1 \ldots v_{n-1}$ belong to $\mathcal{C}$.

The following result is Proposition 3 in section II-2.6 page 54 of [51]. This result was never published. For completeness we include its proof here.

Theorem 3.13. *Let μ be an infinite periodic word of period $p > 0$. Then the bounds of* Fac(μ) *have length at most p.*

Proof. The proof is based on the following remark due to Roland Assous. Namely, a word v is periodic, with period $p > 0$, if and only if every two factors w' and w'' of v, both having length p, contain the same letters and each of these letters occur the same number of times in w' and w''.

We now prove the theorem. Let v be a finite word of length at least $p+1$ so that each factor of length p is a factor of the word μ. It follows from the above remark that the word v is periodic with period p, hence v is of the form $w \ldots w w' = (wn)w'$ where w' is a prefix of w. Since μ is periodic of period p and w is a factor of μ we infer that $w(n+1)$ is a factor of μ, hence v is a factor of μ. Thus v is not a bound of $\mathrm{Fac}(\mu)$.   □

The following is a consequence of Proposition 6 in II-2.6. page 60 of [51]. This result was never published. For completeness we include a proof (in fact two) here.

Theorem 3.14. *Let μ be a uniformly recurrent and nonperiodic word. Then $\mathrm{Fac}(\mu)$ has infinitely many bounds.*

Proof. We give two proofs.
1) Let μ be a uniformly recurrent and nonperiodic word. Suppose for a contradiction that $\mathrm{Fac}(\mu)$ has finitely many bounds and let m be the maximum length of bounds of $\mathrm{Fac}(\mu)$. Let $v \in \mathrm{Fac}(\mu)$ of length at least m. Since μ is uniformly recurrent, $\mathrm{Fac}(\mu)$ is inexhaustible (Theorem 3.11) it contains a word of the form vwv. Since the bounds of $\mathrm{Fac}(\mu)$ have lengths at most m we infer that $\mathrm{Fac}(\mu)$ contains all periodic words of the form $(vw) \ldots (vw)$ and hence contains the set of factors of the infinite periodic word $\mu' := (vw)vw \ldots$. Since $\mathrm{Fac}(\mu)$ is Jónsson (Theorem 3.11) and $\mathrm{Fac}(\mu')$ is infinite, $\mathrm{Fac}(\mu) = \mathrm{Fac}(\mu')$. It follows that μ is periodic. A contradiction.
2) Our second proof is based on the following properties of regular languages.

Claim 3.15. *Let $\mathcal{C}$ be a hereditary class of finite words (ordered by the factor relation). The following two statements are true.*
(i) If $\mathcal{C}$ has a finite number of bounds, then $\mathcal{C}$ is a regular language.
(ii) If $\mathcal{C}$ is an infinite regular language it contains $\mathrm{Fac}(\mu)$ where μ is an infinite periodic word.

Proof of Claim 3.15. Implication (i) is immediate. Indeed, for every finite word v, the set $\uparrow v := \{w : v \text{ is a factor of } w\}$ is regular language (in fact $\uparrow v = \Sigma^* v \Sigma^*$, where Σ is the alphabet). It follows from Kleene's Theorem that the complement of $\uparrow v$, that is $\mathcal{C} \setminus \uparrow v$ is a regular language. If $\mathcal{C}$ has a finite number of bounds, then

$$\mathcal{C} = \bigcup \{\Sigma^* \setminus \uparrow v : v \text{ bound of } \mathcal{C}\}$$

is a finite union of regular languages and is therefore regular.
To prove implication (ii) we use the Pumping Lemma for regular languages [7]. Since $\mathcal{C}$ is a regular language, there are finite words u, v_1, v_2 such that for all $n \in \mathbb{N}$, $v_1 u^n v_2 \in \mathcal{C}$. Let $\mu := uuuu \ldots$. Then $\mathrm{Fac}(\mu) \subseteq \mathcal{C}$.   □

The existence of infinitely many bounds to a nonperiodic and uniformly recurrent word follows from Claim 3.15. Indeed, let μ be a nonperiodic uniformly recurrent word. Then $\mathrm{Fac}(\mu)$ cannot be regular. Otherwise, it follows from the second part of the claim that $\mathrm{Fac}(\mu)$ contains the set of factor of

an infinite periodic word w. Since μ is uniformly recurrent $\mathrm{Fac}(\mu) = \mathrm{Fac}(w)$ and therefore μ is periodic, contradicting our assumption. The required conclusion now follows from the first part of the claim. $\square$

4. Minimal prime hereditary classes

In this section we present the definition and properties of minimal prime hereditary classes of finite binary structures. We introduce the notion of minimal prime structure and we conclude with the notion of almost chainability. Results for graphs, given in the subsequent sections, are more specific. Most of the results presented here were included in chapter 5 of the thesis of the first author [47].

We start with the notion of a module.

Definition 4.1. *Let* $R := (V, (\rho_i)_{i \in I})$ *be a binary relational structure. A* module *of* R *is any subset* A *of* V *such that*

$(x\rho_i a \Leftrightarrow x\rho_i a')$ *and* $(a\rho_i x \Leftrightarrow a'\rho_i x)$ *for all* $a, a' \in A$ *and* $x \notin A$ *and* $i \in I$.

The empty set, the singletons in V and the whole set V are modules and are called *trivial,* (sometimes in the literature, modules are called *interval, autonomous* or *partitive sets*). If R has no nontrivial module, it is called *prime* or *indecomposable.*

For example, if $R := (V, \leq)$ is a chain, its modules are the ordinary intervals of the chain. If $R := (V, (\leq, \leq'))$ is a bichain, then A is a module of R if and only if A is an interval of $(V, \leq)$ and $(V, \leq')$.

The notion of module goes back to Fraïssé [23] and Gallai [26], see also [24]. A fundamental decomposition result of a binary structure into modules was obtained by Gallai [26] for finite binary relations (see [21] for further extensions). We recall the compactness result of Ille [29].

Theorem 4.2. *A binary structure* R *is prime if and only if every finite subset* F *of its domain extends to a finite set* F' *such that* $R_{\restriction F'}$ *is prime.*

We consider the class $\mathrm{Prim}_s := \mathrm{Prim}(\Omega_s)$ of finite binary structures of signature s which are prime. We set $\mathrm{Prim}(\mathcal{C}) := \mathrm{Prim}_s \cap \mathcal{C}$ for every $\mathcal{C} \subseteq \Omega_s$.

We say that a subclass $\mathcal{D}$ of Prim_s is *hereditary* if it contains every member of Prim_s which can be embedded into some member of $\mathcal{D}$.

4.1. **Hereditary classes containing finitely many prime structures.** The following result (see Proposition 5.2 of [48]) improves a result of [1] for hereditary classes of finite permutations.

Theorem 4.3. *Let* $\mathcal{C}$ *be a hereditary class of finite binary structures containing only finitely many prime structures. Then* $\mathcal{C}$ *is hereditarily well-quasi-ordered. In particular,* $\mathcal{C}$ *has finitely many bounds.*

The following result, due to McKay [44] extends Thomassé's result on the well-quasi-order character of the class of countable series-parallel posets [67] which extends the famous Laver's theorem [38] on the well-quasi-order character of the class of countable chains.

Theorem 4.4. *Let $\mathcal{C}$ be a hereditary class of Ω_s. If $\mathrm{Prim}(\mathcal{C})$ is finite, then the collection $\mathcal{C}^{\leq \omega}$ of countable R such that $\mathrm{Age}(R) \subseteq \mathcal{C}$ is well-quasi-ordered by embeddability.*

In fact, McKay obtains a stronger conclusion of Theorem 4.4. If $\mathrm{Prim}(\mathcal{C})$ is finite, and Q is a better-quasi-order, then the class of members of $\mathcal{C}^{\leq \omega}$ labelled by Q is better-quasi-ordered (this implication is false if b.q.o. is replaced by w.q.o.). In particular, if Q is finite, this class is w.q.o. This case follows from Theorem 4.4 above. Indeed, we may view structures labelled by Q as binary structures. In this new class, say $\mathcal{D}$, modules are unchanged, hence there are only finitely many primes and thus the class $\mathcal{D}^{\leq \omega}$ is w.q.o. We will use this observation in the proof of Theorem 4.22 below.

4.2. **Hereditary classes containing infinitely many prime structures.** In this subsection, we report some results included in [47]. We consider hereditary classes containing infinitely many prime structures. We show that each such a class contains one which is minimal with respect to inclusion.

Definition 4.5. *A hereditary class $\mathcal{C}$ of Ω_s is minimal prime if it contains infinitely many prime structures, while every proper hereditary subclass contains only finitely many prime structures.*

This notion appears in the thesis of the first author [47] (see Theorem 5.12, p. 92, and Theorem 5.15, p. 94 of [47]).

Minimal prime classes, when ordered by set inclusion, form an antichain. We have immediately (cf. Théorème 5.14 p. 93 of [47]).

Theorem 4.6. *A hereditary class $\mathcal{C}$ of Ω_s is minimal prime if and only if $\mathrm{Prim}(\mathcal{C})$ is a Jónsson poset that is cofinal in $\mathcal{C}$.*

Proof. Let $\mathcal{C}$ be a minimal prime class. By definition, $\mathrm{Prim}(\mathcal{C})$ is infinite. Let $\mathcal{I}$ be a proper hereditary subclass of $\mathrm{Prim}(\mathcal{C})$. The initial segment $\downarrow \mathcal{I}$ in Ω_s is a proper subclass of $\mathcal{C}$. Hence $\mathcal{I}$ is finite. Thus $\mathrm{Prim}(\mathcal{C})$ is Jónsson. Let $\mathcal{C}' := \downarrow \mathrm{Prim}(\mathcal{C})$. If $\mathcal{C}' \neq \mathcal{C}$, then since $\mathcal{C}$ is minimal prime, $\mathrm{Prim}(\mathcal{C}') = \mathrm{Prim}(\mathcal{C})$ is finite, which is impossible. This proves that the forward implication holds

Conversely, suppose that $\mathrm{Prim}(\mathcal{C})$ is a Jónsson poset which is cofinal in $\mathcal{C}$. Then $\mathcal{C}$ is infinite. If $\mathcal{C}$ is not minimal prime there is a proper hereditary subclass $\mathcal{C}'$ of $\mathcal{C}$ such that $\mathrm{Prim}(\mathcal{C}')$ is infinite. Since $\mathrm{Prim}(\mathcal{C})$ is Jónsson, $\mathrm{Prim}(\mathcal{C}') = \mathrm{Prim}(\mathcal{C})$. Since $\mathcal{C}' = \downarrow \mathrm{Prim}(\mathcal{C}')$ and $\mathrm{Prim}(\mathcal{C})$ is cofinal in $\mathcal{C}$, this yields $\mathcal{C}' = \mathcal{C}$. A contradiction. $\square$

We have the following:

Theorem 4.7. *Every hereditary class of finite binary structures of finite signature, which contains infinitely many prime structures contains a minimal prime hereditary subclass.*

To prove Theorem 4.7 we will need the following lemma that is a special case of Theorem 4.6 of [6].

Lemma 4.8. Prim_s *is level finite.*

Proof. Suppose for a contradiction that there exists an integer $n \geq 0$ such that the level $(\mathrm{Prim}_s)_n$ of Prim_s is infinite and choose n smallest with this property. Define

$$\mathcal{C} := \{R \in \Omega_s : R < S \text{ for some } S \in (\mathrm{Prim}_s)_n\}.$$

Then $\mathcal{C}$ is a hereditary class of Ω_s containing only finitely many prime structures. It follows from Theorem 4.3 that $\mathcal{C}\rceil\nabla$ is hereditary well-quasi-ordered and hence has finitely many bounds. This is not possible since the elements of $\mathrm{Prim}_s(n)$ are bounds of $\mathcal{C}$. □

The proof of Theorem 4.7 goes as follows. Let $\mathcal{C}$ be a hereditary class of Ω_s such that $J := \mathrm{Prim}_s(\mathcal{C})$ is infinite. Since Prim_s is level finite, Lemma 3.5 ensures that J contains an initial segment D which is Jónsson. According to Theorem 4.6, $\downarrow D$ is minimal prime. This completes the proof. □

4.2.1. *Another proof of Lemma 4.8.* We prove the finiteness of the levels of Prim_s via the properties of critical primality. A binary structure R is *critically prime* if it is prime and $R_{\restriction V(R)\setminus\{x\}}$ is not prime for every $x \in V(R)$. Note that $|V(R)|$ has at least four elements. This notion of critical primality was introduced by Schmerl and Trotter [63]. Among results given in their paper, we have the following theorem (this is Theorem 5.9, page 204):

Theorem 4.9. *Let R be a prime binary structure of order $n \geqslant 7$. Then there are distinct $c, d \in V(R)$ such that $R_{\restriction V(R)\setminus\{c,d\}}$ is prime.*

In their paper [63], Schmerl and Trotter characterize critically prime structures within the class of graphs, the class of posets, the class of tournaments, the class of oriented graphs and the class of binary relational structures. The set of critical prime structures ordered by embeddability within each of these classes is a finite union of chains.

Decompose $Prim_s$ into levels; in level i, with $i \leq 2$, are the structures of order zero, one or two.

For structures R in $Prim_s$ of order at least 2, we have the following relationship between the height $h(R)$ in $Prim_s$ and its order, $|V(R)|$ (which is the height of R in Ω_s).

$$(4.1) \qquad\qquad h(R) \leqslant |V(R)| \leqslant 2(h(R) - 1).$$

The first inequality is obvious. For the second, we use induction on $n := h(R) \geq 2$. The basis step $n = 2$ is trivially true. Suppose $n > 2$. Let S be prime such that S embeds in R with $h(S) = n - 1$. From the induction hypothesis, $|V(S)| \leq 2(h(S) - 1) = 2(n - 2)$. According to Theorem 4.9, $|V(R)| - 2 \leq |V(S)|$. Hence $|V(R)| - 2 \leq 2(n - 2)$. Therefore $|V(R)| \leq 2(n - 1)$.

Lemma 4.8 follows from the second inequality in (4.1) since there are only finitely many structures of a given order.

With Theorem 4.2 and 4.7, one gets:

Corollary 4.10. *The age of any infinite prime structure contains a minimal prime age.*

With Lemma 3.5 and Theorem 4.3 we get:

Theorem 4.11. *Every minimal prime hereditary class is the age of some prime structure; furthermore this age is well-quasi-ordered.*

Proof. Let $\mathcal{C}$ be a minimal prime hereditary class. We first prove that it is the age of a prime structure. It follows from Theorem 4.6 that $\mathcal{C} =\downarrow D$ where D is Jónsson. Since D is Jónsson, it is up-directed. Thus $\mathcal{C}$ is an age. Since D is up-directed and countable, it contains a cofinal sequence $R_0 \leq R_1 \leq \ldots < R_n \leq \ldots$. We may define the limit R of these R_n. Since the R_n's are prime, R is prime and $\mathrm{Age}(R) = \mathcal{C}$.

Next we prove that $\mathcal{C}$ is w.q.o. Since D is Jónsson, it is w.q.o. To prove that $\mathcal{C}$ is w.q.o., let $R \in \mathcal{C}$ and consider $\mathcal{C} \setminus (\uparrow \{R\})$. In order to prove that $\mathcal{C}$ is w.q.o. it is enough to prove that $\mathcal{C} \setminus (\uparrow \{R\})$ is w.q.o. by embeddability. Indeed, an antichain that contains R must be in $\mathcal{C} \setminus (\uparrow \{R\})$. Now to prove that $\mathcal{C} \setminus (\uparrow \{R\})$ is w.q.o. we note that since $\mathcal{C} \setminus (\uparrow \{R\})$ is a proper hereditary class in $\mathcal{C}$, it contains only finitely many primes. It follows from Theorem 4.3 that $\mathcal{C} \setminus (\uparrow \{R\})$ is w.q.o. $\qquad\square$

As mentioned in Subsection 3.2.2, it is not known if a w.q.o. hereditary class of finite graphs is b.q.o.

Problem 4.12. *Is every minimal prime hereditary class of finite binary structures b.q.o.?*

It is known that Jónsson posets are b.q.o. [51, 18] but the argument in the proof of Theorem 4.11 does not give the b.q.o. character of the class $\mathcal{C}$. In the case of graphs, minimal prime hereditary classes split into two types. Those which are almost multichainable and those which are the ages of some special graphs. The b.q.o. character of these classes can be obtained by an extension of Higman's theorem to b.q.o. We give below an improvement of Theorem 4.11 based on properties of the kernel of a relational structure.

4.3. **Inexhaustibility, kernel and minimality.** The *kernel* of a relational structure R with domain V is the set

$$\ker(R) := \{x \in V : \mathrm{Age}(R_{\restriction V \setminus \{x\}}) \neq \mathrm{Age}(R)\}.$$

The kernel is an invariant of the age in the sense that if R and R' have the same age, then there is an isomorphism f from $\ker(R)$ onto $\ker(R')$ such that (a) every restriction of f to every finite subset F of $\ker(R)$ extends to every finite superset $\overline{F}$ of F to an embedding of $R_{\restriction \overline{F}}$ in R' and (b) the same property holds for f^{-1}.

An age $\mathcal{A}$ is *inexhaustible*, or has the *disjoint embedding property*, if two arbitrary members of the age can be embedded into a third member in such a way that their domains are disjoint. As it is easy to see, the kernel of a relational structure R is empty if and only if $\mathrm{Age}(R)$ is inexhaustible. We say that an age $\mathcal{C}$ which is not inexhaustible is *exhaustible*. It is *almost inexhaustible* if the kernel of some R with age $\mathrm{Age}(R) = \mathcal{C}$ is finite.

The notion of inexhaustibility was introduced by Fraïssé in the sixties. The notion of kernel was introduced in [51] and studied in several papers [52], [53], and [54] (see Lemme IV-3.1 p. 37), first for structures with finite signature. The general case was considered in [58]. We prove:

Theorem 4.13. *If $\mathcal{C}$ is a minimal prime class of binary structures, then $\mathcal{C}$ is almost inexhaustible.*

In order to prove Theorem 4.13 we recall two facts below. The first one is in [52] see III.1.3, p. 323.

Lemma 4.14. *An element $a \in V(R)$ belongs to $\ker(R)$ if and only if there is some finite subset A of $V(R)$ containing all the images of a by the local automorphisms defined on A.*

We extract the second fact from [51] Corollaire p. 6 in "Caractérisation combinatoire et topologique des âges les plus simples". For reader's convenience, we give a proof.

Lemma 4.15. *Let $R := (V, (\rho_i)_{i \in I})$ be a relational structure made of finitely many binary relations. If $\mathrm{Age}(R)_{1^-} := \{(S, a) : S \in \mathrm{Age}(R), a \in V(S)\}$ is well-quasi-ordered, then $\ker(R)$ is finite.*

Proof. Suppose that $\ker(R)$ is infinite. We built a sequence $(R_{\restriction A_n}, a_n)$ of elements of $\mathrm{Age}(R)_{1^-}$ such that no two members of the sequence have a common extension belonging to $\mathrm{Age}(R)_{1^-}$. In particular these members form an infinite antichain of $\mathrm{Age}(R)_{1^-}$. We pick $a_0 \in \ker(R)$ and A_0 given by Lemma 4.14. Suppose (A_n, a_n) defined for $n < m$, pick $a_m \in V \setminus \cup \bigcup_{n<m} A_n$, select A given by Lemma 4.14 and set $A_m := A \cup \bigcup_{n<m} A_n$. $\qquad\square$

Let $\mathcal{C}$ be a class of finite binary structures $\mathcal{S} := (F, (\rho_i)_{i \in I})$ with a finite signature s. Denote by $\mathcal{C}^{+1}$ the class of $S := (F, (\rho_i)_{i \in I})$ such that there is some $a \in F$ such that $S_{\restriction F \setminus \{a\}} \in \mathcal{C}$.

The following lemma is Proposition 5.32 p. 105 of [47] and Theorem 4.5 page 20 of [16]. A similar fact, but nonexplicit, appears in the proof of Theorem 4.24 p. 267 of [55]. For the reader's convenience, we give a proof.

Lemma 4.16. *Let $\mathcal{C}$ be a hereditary class of binary structures. If the members of $\mathcal{C}$ are not necessarily finite and if these members when labelled by any better-quasi-order form a well-quasi-order, then $\mathcal{C}^{+1}$ has the same property. If $\mathcal{C}$ is made of finite structures and is hereditarily well-quasi-ordered, then $\mathcal{C}^{+1}$ is hereditarily well-quasi-ordered.*

Proof. Let I be such that each $\mathcal{S} \in \mathcal{C}$ is of the form $\mathcal{S} := (F, (\rho_i)_{i \in I})$. Let W be a w.q.o. By hypothesis, the set $(2 \times 2 \times 2)^I$ is finite, hence with the equality ordering it is w.q.o. The direct product W' of W with $(2 \times 2 \times 2)^I$ is w.q.o. We code members of $\mathcal{C}^{+1}$ labelled by W by members of $\mathcal{C}$ labelled by W'. Indeed, for each $\mathcal{S} := (F, (\rho_i)_{i \in I}) \in \mathcal{C}^{+1}$ we select $a \in F$ such that $\mathcal{S}_{\restriction F \setminus \{a\}} \in \mathcal{C}$ and we label $\mathcal{S}_{\restriction F \setminus \{a\}}$ by the map g_a defined for $x \in F \setminus \{a\}$ by $g_a(x) := (\rho_i(a, x), \rho_i(x, a), \rho_i(a, a))_{i \in I}$. Now if f is a labelling of F in W, we associate the labelling f' of $F \setminus \{a\}$ by setting $f' := (f_{\restriction F \setminus \{a\}}, g_a)$. By construction, if $\mathcal{S}, \mathcal{S}' \in \mathcal{C}$, an embedding h from the labelled structure $\mathcal{S}_{\restriction F \setminus \{a\}}$ in the labelled $\mathcal{S}'_{\restriction F' \setminus \{a'\}}$ will extend to an embedding of the labelled structure $\mathcal{S}$ in the labelled structure $\mathcal{S}'$ with a mapped to a'. The conclusion follows. □

As a consequence we obtain:

Corollary 4.17. *Let $R := (V, (\rho_i)_{i \in I})$ be a relational structure made of finitely many binary relations and let $a \in V$. If $\mathrm{Age}(R_{\restriction V \setminus \{a\}})$ is hereditarily well-quasi-ordered, then $\mathrm{Age}(R)$ is hereditarily well-quasi-ordered.*

Proof. If $a \notin \ker(R)$, there is nothing to prove. If $a \in \ker(R)$, we set $\mathcal{C} := \mathrm{Age}(R_{\restriction V \setminus \{a\}})$. We observe that $\mathrm{Age}(R) \subseteq \mathcal{C}^{+1}$ and we apply Lemma 4.16. □

Proof of Theorem 4.13. Let $\mathcal{C}$ be a minimal prime class and R such that $\mathrm{Age}(R) = \mathcal{C}$. Suppose that $\ker(R)$ is nonempty. Let $a \in \ker(R)$. Then $\mathrm{Age}(R_{\restriction V \setminus \{a\}}) \neq \mathrm{Age}(R) = \mathcal{C}$. Since $\mathcal{C}$ is minimal prime, $\mathrm{Age}(R_{\restriction V \setminus \{a\}})$ contains only finitely many primes. Theorem 4.3 asserts that $\mathrm{Age}(R_{\restriction V \setminus \{a\}})$ is hereditarily w.q.o. Corollary 4.17 asserts that $\mathrm{Age}(R)$ is hereditarily w.q.o. Lemma 4.15 asserts that $\ker(R)$ is finite. With that the proof is complete. □

Since each hereditary well-quasi-ordered class has finitely many bounds (Theorem 3.1) we have only countably many exhaustible minimal prime classes.

Corollary 4.18. *There are at most countably many minimal prime classes $\mathcal{C}$ such that $\mathcal{C}$ is exhaustible.*

Problem 4.19. *(1) Is it true that $|\ker(\mathcal{R})| \leq 2$ if $\mathrm{Age}(R)$ is minimal prime?*
(2) Is the number of exhaustible minimal prime ages finite?

As we will see, the answers to questions (1) and (2) are positive if one considers minimal prime classes of graphs. In this case, there are only five examples with a nonempty kernel.

4.4. **Links with an other notion of minimality.**

Definition 4.20. *A binary relational structure R is* minimal prime *if R is prime and R embeds in every induced indecomposable substructure with the same cardinality.*

Several examples of graphs and posets are given in [60].

Problem 4.21. *Is it true that the age of a minimal prime binary structure is necessarily minimal prime?*

Even in the case of graphs we do not know the answer. The converse is false in the sense that there are minimal prime ages of graphs such that no graph with that age is minimal prime. We prove:

Theorem 4.22. *If a hereditary class $\mathcal{C}$ is minimal prime and exhaustible, then every binary prime structure R with $\mathrm{Age}(R) = \mathcal{C}$ embeds a minimal prime structure.*

The proof relies on the following two lemmas which are a consequence of Theorem 4.4 and Lemma 4.16.

Lemma 4.23. *If $\mathcal{C}$ is minimal prime and exhaustible, then $\mathcal{C}^{\leq \omega}$ is well-quasi-ordered.*

Proof. Let R with $\mathrm{Age}(R) = \mathcal{C}$. Pick $a \in \ker(R)$. Let $\mathcal{D} := \mathrm{Age}(R_{\restriction V(R) \setminus \{a\}})$. This age contains only finitely many primes. From Theorem 4.4, $\mathcal{D}^{\leq \omega}$ is well-quasi-ordered. Furthermore, members of $\mathcal{D}^{\leq \omega}$ when labelled by any finite set form a well-quasi-ordered set. According to Lemma 4.16, $(\mathcal{D}^{\leq \omega})^{+1}$ has the same property. Next, $\mathcal{C}^{\leq \omega} \subseteq (\mathcal{D}^{\leq \omega})^{+1}$. Indeed, every member of $\mathcal{C}^{\leq \omega}$ has a copy R' in a countable extension R'' of R having the same age as R hence $\mathrm{Age}(R'_{\restriction V(R') \setminus \{a\}}) \subseteq \mathcal{C}$. Hence, $\mathcal{C}^{\leq \omega}$ is well-quasi-ordered. $\square$

The following lemma is Lemma 5.37 p. 102 of [47].

Lemma 4.24. *Let $\mathcal{C}$ be a hereditary class of Ω_s. If $\mathcal{C}^{\leq \omega}$ is well founded, then every prime member of $\mathcal{C}^{\leq \omega}$, if any, embeds a minimal one.*

Proof of Theorem 4.22. Let R be a prime structure with $\mathrm{Age}(R) = \mathcal{C}$. According to Lemma 4.23, $\mathcal{C}^{\leq \omega}$ is well-quasi-ordered. According to Lemma 4.24, R embeds a minimal prime member. $\square$

4.5. **Primality and almost multichainability.** A relational structure R is *almost multichainable* if its domain V is the disjoint union of a finite set F and a set $L \times K$ where K is a finite set, for which there is a linear order $\leq$ on L, satisfying the following condition:

  • For every local isomorphism h of the chain $C := (L, \leq)$ the map $(h, 1_K)$ extended by the identity on F is a local isomorphism of R (the map $(h, 1_K)$ is defined by $(h, 1_K)(x, y) := (h(x), y)$).

The notion of almost multichainability was introduced in [51] (see [55] for further references and discussions). The special case $|K| = 1$ is the notion of almost chainability introduced by Fraïssé. The use of this notion in relation

with the notion of primality is illustrated in several papers, notably [10], [56].

We recall 1. of Theorem 4.19, p. 265 of [55].

Proposition 4.25. *The age of an almost multichainable structure is hereditarily well-quasi-ordered.*

More is true: the age of every almost multichainable structure is hereditarily b.q.o. In fact:

Proposition 4.26. *If $\mathcal{C}$ is the age of a almost multichainable structure, then the collection $\mathcal{C}^{\leq\omega}$ of countable structures whose ages are included in $\mathcal{C}$ is b.q.o. and is in fact hereditary b.q.o.*

For a complete proof, see [62]. For the reader's convenience, we give a hint. The proof of Proposition 4.25 given in [55] consists to interpret members of the age by words over a finite alphabet and apply Higman's Theorem on words. Here, using the same alphabet, we interpret the countable structures whose ages are included in $\mathcal{C}$ by countable chains labelled by this alphabet. Since the alphabet is b.q.o., Laver's Theorem implies that this collection of labelled chains is b.q.o. Now, if we label these labelled chains by a b.q.o., an other application of Laver's Theorem yields that the resulting class is b.q.o.

With Theorem 3.1, we have:

Theorem 4.27. *If the signature is bounded, the cardinality of bounds of the age of an almost multichainable structure is bounded.*

Applying Lemma 4.24 and Proposition 4.26, we have:

Theorem 4.28. *If a relational structure R is almost multichainable, then $\mathrm{Age}(R)$ is hereditarily well-quasi-ordered. Hence, it has finitely many bounds. Every prime R' with the same age (if any) contains a minimal prime structure.*

Problem 4.29. *If a minimal prime age is 2^--well-quasi-ordered, is it the age of an almost multichainable binary relational structure?*

The answer is positive for graphs. Indeed, the minimal prime ages which are not ages of multichainable graphs are ages of some special graphs, the G_μ's, their ages are not 2^--w.q.o. Some are 1^--w.q.o. (when μ is periodic). For more, see Remark 5.17 and Theorem 5.19.

5. Minimal prime ages of graphs

Our description of minimal prime ages of graphs is based on several results. Firstly a previous description of unavoidable prime graphs in large finite prime graphs due to Chudnovsky, Kim, Oum and Seymour [19], see also Malliaris and Terry [42]. Next a study of graphs associated to 0-1 sequences.

5.1. **Unavoidable prime graphs.** We introduce some finite prime graphs. Fix an integer $n \geq 1$.

- *The half-graph H_n of height n*, is a graph with $2n$ vertices $a_1 \ldots, a_n$, $b_1, \ldots, b_n$ such that a_i is adjacent to b_j if and only if $i \leq j$ and such that $\{a_1 \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ are independent sets (i.e. no two vertices are adjacent).
- *The half split graph H'_n of height n*, is a graph with $2n$ vertices $a_1, \ldots, a_n$, $b_1 \ldots, b_n$ such that a_i is adjacent to b_j if and only if $i \leq j$ and such that $\{a_1 \ldots, a_n\}$ is an independent set (i.e. no two vertices are adjacent) and $\{b_1, \ldots, b_n\}$ is a clique (a graph is a split graph if its vertices can be partitioned into a clique and an independent set).
- Let $H'_{n,I}$ be the graph obtained from H'_n by adding a new vertex adjacent to $a_1, \ldots, a_n$ (and no others). Let H^*_n be the graph obtained from H'_n by adding a new vertex adjacent to a_1 (and no others).
- *The thin spider with n legs* is a graph with $2n$ vertices $a_1, \ldots, a_n$, $b_1 \ldots, b_n$ such that $\{a_1 \ldots, a_n\}$ is an independent set and $\{b_1, \ldots, b_n\}$ is a clique, and a_i is adjacent to b_j if and only if $i = j$. The *thick spider with n legs* is the complement of the thin spider with n legs. In particular, it is a graph with $2n$ vertices $a_1 \ldots, a_n$, $b_1, \ldots, b_n$ such that $\{a_1 \ldots, a_n\}$ is an independent set $\{b_1, \ldots, b_n\}$ is a clique, and a_i is adjacent to b_j if and only if $i \neq j$. A *spider* is a thin spider or a thick spider. In item (4) of Theorem 5.2 we consider the extension of this notion to infinite sets.
- A sequence of distinct vertices $v_0, \ldots, v_m$ in a graph G is called a *chain from a set $I \subseteq V(G)$ to v_m* if $m \geq 2$ is an integer, $v_0, v_1 \in I$, $v_2, \ldots, v_m \notin I$, and for all $i > 0$, v_{i-1} is either the unique neighbor or the unique nonneighbor of v_i in $\{v_0, \ldots, v_{i-1}\}$. The length of a chain $v_0, \ldots, v_m$ is m.

The following is due to Chudnovsky et al. [19]:

Theorem 5.1 (Theorem 1.2 of [19])**.** *For every integer $n \geq 3$ there is N such that every prime graph with at least N vertices contains one of the following graphs or their complements as an induced subgraph.*

(1) *The 1-subdivision of $K_{1,n}$ (denoted by $K^1_{1,n}$).*
(2) *The line graph of $K_{2,n}$.*
(3) *The thin spider with n legs.*
(4) *The half-graph of height n.*
(5) *The graph $H'_{n,I}$.*
(6) *The graph H^*_n.*
(7) *A prime graph induced by a chain of length n.*

In [42], Malliaris and Terry prove an infinitary version of Theorem 5.1 for infinite graphs, then use it to prove Theorem 5.1. Their result is the following.
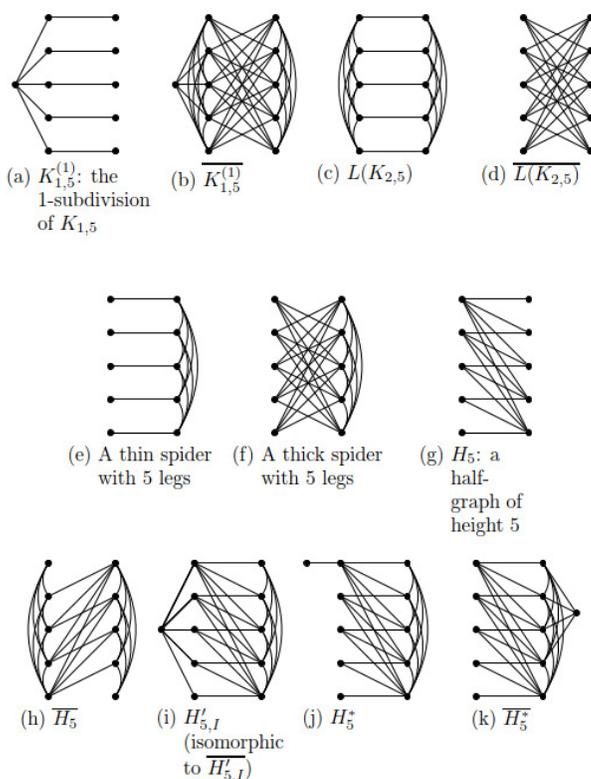
(a) $K_{1,5}^{(1)}$: the 1-subdivision of $K_{1,5}$

(b) $\overline{K_{1,5}^{(1)}}$

(c) $L(K_{2,5})$

(d) $\overline{L(K_{2,5})}$

(e) A thin spider with 5 legs

(f) A thick spider with 5 legs

(g) H_5: a half-graph of height 5

(h) $\overline{H_5}$

(i) $H'_{5,I}$ (isomorphic to $\overline{H'_{5,I}}$)

(j) H_5^*

(k) $\overline{H_5^*}$

FIGURE 1. Unavoidable prime finite graphs (this is Figure 1 from [19])

Theorem 5.2 (Theorem 6.8 of [42]). *An infinite prime graph G contains one of the following.*

(1) *Copies of $H_n, \overline{H_n}, H_n^*, \overline{H_n^*}, H'_{n,I}, \overline{H'_{n,I}}$ for arbitrarily large finite n,*

(2) *Prime graphs induced by arbitrarily long finite chains,*

(3) *$K_{1,\omega}^1$ or its complement,*

(4) *The line graph of $K_{2,\omega}$ or its complement,*

(5) *A spider with ω many legs.*

The graphs mentioned in the last three items and some infinite versions of the graphs in item 1 were considered in [60]. In addition, the following characterization of unavoidable infinite prime graphs without infinite clique (or infinite independent set) was given.

Theorem 5.3 (Theorem 2 of [60]). *An infinite prime graph which does not contain an infinite clique embeds one of the following:*

(1) *The bipartite half-graph of height ω.*

(2) *The infinite one way path.*

(3) *The 1-subdivision of $K_{1,\omega}$.*

(4) *The complement of the line graph of $K_{2,\omega}$.*

The graphs mentioned in Theorem 5.3 are depicted in Figure 2.

5.2. **Eleven almost multichainable graphs and their ages.** Let $\mathcal{M}$ be
the graphs $G_0, G_1, G_3, G_4,$ G_5 and G_6 depicted in Figures 2 and 3. Let $\tilde{\mathcal{M}}$
be the list of these graphs and their complements. Let $\mathcal{L}$ be the set of the
ages of these graphs and of their complements. It should be noted that the
graphs $G_5, \overline{G}_5$ have the same age, hence $\mathcal{L}$ has eleven members.
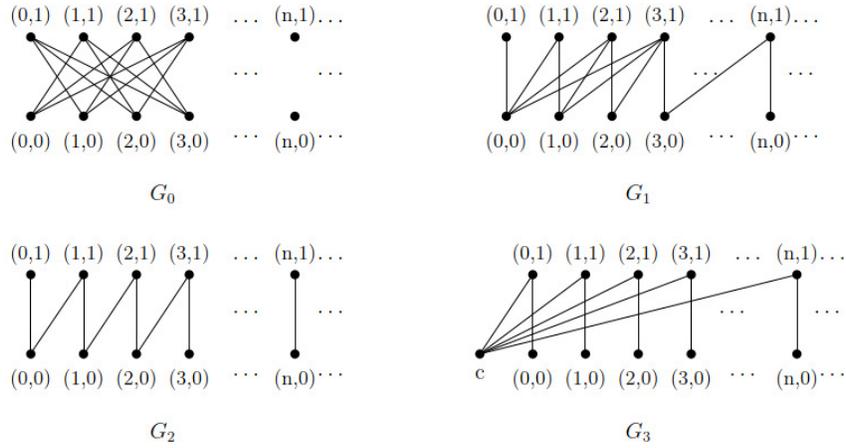

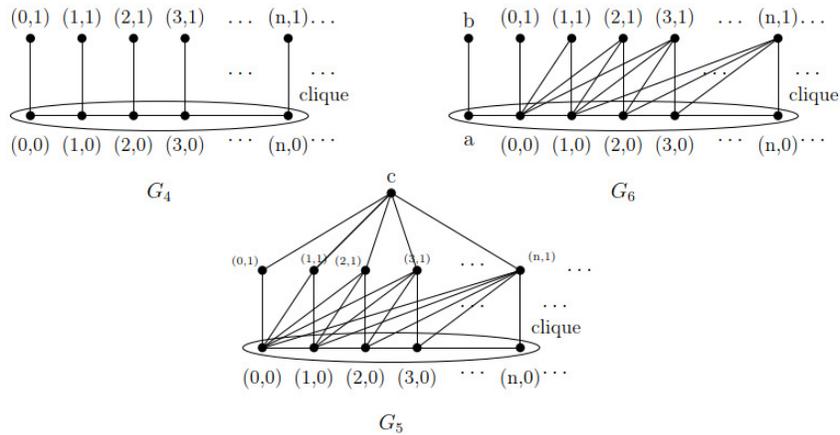FIGURE 2. The graphs listed in Theorem 5.3

FIGURE 3. Prime graphs with large independent sets and
large cliques

Theorem 5.4. *Members of $\mathcal{M}$ and their complements are almost multichainable and minimal prime. Members of $\mathcal{L}$ are distinct and minimal prime, five of them are exhaustible.*

Proof. An inspection of the six members of $\mathcal{M}$ shows that G_0, G_1 and G_4 are multichainable with an empty kernel, the three others are almost multichainable with a one-element kernel, in the case of G_3 and G_5, and a two-element kernel in the case of G_6. This gives three exhaustible ages; with the ages of their complements added (and since G_5 and $\overline{G}_5$ have the same age) this gives five exhaustible ages. The fact that these graphs are minimal prime is given in [60]. The second part of the theorem, notably the fact that the ages are distinct and minimal prime is detailed in chapter 6 page 109 of the first author's thesis [47]. □

The only prime graphs occurring in Theorems 5.1 and 5.2 and not in Theorem 5.4 are chains. Chains can be represented by words on the alphabet $\{0, 1\}$. They will give rise to uncountably many minimal prime ages. We study these graphs and their ages in the next subsection.

5.3. Graphs associated to 0-1 sequences.

Definition 5.5. *To a word μ we associate the graph G_μ whose vertex set $V(G_\mu)$ is $\{-1, 0, \ldots, n-1\}$ if the domain of μ is $\{0, \ldots, n-1\}$, $\{-1\} \cup \mathbb{N}$ if the domain of μ is $\mathbb{N}$, and $\mathbb{N}^*$ or $\mathbb{Z}$ if the domain of μ is $\mathbb{N}^*$ or $\mathbb{Z}$. For two vertices i, j with $i < j$ we let $\{i, j\}$ be an edge of G_μ if and only if*

$$\mu_j = 1 \text{ and } j = i + 1, \text{ or}$$
$$\mu_j = 0 \text{ and } j \neq i + 1.$$

For instance, if μ is the word defined on $\mathbb{N}$ by setting $\mu_i = 1$ for all $i \in \mathbb{N}$, then G_μ is the infinite one way path on $\{-1\} \cup \mathbb{N}$. Note that if μ' is the word defined on $\mathbb{N}$ by setting $\mu'_i = 1$ for all $i \in \mathbb{N} \setminus \{1\}$ and $\mu'_1 = 0$, then $G_{\mu'}$ is also the infinite one way path. In particular, the graphs G_μ and $G_{\mu'}$ have the same age but μ and μ' do not have the same sets of finite factors.

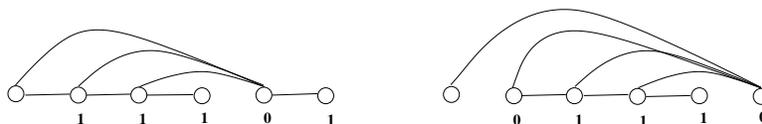FIGURE 4. 0-1 words of length two and their corresponding graphs.

FIGURE 5. Two distinct 0-1 sequences with isomorphic corresponding graphs.

This correspondence between 0-1 words and graphs was first considered in [64], [65]; see also [70] and [19].

Remark 5.6. *If I is an interval of $\mathbb{N}$ and $\mu := (\mu_i)_{i \in I}$ is a 0-1 sequence, then $\overline{G_\mu} = G_{\overline{\mu}}$, where $\overline{\mu} := (\overline{\mu}_i)_{i \in I}$ is the 0-1 sequence defined by $\overline{\mu}(i) := \mu(i) \dot{+} 1$ and $\dot{+}$ is the addition modulo 2.*
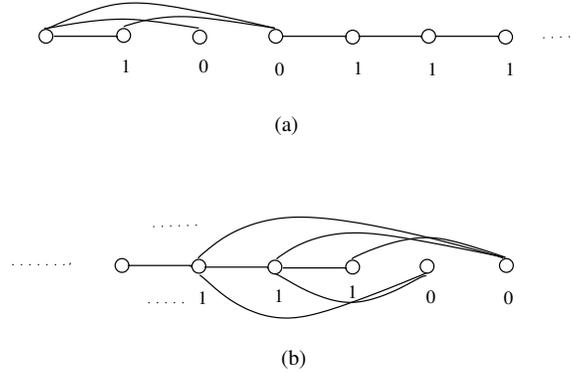

(a)

(b)

FIGURE 6. 0-1 graphs nonrealizable by a sequence on $\mathbb{Z}$.

Remark 5.7. *Given a 0-1 graph defined on $\mathbb{N} \cup \{-1\}$ or on $\mathbb{N}^*$ there does not exist necessarily a 0-1 graph on $\mathbb{Z}$ with the same age.*

Indeed, (a) Let $\mu := 100111\ldots$ be an infinite word on $\mathbb{N}$ (the corresponding graph is depicted in (a) of Figure 6). There does not exist a word μ' on $\mathbb{N}^*$ or $\mathbb{Z}$ such that $\mathrm{Age}(G_\mu) = \mathrm{Age}(G_{\mu'})$.
(b) Let $\nu := \ldots 11100$ be an infinite word on $\mathbb{N}^*$ (the corresponding graph is depicted in (b) of Figure 6). There does not exist a word ν' on $\mathbb{N}$ or $\mathbb{Z}$ such that $\mathrm{Age}(G_\nu) = \mathrm{Age}(G_{\nu'})$.

Proof of (a). Every vertex of the graph G_μ has finite degree. Suppose for a contradiction that there exists a word μ' on $\mathbb{N}^*$ or $\mathbb{Z}$ such that $\mathrm{Age}(G_\mu) = \mathrm{Age}(G_{\mu'})$. Then there exists $i \in \mathbb{Z}$ such that $\mu'(i) = 0$ because otherwise $G_{\mu'}$ would be a path and hence $\mathrm{Age}(G_\mu) \neq \mathrm{Age}(G_{\mu'})$. But then the vertex i of $G_{\mu'}$ would have infinite degree which is impossible since every vertex of the graph G_μ has finite degree. □

Proof of (b). The graph G_ν has two vertices of infinite degree. Suppose for a contradiction that there exists a word ν' on $\mathbb{N}$ or $\mathbb{Z}$ such that $\mathrm{Age}(G_\nu) = \mathrm{Age}(G_{\nu'})$. If ν' is a word on $\mathbb{N}$, then ν' must take the value 0 on an infinite subset I of $\mathbb{N}$ because otherwise every vertex of $G_{\nu'}$ would have finite degree which is impossible since $\mathrm{Age}(G_\nu) = \mathrm{Age}(G_{\nu'})$. Let $I' \subseteq I$ be an infinite set of nonconsecutive integers. Then $G_{\nu'}$ induces an infinite clique on I'. This is not possible since the only cliques of G_ν have cardinality 3. If ν' is a word on $\mathbb{Z}$, then, if it takes the value 0 finitely many times, it must have $u = 0111$

as a factor. Let J be a set of four consecutive integers $j_1 < j_2 < j_3 < j_4$ such that $\nu'_{\restriction J} = u$ and $J' = \{i, j\}$ with $j = i + 1 < j_1$ and $\nu'_{\restriction J'} = 11$. Then $G_{\nu'}$ induces on $J' \cup J$ a subgraph which doesn't belong to $\mathrm{Age}(G_\mu)$. And if ν' takes the value 0 infinitely many times, we have the same conclusion as for the word on $\mathbb{N}$. $\qquad\square$

Remark 5.8. *Given a word ν we associate the graph G^ν whose vertex set $V(G^\nu)$ is $\{-n+1, \ldots, 0, 1\}$ if the domain of ν is $\{-n+1, \ldots, 0\}$, $\mathbb{N}$ or $\mathbb{Z}$ if the domain of ν is $\mathbb{N}$ or $\mathbb{Z}$ respectively, and $\mathbb{N}^* \cup \{1\}$ if the domain of ν is $\mathbb{N}^*$. For two vertices i, j with $i < j$ we let $\{i, j\}$ be an edge of G^ν if and only if*

$$\nu_i = 1 \ \text{ and } \ j = i + 1, \ \text{or}$$
$$\nu_i = 0 \ \text{ and } \ j \neq i + 1.$$

If ν is of domain $\{0, \ldots, n-1\}$, $\mathbb{N}$, $\mathbb{N}^*$ or $\mathbb{Z}$ define ν^* to be the sequence of domain $\{-n+1, \ldots, 0\}$, $\mathbb{N}^*$, $\mathbb{N}$ or $\mathbb{Z}$ respectively by setting $\nu^*(i) := \nu(-i)$. Then G^{ν^*} and G_ν are isomorphic.

Remark 5.9. *For every infinite word μ the graph G_μ is the union of at most two infinite cliques and at most two infinite independent sets.*
   To see that, let μ be a 0-1 sequence on an infinite interval J of $\mathbb{Z}$.

(1) *If μ takes the value 0 (resp. the value 1) finitely many times, then there exists a finite interval K of J, which contains all the integers for which μ takes the value 0 (resp. μ takes the value 1), such that $G_\mu{\restriction}_{J \setminus K}$ has at most two connected components each one being an infinite path (resp. $G_\mu{\restriction}_{J \setminus K}$ is the complement of a graph which is a union of at most two infinite paths).*

(2) *If μ takes the values 0 and 1 infinitely many times, let $J_0 := \{j \in J : \mu(j) = 0\}$ and $J_1 := \{j \in J : \mu(j) = 1\}$. If there are no consecutive integers in J_0 (resp. in J_1), then the graph G_μ induces an infinite clique on J_0 (resp. an infinite independent set on J_1). Otherwise the set J_0 (resp. J_1) can be partitioned into two sets. Indeed, the sets J_0 and J_1, being infinite, they can be relabelled with integers (of $\mathbb{N}$ or $\mathbb{N}^*$ depending on whether J is a final segment or an initial segment of $\mathbb{Z}$) and we can set $J_0 := \{j_k : k \in L\}$ (resp. $J_1 := \{j'_k : k \in L\}$) where L is $\mathbb{N}$ or $\mathbb{N}^*$. Let $C_0 := \{j_{2k} : k \in L\}$ and $C_1 := J_0 \setminus C_0$ (resp. $I_0 := \{j'_{2k} : k \in L\}$ and $I_1 := J_1 \setminus I_0$). Then $\{C_0, C_1\}$ is a partition of J_0 and G_μ induces a clique on C_0 and on C_1 (resp. $\{I_0, I_1\}$ is a partition of J_1 and G_μ induces an independent set on I_0 and on I_1). For an example consider the periodic sequence $\mu := 011011 \ldots$. Then G_μ induces an infinite clique on J_0 and an infinite independent set on I_0 and on I_1. If we consider the periodic sequence $\mu := 100100 \ldots$, then G_μ induces an independent set on J_1 and a clique on C_0 and on C_1.*

   Here is our first result.

Theorem 5.10. *For every* 0-1 *word* μ *the age* $\mathrm{Age}(G_\mu)$ *consists of permutation graphs.*

The proof of Theorem 5.10 is given in Section 6. It follows from the Compactness Theorem of First Order Logic and Lemma 6.1. It was brought to us by Brignall [13] that chains are the same objects as *pin sequences* (see [11] Subsection 2.6. p. 41).

The next result is about the number of hereditary classes of finite permutation graphs. It is easy to prove and well known that there are $2^{\aleph_0}$ such classes. This is due to the existence of infinite antichains among finite permutation graphs.

In general, it is not true that two words with different sets of finite factors give different ages. But, we prove:

Theorem 5.11. *Let* μ *and* μ' *be two* 0-1 *words. If* μ *is recurrent and* $\mathrm{Age}(G_\mu) \subseteq \mathrm{Age}(G_{\mu'})$, *then* $Fac(\mu) \subseteq Fac(\mu')$.

Using this result and the fact that there are $2^{\aleph_0}$ 0-1 recurrent words with distinct sets of factors, we obtain the following.

Corollary 5.12. *There are* $2^{\aleph_0}$ *ages of permutation graphs.*

The ages we obtain in Corollary 5.12 are not necessarily well-quasi-ordered. To obtain well-quasi-ordered ages, we consider graphs associated to uniformly recurrent sequences.

Theorem 5.13. *Let* μ *be a* 0-1 *sequence on an infinite interval of* $\mathbb{Z}$. *The following propositions are equivalent.*

 (*i*) μ *is uniformly recurrent.*
 (*ii*) μ *is recurrent and* $\mathrm{Age}(G_\mu)$ *is minimal prime.*

The proofs of Theorems 5.11 and 5.13 are given in Sections 9 and 10.

As it is well known, there are $2^{\aleph_0}$ uniformly recurrent words with distinct sets of factors (e.g. Sturmian words with different slopes, see chapter 6 of [22]). With Theorem 5.11 we get:

Corollary 5.14. *There are* $2^{\aleph_0}$ *ages of permutation graphs which are minimal prime.*

Theorem 4.11 asserts that minimal prime ages are well-quasi-ordered. Since minimal prime ages are incomparable when ordered by set-inclusion, it follows from Corollary 5.14 that the set of well-quasi-ordered ages of permutation graphs, when ordered by set inclusion, has an uncountable antichain. On the other hand, observe that the chains are countable.

Problem 5.15. *Does every uncountable set of ages of permutation graphs, when ordered by set inclusion, contain an uncountable antichain of ages?*

When μ is uniformly recurrent, the age $\mathrm{Age}(G_\mu)$ is w.q.o. since it is minimal prime. In fact:

Theorem 5.16. *If μ is a uniformly recurrent word, then $\mathrm{Age}(G_\mu)$ is better-quasi-ordered.*

Proof. Let μ is a uniformly recurrent word. Then $\mathrm{Fac}(\mu)$ is Jónsson and hence is b.q.o. (see [51, 18]). From the extension of Higman's Theorem to b.q.o., the set $(\mathrm{Fac}(\mu))^*$ of finite sequences of members of $\mathrm{Fac}(\mu)$, once equipped with Higman's ordering of finite sequences, is b.q.o. If $s := (u_0, \ldots, u_k) \in (\mathrm{Fac}(\mu))^*$, we may represent it by a sequence $u'_0, \ldots, u'_k$ of factors of μ in such a way that u'_i is before u'_{i+1} and not contiguous to it. The graph induced by G_μ on this union of factors does not depend of the representation. Denote it by $G(s)$. Observe that the map which associate to each s the graph $G(s)$ is order preserving. It follows that its image is b.q. o. Then observe that this image is $\mathrm{Age}(G_\mu)$. The required result follows. $\square$

Remark 5.17. *If μ is an infinite word, then $\mathrm{Age}(G_\mu)$ is not 2^--w.q.o. To see that, consider the labelled sequence $(G_n, -1, n)$ where $G_n := G_\mu\restriction_{\{-1,\ldots,n\}}$. This is an antichain. Suppose for a contradiction that the labelled graph $(G_n, -1, n)$ embeds in the labelled graph $(G_{n'}, -1, n')$. Then the labelled graph (G_n, n) embeds in the labelled graph $(G_{n'}, n')$. Since $n-1$ is the unique neighbour or nonneighbour of n in G_n we infer that $n-1$ is mapped to the unique neighbour or nonneighbour of n' in $G_{n'}$. Hence, the labelled graphs obtained by deleting n and n' and labelling $n-1$ and $n'-1$ embed in each other. Since -1 must be mapped on -1, this yields a contradiction.*

Remark 5.18. *If μ is not eventually periodic, then the class $\mathrm{Age}(G_\mu)^{\leq \omega}$ of countable G such that $\mathrm{Age}(G) \subseteq \mathrm{Age}(G_\mu)$ is not well founded. Indeed, the sequence of $G_n := (G_{\mu\restriction[n, \to[}))$, $n \in \mathbb{N}$, is strictly decreasing. This simple fact is a reason for introducing uniformly recurrent sequences in the theory of relations.*

Theorem 5.19. *Let μ be a word with domain $\mathbb{N}$. The following propositions are equivalent.*

(*i*) *μ is periodic;*
(*ii*) *$\mathrm{Age}(G_\mu)$ is 1^--well-quasi-ordered;*
(*iii*) *$\mathrm{Age}(G_\mu)$ is 1^--better-quasi-ordered;*
(*iv*) *The class $\mathrm{Age}(G_\mu)^{\leq \omega}$ of countable G such that $\mathrm{Age}(G) \subseteq \mathrm{Age}(G_\mu)$ is 1^--better-quasi-ordered.*

Proof. $(iv) \Rightarrow (iii)$ and $(iii) \Rightarrow (ii)$ are obvious.
$(ii) \Rightarrow (i)$. Suppose that $\mathrm{Age}(G_\mu)$ is 1^--w.q.o. We claim that $\mathrm{Fac}(\mu)$ is w.q.o. by the suffix order. According to Theorem 3.12 this implies that μ is periodic. The proof of our claim is based on the following observation. Let $w := w_0 \ldots w_n$ and $w' := w'_0 \ldots w'_{n'}$ be two finite words. Then w is a suffix of w' if and only if the labelled graph (G_w, n) embeds in the labelled graph $(G_{w'}, n')$. Indeed, if n is mapped to n', then since $n-1$ is the unique neighbour or nonneighbour of n in G_w we infer that $n-1$ is mapped to the unique neighbour or nonneighbour of n' in $G_{w'}$. Hence, the labelled graphs

obtained by deleting n and n' and labeling them $n-1$ and $n'-1$ embed in each other.

$(i) \Rightarrow (iv)$. Suppose that μ is periodic. Then the collection $\mathrm{Fac}(\mu)^{\leq\omega}$ of words w such that $\mathrm{Fac}(w) \subseteq \mathrm{Fac}(\mu)$, when ordered by the factor ordering, is covered by finitely many chains. Those chains are well founded, hence this collection is b.q.o. According to Laver's Theorem, the collection of countable chains labelled by $\mathrm{Fac}(\mu)^{\leq\omega}$ is b.q.o. To conclude that the class $\mathrm{Age}(G_\mu)^{\leq\omega}$ is b.q.o. it suffices to show that this is the image of $\mathrm{Fac}(\mu)^{\leq\omega}$ by an order preserving map. Let $(w_x)_{x\in C}$ be a countable chain, set $\oplus_{x\in C} w_x$ be the graph G defined on the disjoint union of the domains of the w_x's such that for $(x, w_x(i)), (y, w_y(j)) \in V(G)$, if $x < y$ in C, $G((x, w_x(i)), (y, w_y(j))) = w_y(j) \oplus 1$, whereas if $x = y$, $G((x, w_x(i)), (y, w_y(j))) = G_{w_x}(i, j)$. Observe that if a labeled chain is smaller that another the results are in the same order. And note that every member of $\mathrm{Fac}(\mu)^{\leq\omega}$ is the image of such a process. In order to prove that $\mathrm{Age}(G_\mu)^{\leq\omega}$ is 1^--chapter, prove first that the collection of (G_w, a_w) where w is a word such that $\mathrm{Fac}(w) \subseteq \mathrm{Fac}(\mu)$ and a_w is a constant is b.q.o. Decompose each G_w into an initial part and a final part containing only the label a_w. Due to the periodicity of μ, the final parts and the initial parts belong to finitely many chains, thus form a b.q.o. Since the product of two b.q.o. is b.q.o. the collection of (G_w, a_w) forms a b.q.o. Replacing in the construction above one G_w by one (G_w, a_w) we get the same conclusion. □

Permutation graphs come from posets and from bichains. Let us recall that a *bichain* is relational structure $R := (V, (\leq', \leq''))$ made of a set V and two linear orders $\leq'$ and $\leq''$ on V. If V is finite and has n elements, there is a unique permutation σ of $\{1, \ldots, n\}$ for which R is isomorphic to the bichain $C_\sigma := (\{1, \ldots, n\}, \leq, \leq_\sigma)$ where $\leq$ is the natural order on $\underline{n} := \{1, \ldots, n\}$ and $\leq_\sigma$ is the linear order defined by $i \leq_\sigma j$ if $\sigma(i) \leq \sigma(j)$. If we represent bichains by permutations, embeddings between bichains is equivalent to the *pattern containment* between the corresponding permutations, see Cameron [17]. To a bichain $R := (V, (\leq', \leq''))$, we may associate the intersection order $o(R) := (V, \leq' \cap \leq'')$ and to $o(R)$ its comparability graph.

The following is Theorem 67 from [62].

Theorem 5.20. (1) *Let* $P := (V, \leq)$ *be a poset. Then* $\mathrm{Age}(\mathrm{Inc}(P))$ *is minimal prime if and only if* $\mathrm{Age}(\mathrm{Comp}(P))$ *is minimal prime. Furthermore,* $\mathrm{Age}(P)$ *is minimal prime if and only if* $\mathrm{Age}(\mathrm{Inc}(P))$ *is minimal prime and* $\downarrow \mathrm{Prim}(\mathrm{Age}(P)) = \mathrm{Age}(P)$.

(2) *Let* $B := (V, (\leq_1, \leq_2))$ *be a bichain and* $o(B) := (V, \leq_1 \cap \leq_2)$. *Then* $\mathrm{Age}(B)$ *is minimal prime if and only if* $\mathrm{Age}(o(B))$ *is minimal prime and* $\downarrow \mathrm{Prim}(\mathrm{Age}(B)) = \mathrm{Age}(B)$.

With (2) of Theorem 5.20 and Corollary 5.14 we have:

Theorem 5.21. *There are* $2^{\aleph_0}$ *ages of bichains and permutation orders which are minimal prime.*

5.4. **A complete characterization of minimal prime ages of graphs.**

Theorem 5.22. *A hereditary class $\mathcal{C}$ of finite graphs is minimal prime if and only if $\mathcal{C} = \mathrm{Age}(G_\mu)$ for some uniformly recurrent word on $\mathbb{N}$, or $\mathcal{C} \in \mathcal{L}$.*

Proof. The sufficient condition follows from Theorems 5.4 and 5.13 and chapter 6 page 109 of the first author's thesis [47].

The necessary condition follows essentially from Theorem 5.1. Let $\mathcal{C}$ be a minimal prime age. Then $\mathcal{C}$ contains infinitely many prime graphs of one of the types given in Theorem 5.1. If for an example, $\mathcal{C}$ contains infinitely many chains, that is graphs of the form G_μ for μ finite, then, since it is minimal prime, we claim that this is the age of some G_μ with μ uniformly recurrent. Indeed, let $\mathcal{A}$ be an age containing 0-1 graphs G_w for arbitrarily long finite words w. We prove that $\mathcal{A}$ contains the age of a graph G_μ where μ is a uniformly recurrent word. Indeed, let W be the set of finite words w such that $G_w \in \mathcal{A}$. Clearly, W is an infinite hereditary set of finite words. It follows from Lemma 3.5 that W contains an initial segment U which is Jónsson. It follows from the equivalence $(i) \Leftrightarrow (iii)$ of Theorem 3.11 that $U = \mathrm{Fac}(\mu)$ where μ is a uniformly recurrent word. We now prove that $\mathrm{Age}(G_\mu) \subseteq \mathcal{A}$. Let $H \in \mathrm{Age}(G_\mu)$. There exists then $w \in \mathrm{Fac}(\mu)$ such that H is an induced subgraph of G_w. But $w \in \mathrm{Fac}(\mu) \subseteq W$. Thus $G_w \in \mathcal{A}$ as required. For the other cases, use the structure of the infinite graphs described in Figures 2 and 3. $\square$

Theorem 5.23. (1) *A minimal prime hereditary class $\mathcal{C}$ of finite graphs is hereditarily well-quasi-ordered if and only if $\mathcal{C} \in \mathcal{L}$.*

(2) *A minimal prime hereditary class $\mathcal{C}$ of finite graphs remains well-quasi-ordered when just one label is added if and only if $\mathcal{C} = \mathrm{Age}(G_\mu)$ for some periodic 0-1 word on $\mathbb{N}$, or $\mathcal{C} \in \mathcal{L}$.*

Proof. (1) The sufficient condition follows from Proposition 4.25 and Theorem 5.4.

To prove the necessary condition, let $\mathcal{C}$ be a minimal hereditary class of finite graphs which is hereditarily well-quasi-ordered. From Theorem 5.22, $\mathcal{C} \in \mathcal{L}$ or $\mathcal{C} = \mathrm{Age}(G_\mu)$ for some uniformly recurrent word on $\mathbb{N}$. According to Remark 5.17, $\mathrm{Age}(G_\mu)$ is not hereditarily well-quasi-ordered. It follows that $\mathcal{C} \in \mathcal{L}$.

(2) Follows from item (1) and Theorem 5.19.

$\square$

The corresponding characterization of minimal prime ages of posets and bichains will follow from Theorem 5.20 and a careful examination of our list of graphs to decide which graphs are comparability graphs.

Corollary 5.24. (1) *A hereditary class $\mathcal{C}$ of finite comparability graphs is minimal prime if and only if $\mathcal{C} = \mathrm{Age}(G_\mu)$ for some uniformly recurrent word on $\mathbb{N}$, or*

$\mathcal{C} \in \{\mathrm{Age}(G_0), \mathrm{Age}(G_1), \mathrm{Age}(\overline{G}_1), \mathrm{Age}(G_3), \mathrm{Age}(G_5), \mathrm{Age}(G_6), \mathrm{Age}(\overline{G}_6)\}.$
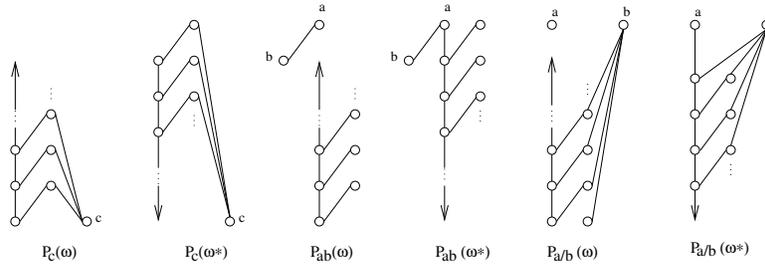
FIGURE 7. Transitive orientations of the graphs G_5, $\overline{G}_5$, G_6 and $\overline{G}_6$

(2) *A hereditary class $\mathcal{C}$ of finite permutation graphs is minimal prime if and only if $\mathcal{C} = \mathrm{Age}(G_\mu)$ for some uniformly recurrent word on $\mathbb{N}$, or $\mathcal{C} \in \{\mathrm{Age}(G_1), \mathrm{Age}(\overline{G}_1), \mathrm{Age}(G_5), \mathrm{Age}(G_6), \mathrm{Age}(\overline{G}_6)\}$.*

We end this section with the following conjecture.

Conjecture. *Every infinite prime graph embeds one of the graphs depicted in Figures 2 and 3, or a graph G_μ for some 0-1 sequence μ on an interval of $\mathbb{Z}$.*

5.5. **Bounds of minimal prime hereditary classes.** We recall that a *bound* of a hereditary class $\mathcal{C}$ of finite structures (e.g. graphs, ordered sets) is any structure $R \notin \mathcal{C}$ such that every proper induced substructure of R belongs to $\mathcal{C}$.

As we have seen in Theorem 5.22 minimal prime ages of graphs belong either to $\mathcal{L}$, in which case they have finitely many bounds since they are ages of multichainable graphs (Theorem 4.28), or they are of the form $\mathrm{Age}(G_\mu)$ with μ uniformly recurrent.

If μ is a 0-1 periodic word, $\mathrm{Age}(G_\mu)$ may have infinitely many bounds. This is the case if μ is constant. For the remaining cases within uniformly recurrent sequences, we have the following theorem.

Theorem 5.25. *Let μ be a 0-1 uniformly recurrent word.*

(1) *If μ is nonperiodic, then $\mathrm{Age}(G_\mu)$ has infinitely many bounds;*
(2) *If μ is periodic and nonconstant, then $\mathrm{Age}(G_\mu)$ has finitely many bounds.*

The proof of Theorem 5.25 is given in Section 11. In [14], Brignall et al. provided an example of a hereditary class of permutation graphs which is w.q.o. , has finitely many bounds, but is not labelled w.q.o. solving negatively a conjecture of Korpelainen et al. [37].

As stated in (2) of Theorem 5.25, ages of 0-1 graphs corresponding to periodic and nonconstant words provide infinitely many examples of such classes. Note that these classes are 1^--w.q.o.

6. A proof of Theorem 5.10 and a characterization of order types of realizers of transitive orientations of 0-1 graphs

Let $P := (V, \leq)$ be a poset. An element $x \in V$ is *extremal* if it is maximal or minimal.

Lemma 6.1. *Let* $w := w_0 \ldots w_{n-1}$ *be a finite word with* $n \geq 2$ *and* $w' := w_0 \ldots w_{n-2}$. *Then every realizer* $(L_{w'}, M_{w'})$ *of a transitive orientation of* $G_{w'}$ *on* $\{-1, 0, \ldots, n-2\}$ *(if any) such that* $n-2$ *is extremal in* $L_{w'}$ *or in* $M_{w'}$ *extends to a realizer* (L_w, M_w) *of a transitive orientation of* G_w *on* $\{-1, 0, \ldots, n-1\}$ *such that* $n-1$ *is extremal in* L_w *or in* M_w.

Proof. Let $(L_{w'}, M_{w'})$ be a realizer of a transitive orientation $P_{w'}$ of $G_{w'}$ on $\{-1, 0, \ldots, n-2\}$ such that $n-2$ is extremal in $L_{w'}$ or in $M_{w'}$. We may assume without loss of generality that $n-2$ is maximal in $L_{w'}$ or in $M_{w'}$. Otherwise, consider $P_{w'}^*$ and the pair $(L_{w'}^*, M_{w'}^*)$. Note that $P_{w'}^*$ is a transitive orientation of $G_{w'}$, the pair $(L_{w'}^*, M_{w'}^*)$ is a realizer of $P_{w'}^*$, and $n-2$ is maximal in $L_{w'}^*$ or in $M_{w'}^*$ (this is because $n-2$ is minimal in $L_{w'}$ or in $M_{w'}$). We then extend $(L_{w'}^*, M_{w'}^*)$ to a realizer of $P_{w'}^*$ with the desired property. The dual of this realizer is a realizer of P_w with the required property. We may also suppose that $n-2$ is maximal in $L_{w'}$, because otherwise, we interchange the roles of $L_{w'}$ and $M_{w'}$.
• If $w_{n-1} = 1$, then $\{n-2, n-1\}$ is the unique edge of G_w containing $n-1$. Clearly $P_w := P_{w'} \cup \{(n-1, n-2)\}$ is a transitive orientation of G_w. Let L_w be the linear order obtained from $L_{w'}$ so that $n-1$ appears immediately before $\max(L_{w'}) = n-2$ and larger than all other elements and let M_w be the linear order obtained from $M_{w'}$ by letting $n-1$ smaller than all elements of $M_{w'}$. Clearly, (L_w, M_w) is a realizer of P_w and by construction $n-1$ is minimal in P_w and in M_w.
• Else if $w_{n-1} = 0$, then $\{n-2, n-1\}$ is the unique non edge of G_w containing $n-1$. Since $n-2$ is maximal in $P_{w'}$ we infer that $P_w := P_{w'} \cup \{(x, n-1) : x \in \{-1, 0, \ldots, n-3\}\}$ is a transitive orientation of G_w in which $n-1$ and $n-2$ are incomparable. Let L_w be the linear order obtained from $L_{w'}$ so that $n-1$ appears immediately before $\max(L_{w'}) = n-2$ and larger than all other elements and let M_w be the linear order obtained from $M_{w'}$ by letting $n-1$ larger than all elements of $M_{w'}$. Clearly, (L_w, M_w) is a realizer of P_w (indeed, $n-2$ and $n-1$ are incomparable in $L_w \cap M_w$ and for all $x \in \{-1, 0, \ldots, n-3\}$, $x < n-1$ in $L_w \cap M_w$ proving that $\{L_w, M_w\}$ is a realizer of P_w). By construction $n-1$ is maximal in P_w and M_w. The proof of the lemma is now complete. $\square$

Lemma 6.2. *Let* μ *be a 0-1 sequence defined on an interval* I *of* $\mathbb{Z}$. *Then* G_μ *is a comparability graph and an incomparability graph. In particular, if* I *is finite, then* G_μ *is a permutation graph.*

Proof. We consider two cases.
(a) I is finite or I is a final segment of $\mathbb{Z}$ bounded below. Write $I := \{i_0, \ldots, i_n, \ldots\}$ and define for every n a realizer (L_n, M_n) of a transitive

orientation of the restriction of G_μ to $\{i_0 - 1, i_0, \ldots, i_n\}$. For that, use Lemma 6.1 and induction on n. Note that for $n = 0$, the restriction of G_μ to $\{i_0 - 1, i_0\}$ is either a 2-element independent set or a 2-element clique, and these are permutation graphs. Then (L_μ, M_μ) where $L_\mu := \bigcup_{n \in I} L_n$ and $M_\mu := \bigcup_{n \in I} M_n$ is a realizer of a transitive orientation of G_μ. Hence G_μ is a comparability and an incomparability graph, and is a permutation graph if I is finite.

(b) I is an initial segment of $\mathbb{Z}$. In this case, if F is any finite subset of I, let J be a finite interval of I containing F. Let w be the restriction of μ to $J \setminus \{\min(J)\}$. Then the graph induced by G_μ on J is G_w. It follows from (a) that G_w is a permutation graph, hence $G_{\mu \restriction F}$ is a permutation graph. It follows from the Compactness Theorem of First Order Logic that G_μ is a comparability and an incomparability graph. □

Theorem 5.10 readily follows from Lemma 6.2. The remainder of this section is devoted to characterizing the order types of linear extensions in a realizer of a transitive orientation of the graph G_μ in the case of μ is a 0-1 word on $\mathbb{N}$.

Lemma 6.3. *Let $w := w_0 \ldots w_{n-1}$ be a finite word with $n \geq 3$. If (L_w, M_w) is a realizer of a transitive orientation of G_w on $\{-1, 0, \ldots, n - 1\}$ constructed step by step by means of Lemma 6.1, then for all $0 \leq k \leq n - 3$ the set $\{k + 2, \ldots, n - 1\}$ does not meet the intervals of L_w and M_w generated by $\{-1, 0, \ldots, k\}$.*

Proof. Let $k \in \{0, \ldots, n - 3\}$ and $j \in \{k + 2, \ldots, n - 1\}$.

CASE 1: $j > k + 2$.
Suppose $w_j = 0$. Then j is adjacent to all vertices of $\{-1, 0, \ldots, k + 1\}$. It follows from the algorithm described in Lemma 6.1 that in a transitive orientation of G_w the vertex j is larger than all elements of $\{-1, 0, \ldots, k+1\}$ or the vertex j is smaller than all elements of $\{-1, 0, \ldots, k + 1\}$. Hence, if $\{L_w, M_w\}$ is a realizer of a transitive orientation of G_w, then j, in both L_w and M_w, is either above all elements of $\{-1, 0, \ldots, k + 1\}$ or is below all elements of $\{-1, 0, \ldots, k + 1\}$. Hence, $j \notin I$. We now consider the case $w_j = 1$. Then j is not adjacent to any vertex of $\{-1, 0, \ldots, k + 1\}$. Hence, if $\{L_w, M_w\}$ is a realizer of a transitive orientation of G_w, then j is either above all elements of $\{-1, 0, \ldots, k + 1\}$ in L_w and below all elements of $\{-1, 0, \ldots, k + 1\}$ in M_w, or j is below all elements of $\{-1, 0, \ldots, k + 1\}$ in L_w and above all elements of $\{-1, 0, \ldots, k + 1\}$ in M_w. Hence, $j \notin I$.

CASE 2: $j = k + 2$.
We may assume without loss of generality that $k + 1$ is maximal in the restriction of a transitive orientation P of G_w and L_w to $\{-1, 0, \ldots, k + 1\}$ (otherwise consider the dual of P which is a transitive orientation of the restriction of G_w to $\{-1, 0, \ldots, k + 1\}$). It follows from the algorithm described in Lemma 6.1 that $k + 2 \notin I$ as required. □

As it is customary, we denote by ω the order type of $\mathbb{N}$, by ω^* the order type of its dual and by $\omega^* + \omega$ the order type of $\mathbb{Z}$.

The proof of the following lemma is easy and is left to the reader.

Lemma 6.4. (1) *The intersection of two linear orders of order type ω is a w.q.o.*

(2) *The intersection of two linear orders of order types ω and ω^* has no infinite chains.*

(3) *The intersection of two linear orders of order types ω and $\omega^* + \omega$ is well founded.*

Corollary 6.5. *Let μ be a word on $\mathbb{N}$. If (L, M) is a realizer of a transitive orientation of G_μ, then the order types of L and M embed into $\omega^* + \omega$. Furthermore, if μ has finitely many 0's or 1's, then the order types of L and M embed into ω or ω^*, else at least one of L and M have order type $\omega^* + \omega$.*

Proof. Let (L, M) be a realizer of a transitive orientation of G_μ. According to Lemma 6.3, for every $k \in \mathbb{N}$, the least interval of L containing $\{-1, 0, \dots, k\}$ is included in $\{-1, 0, \dots, k+1\}$. Hence L is a countable increasing union of finite intervals, proving that L embeds in $\mathbb{Z}$.

If μ has finitely many 0's or 1's, then there exists a final interval I of $\mathbb{N}$ such that the restriction of G_μ to I is an infinite one way path or the complement of an infinite one way path. It can be easily seen that the order types in a realizer of transitive orientations of an infinite one way path or its complement are $\{\omega, \omega\}$ or $\{\omega, \omega^*\}$ or $\{\omega^*, \omega^*\}$. Since $\mathbb{N} \setminus I$ is an initial segment of $\mathbb{N}$ we have that the order types of a linear extension in a realizer P_μ are $\{\omega, \omega\}$ or $\{\omega, \omega^*\}$ or $\{\omega^*, \omega^*\}$. Next we suppose that μ has infinitely many 0's and 1's. There exists then two infinite subsets of nonconsecutive integers J and K so that μ is constant on J and K, and μ takes the value 1 on J and takes the value 0 on K. Then P_μ has an infinite antichain, induced by the set J, and an infinite chain, induced by the set K. The order types of a linear extension in a realizer of P_μ cannot be $\{\omega, \omega\}$ or $\{\omega^*, \omega^*\}$ because otherwise P_μ or its dual is w.q.o and hence has no infinite antichains. The order types of a linear extension in a realizer of P_μ cannot be $\{\omega, \omega^*\}$ either because otherwise all chains of P_μ would be finite. $\qquad \square$

We now provide examples of P_μ that have realizers of type $(\omega, \omega^* + \omega)$ and $(\omega^* + \omega, \omega^* + \omega)$.

Example 6.6. *Let $\mu := 001100110011\dots$. The order types of a linear extension in a realizer of P_μ are ω and $\omega^* + \omega$. Indeed, an embedding of P_μ into $\mathbb{N} \times \mathbb{Z}$ is depicted in Figure 8. It follows easily that P_μ has a realizer of type $(\omega, \omega^* + \omega)$. Since P_μ is prime it has a unique realizer up to a transposition.*

Example 6.7. *Let $\mu := 011011011\dots$. The order types of a linear extension in a realizer of P_μ is $\mathbb{Z}$. Indeed, an embedding of P_μ into $\mathbb{Z} \times \mathbb{Z}$ is depicted in Figure 9. It follows easily that P_μ has a realizer of type $(\omega^* + \omega, \omega^* + \omega)$. Since P_μ is prime it has a unique realizer up to a transposition.*
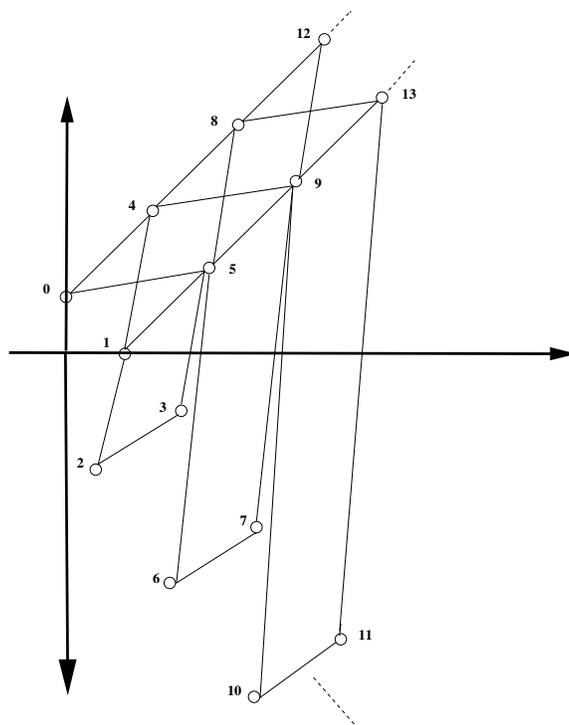
FIGURE 8. An embedding into $\mathbb{N} \times \mathbb{Z}$ of a transitive orientation of the graph corresponding to the periodic 0-1 sequence $\mu := 001100110011\ldots$.

We should mention that in the first example neither G_μ nor its complement are permutation graphs, while in the second example both are.

7. MODULES IN G_μ

The aim of this section is to characterize the modules of a 0-1 graph. We prove among other things, that if G_μ is not prime, then μ contains large factors of 0's or 1's. Results of this section will be used in Section 8 to derive properties of embeddings between 0-1 graphs.

We recall that if $G := (X, E)$ is a graph, then a subset M of X is called a *module* in G if for every $x \notin M$, either x is adjacent to all vertices of M or x is not adjacent to any vertex of M.

The following lemma will be useful.

Lemma 7.1. *A graph and its complement have the same set of modules. In particular, G_μ and $G_{\overline{\mu}}$ have the same modules.*

Lemma 7.1 and Remark 5.6 of subsection 5.3 combined together will allow us to simplify proofs. Indeed, if we are arguing on the value of μ on a particular integer i we may only consider the case $\mu(i) = 0$ (or $\mu(i) = 1$).
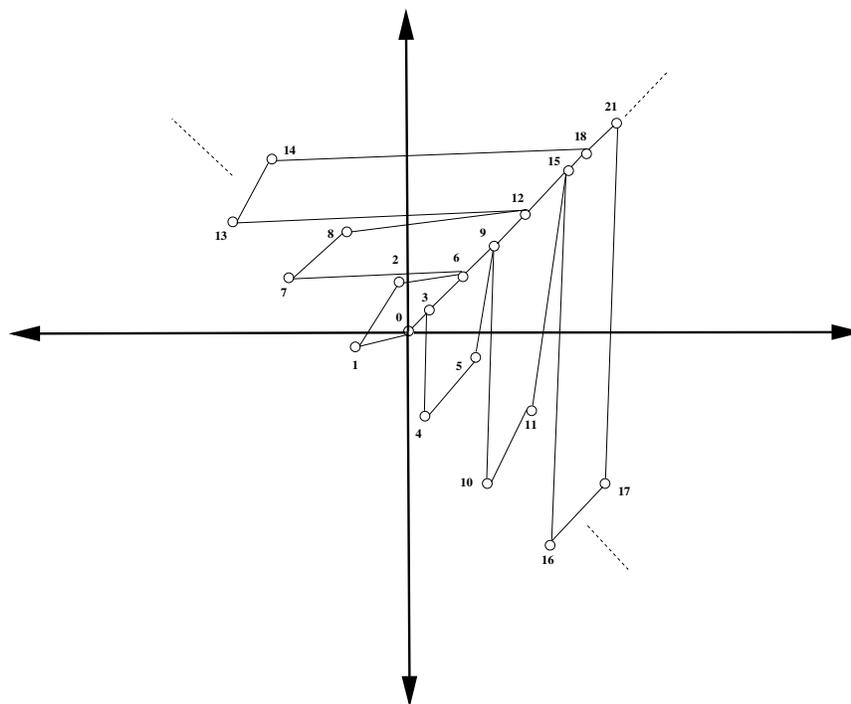
FIGURE 9. An embedding into $\mathbb{Z} \times \mathbb{Z}$ of a transitive orienta-
tion of the graph corresponding to the periodic 0-1 sequence
$\mu := 011011011 \dots$.

We recall some properties of modules in a graph. The proof of the fol-
lowing lemma is easy and is left to the reader (see [24]).

Lemma 7.2. *Let $G = (V, E)$ be a graph. The following propositions are
true.*

(1) *The intersection of a nonempty set of modules is a module (possibly
empty).*
(2) *The union of two modules with nonempty intersection is a module.*
(3) *For two modules M and N, if $M \setminus N \neq \varnothing$, then $N \setminus M$ is a module.*

Let $G := (X, E)$ be a graph and $\{x, y, z\} \subseteq X$. We say that z *separates*
x and y if $\{z, x\}$ is an edge and $\{z, y\}$ is not an edge, or vice versa. For
instance,

- if I is an interval of $\mathbb{N}$, μ is a 0-1 sequence on I and $i \in I$, then i
  separates $i - 1$ and j for all $j < i - 1$ in G_μ. (Indeed, $\{j, i\}$ is an
  edge if and only if $\{i - 1, i\}$ is not an edge).

Lemma 7.3. *Let G be a graph and $\{x, y, z\} \subseteq V(G)$. If z separates x and
y and if x and y belong to a module in G, then z belongs to that module.*

Lemma 7.4. *Let μ be a 0-1 sequence on an interval $I := \{i_1, \ldots, i_n, \ldots\}$ of $\mathbb{N}$ and let $i_0 := i_1 - 1$. Let $J \subseteq \{i_0\} \cup I$ be a nonempty subset. Let J^- be the maximal initial segment of J which is an interval of $\{i_0\} \cup I$. If J is not an interval of $\{i_0\} \cup I$, then J^- is a module of $G_{\mu \restriction J}$. In particular, if J^- is not a singleton, then $G_{\mu \restriction J}$ is not prime.*

Proof. If J is not an interval of $\{i_0\} \cup I$, $J \setminus J^-$ is nonempty. Furthermore, no element of $J \setminus J^-$ separates two elements of J^- (indeed, the element $i_k := \max(J^-) + 1$ does not belong to J^- and is the only element of I that separates two elements of J^-). Therefore J^- is a module of $G_{\mu \restriction J}$. If J^- is not a singleton, then since it is distinct from J, it is a nontrivial module of $G_{\mu \restriction J}$ and therefore $G_{\mu \restriction J}$ is not prime. $\square$

Corollary 7.5. *Let μ be a 0-1 word on $\mathbb{N}$ and let $F \subseteq \{-1\} \cup \mathbb{N}$ be such that $G_{\mu \restriction F}$ is prime. Then $F \setminus \{\min(F)\}$ is an interval of $\mathbb{N}$.*

Proof. We apply Lemma 7.4 with $J := F \setminus \{\min(F)\}$. It follows that if J is not an interval of $\mathbb{N}$, then J^- is a module of $G_{\mu \restriction J}$. Since no element of $J \setminus J^-$ separates two elements of $\{\min(F)\} \cup J^-$ we infer that $\{\min(F)\} \cup J^-$ is a module of $G_{\mu \restriction F}$ which is prime. Hence, if J^- is not empty $\{\min(F)\} \cup J^-$ is a nontrivial module of $G_{\mu \restriction F}$ which is impossible. This proves that $F \setminus \{\min(F)\}$ is an interval of $\mathbb{N}$ as required. $\square$

Lemma 7.6. *Let $I := \{i_1, \ldots, i_n, \ldots\}$ be an interval of $\mathbb{N}$ of cardinality at least 3, μ be a 0-1 sequence on I and $i_0 = i_1 - 1$. For $k \geq 2$, $\{i_0, \ldots, i_{k-1}\}$ is a module of $G_\mu \setminus \{i_j\}$ if and only if $k = j$.*

Proof. We only need to prove the forward implication. Suppose that $\{i_0, \ldots, i_{k-1}\}$ is a module of $G_\mu \setminus \{i_j\}$. Then $i_j \notin \{i_0, \ldots, i_{k-1}\}$ and hence $k \leq j$. Since i_k is the only vertex that separates i_{k-1} and i_{k-2} (recall that $k \geq 2$) we infer that $j = k$. $\square$

Corollary 7.7. *Let $I := \{i_1, \ldots, i_n\}$ be an interval of $\mathbb{N}$ of cardinality at least 3, μ be a 0-1 sequence on I and $i_0 = i_1 - 1$. We suppose G_μ is prime and let $x \in \{i_0\} \cup I$. If $G_\mu \setminus \{x\}$ is prime, then $x \in \{i_0, i_1, i_n\}$.*

In the next lemma we state some properties of modules of G_μ when μ is a word on $\mathbb{N}$. It follows that a nontrivial module of G_μ with at least three elements is necessarily the whole domain of G_μ minus a singleton.

Lemma 7.8. *Let $I := \{i_1, \ldots, i_n, \ldots\}$ be an interval of $\mathbb{N}$, $i_0 = i_1 - 1$ and let μ be a 0-1 sequence on I. Let M be a nontrivial module of G_μ.*
(1) *Let $i_j, i_k \in M$ with $j < k$.*
   (a) *If $i_{k+1} \in I$, then $i_{k+1} \in M$.*
   (b) *$\{m \in I : i_k \leq m\} \subseteq M$.*
   (c) *exactly one of i_0 and i_1 is in M.*
(2) *The largest final segment F of I included in M is nonempty.*
(3) *Assume F has at least two elements. Then*

    (a) μ *is constant on* $F \setminus \{\min(F)\}$ *and* $\mu(\min(F)) \neq \mu(\min(F) + 1)$.
    (b) $\{m \in I : m \leq \min(F) - 2\} \subseteq M$.
    (c) $F = I$ *or* $F = I \setminus \{i_1\}$.

Proof. (1) Let $i_j, i_k \in M$ with $j < k$.

    (a) Suppose $i_{k+1} \in I$. Since $i_j, i_k \in M$ and i_{k+1} separates i_j and i_k we infer that $i_{k+1} \in M$. This proves item $(1)(a)$.

    (b) Since M contains at least two distinct elements, item $(1)(b)$ now follows by repeatedly applying item $(1)(a)$.

    (c) Suppose for a contradiction that $\{i_0, i_1\} \subseteq M$. It follows from item $(1)(b)$ that $I \subseteq M$. This is impossible since M is nontrivial. This proves that at least one of i_0 or i_1 is not in M. Suppose that $M \cap \{i_0, i_1\} = \emptyset$. Let k be the smallest positive integer such that $i_k \in M$ (note that $k \geq 2$). Since M is nontrivial there exists some $j > k$ such that $i_j \in M$. Since $i_{k-1} \notin M$, i_{k-1} cannot separate i_k and i_j. Since i_{k-1} and i_k are consecutive, $\mu(i_k) \neq \mu(i_j)$. Hence, i_0 separates the two elements i_k and i_j of M. Since $i_0 \notin M$ and M is a module, we obtain a contradiction. This proves item $(1)(c)$.

(2) Let F be the largest final segment of I included in M. We prove that F is nonempty. Indeed, since M is nontrivial, it has at least two elements i_j, i_k with $j < k$. From item $(1)(b)$ it follows that either $i_{k+1} \in I$ and hence the final segment $\{m \in I : i_k \leq m\}$ is nonempty and is a subset of M. Or, $\max(I) = i_k$ and $\{i_k\}$ is a final segment of I and belongs to M.

(3) (a) Let $l, m \in F \setminus \{\min(F)\}$. Since $\min(F) - 1 \notin M$ and M is a module, we infer that the vertex $\min(F) - 1$ of G_μ is either adjacent to both l and m or not adjacent to both l and m. Thus $\mu(l) = \mu(m)$ and μ is constant on $F \setminus \{\min(F)\}$ as required. Since $\min(F)$ and $\min(F) + 1$ are elements of M and since $\min(F) - 1$ and $\min(F)$ are consecutive in I we must have $\mu(\min(F)) = \mu(\min(F) + 1) + 1$, that is $\mu(\min(F)) \neq \mu(\min(F) + 1)$, proving item $(3)(a)$.

    (b) It follows from $(3)(a)$ that every element $m \in I \cup \{i_0\}$ such that $m < \min(F) - 1$ is adjacent to $\min(F)$ but not adjacent to $\min(F) + 1$ or vice versa. Since $\min(F)$ and $\min(F) + 1$ are elements of M and M is a module we infer that $m \in M$. Hence, $\{m \in I : m \leq \min(F) - 2\} \subseteq M$ proving item $(3)(b)$.

    (c) It follows from $(1)(c)$ and $(3)(b)$ that $\min(F) \in \{i_1, i_2\}$. If $\min(F) = i_1$, then $F = I$. Else if $\min(F) = i_2$, then $F = I \setminus \{i_1\}$. This completes the proof of $(3)(c)$ and of the lemma.

    The proof of the lemma is now complete.   □

    Let I be an interval of $\mathbb{N}$ and G_μ be the graph on $\{\min(I) - 1\} \cup I$ associated to a sequence μ defined on I. In the following proposition we characterize the modules of G_μ. We prove that if $\mu \notin \{011, 100, 001, 110\}$, then G_μ has at most one nontrivial module. Also, if I is finite, then a nontrivial module of G_μ has necessarily cardinality 2 or $|I|$.

Proposition 7.9. *Let $I := \{i_1, \ldots, i_n, \ldots\}$ be an interval of $\mathbb{N}$, $i_0 := i_1 - 1$ and let μ be a 0-1 sequence on I.*

(1) *If $\mu \notin \{011, 100, 001, 110\}$, then G_μ has at most one nontrivial module.*
(2) *If M is a nontrivial module of G_μ, then <u>either</u> $M = I$ or $M = \{i_0\} \cup I \setminus \{i_1\}$, <u>or</u> I is finite, $I = \{i_1, \ldots, i_n\}$ and $M = \{i_0, i_n\}$ or $M = \{i_1, i_n\}$.*

Proof. We recall that a graph on at most two vertices is prime. Hence, if G_μ has a nontrivial module, then $|I| \geq 2$.

CLAIM. If $|I| = 2$, then G_μ has exactly one nontrivial module.

Proof of claim. We only consider the case $\mu(i_1) = 0$ and deduce the other case by considering $\overline{\mu}$. By inspection, if $\mu(i_2) = 0$, then $\{i_0, i_2\}$ is the only nontrivial module of G_μ. if $\mu(i_2) = 1$, then $\{i_1, i_2\}$ is the only nontrivial module of G_μ. $\square$

(1) We use the characterisation of modules of G_μ found in item (2). We consider all possible pairs of such modules.

CASE 1: $\{i_0, i_n\}$ and $\{i_1, i_n\}$ are both modules of G_μ.
Since $\{i_0, i_n\} \cap \{i_1, i_n\} \neq \varnothing$ and the union of two modules with nonempty intersection is a module (see item (2) of Lemma 7.2) we infer that $A := \{i_0, i_n\} \cup \{i_1, i_n\} = \{i_0, i_1, i_n\}$ is a module of G_μ. It follows from item (2) that A is trivial. Since A has 3 elements we infer that $A = \{i_0\} \cup I$. This implies that $|I| = 2$ and hence $n = 2$. We derive a contradiction from the claim.

CASE 2: I and $\{i_0\} \cup I \setminus \{i_1\}$ are both modules of G_μ.
Since the intersection of two modules is a module we infer that $A := I \cap (\{i_0\} \cup I \setminus \{i_1\}) = I \setminus \{i_1\}$ is a module of G_μ. It follows from item (2) that A is trivial. Hence, $A = \varnothing$ or A is a singleton or $A = \{i_0\} \cup I$. This last case is not possible. The case $A = \varnothing$ is also not possible because otherwise $I = \{i_1\}$, which contradicts $|I| \geq 2$. We are left with the case A is a singleton, that is I has two elements. We derive a contradiction from the claim.

CASE 3: $\{i_0, i_n\}$ and I are both modules in G_μ.
We apply item (3) of Lemma 7.2 with $M := \{i_0, i_n\}$ and $N := I$. Since $M \setminus N \neq \varnothing$, then $A := N \setminus M = I \setminus \{i_n\}$ is a module of G_μ. It follows from item (2) that A is trivial. Hence, $A = \varnothing$ or A is a singleton or $A = \{i_0\} \cup I$. This last case is not possible. The case $A = \varnothing$ is also not possible because otherwise $I = \{i_n\}$, which contradicts $|I| \geq 2$. We are left with the case A is a singleton, that is I has two elements. We derive a contradiction from the claim.

CASE 4: $\{i_1, i_n\}$ and $\{i_0\} \cup I \setminus \{i_1\}$ are both modules of G_μ.
We apply item (3) of Lemma 7.2 with $M := \{i_1, i_n\}$ and $N := \{i_0\} \cup I \setminus \{i_1\}$. Since $M \setminus N \neq \varnothing$, then $A := N \setminus M = \{i_0\} \cup I \setminus \{i_1, i_n\}$ is a module of G_μ. It follows from item (2) that A

is trivial. Hence, $A = \varnothing$ or A is a singleton or $A = \{i_0\} \cup I$. This last case and the case $A = \varnothing$ are not possible. We are left with the case A is a singleton. Since $i_0 \notin I \setminus \{i_1, i_n\}$ we infer that $I \setminus \{i_1, i_n\} = \varnothing$ and hence I has at most two elements. We derive a contradiction from $|I| \geq 2$ in the case I is a singleton, and from the claim in the case $|I| = 2$.

CASE 5: $\{i_1, i_n\}$ and I are both modules in G_μ.

Since I is a module it follows from $(3)(c)$ of Lemma 7.8 that μ is constant on $I \setminus \{i_1\}$ and $\mu(i_1) \neq \mu(i_n)$. Then $n \leq 3$ because otherwise i_2 separates i_1 and i_n contradicting our assumption that $\{i_1, i_n\}$ is a module in G_μ. It follows that $\mu = 100$ or $\mu = 011$.

CASE 6: $\{i_0, i_n\}$ and $\{i_0\} \cup I \setminus \{i_1\}$ are both modules in G_μ.

Since $\{i_0\} \cup I \setminus \{i_1\}$ is a module it follows from $(3)(c)$ of Lemma 7.8 that μ is constant on $I \setminus \{i_1, i_2\}$ and $\mu(i_2) \neq \mu(i_n)$. Then $n \leq 3$ because otherwise i_{n-1} separates i_0 and i_n contradicting our assumption that $\{i_0, i_n\}$ is a module in G_μ. It follows that $\mu = 110$ or $\mu = 001$.

(2) Let M be nontrivial module of G_μ. Suppose first that M has cardinality at least 3. Let F be the largest final segment of I included in M. From (2) of Lemma 7.8, F is nonempty. Since M has at least 3 elements, it follows from item $(1)(b)$ and $(1)(c)$ of Lemma 7.8 that F has at least two elements. It follows from $(3)(c)$ of Lemma 7.8 that $F = I$ or $F = I \setminus \{i_1\}$. Since $F \subseteq M \subset I \cup \{i_0\}$, if $F = I$, then $M = I$. Else, it follows from $(1)(c)$ of Lemma 7.8 that $i_0 \in M$. Hence, $M = \{i_0\} \cup I \setminus \{i_1\}$.

We now consider the case M has exactly two elements. It follows from item $(1)(b)$ of Lemma 7.8 that $i_n \in M$. It follows from item $(1)(c)$ of Lemma 7.8 that exactly one of i_0 and i_1 is in M. Hence, $M = \{i_0, i_n\}$ or $M = \{i_1, i_n\}$.

The proof of the proposition is now complete. $\square$

It should be mentioned that the result of Proposition 7.9 for the particular case where the interval I is finite, can be deduced from Lemma 3.4 (see Figure 7) of [5] which specifies the enumeration of the vertices of the decomposable chains which were described in Figure 3 of [19] as a consequence of Proposition 2.2 of [19].

Several corollaries will now follow. Let μ_1 be the 0-1 word on $\mathbb{N}$ such that $\mu_1(0) = 0$ and $\mu_1(i) = 1$ for all $i \geq 1$. Let μ_2 be the 0-1 word on $\mathbb{N}$ such that $\mu_2(0) = \mu_2(1) = 0$ and $\mu_1(i) = 1$ for all $i \geq 2$.

Corollary 7.10. *Let μ be a 0-1 word on $\mathbb{N}$. The graph G_μ is prime if and only if $\mu \notin \{\mu_1, \overline{\mu_1}, \mu_2, \overline{\mu_2}\}$.*

Proof. We prove the following equivalence: the graph G_μ is not prime if and only if $\mu \in \{\mu_1, \overline{\mu_1}, \mu_2, \overline{\mu_2}\}$.

To prove the necessary condition, let be a nontrivial module of G_μ. It follows

from (1) of Proposition 7.9 that $M = \mathbb{N}$ or $M = \{-1\} \cup \mathbb{N} \setminus \{0\}$. It follows
from (3) (a) of Lemma 7.8 that $\mu \in \{\mu_1, \overline{\mu_1}, \mu_2, \overline{\mu_2}\}$ as required.
The proof of the sufficient condition is easy.   □

Since all of the 0-1 sequences in $\{\mu_1, \overline{\mu_1}, \mu_2, \overline{\mu_2}\}$ are not recurrent we get
this.

Corollary 7.11. *Let μ be a recurrent 0-1 word on $\mathbb{N}$. Then the graph G_μ
is prime.*

We have a similar conclusion to the corollary if we consider words on $\mathbb{N}^*$
or $\mathbb{Z}$ but not necessarily recurrent.

Lemma 7.12. *Let μ be a 0-1 word on $\mathbb{N}^*$ or on $\mathbb{Z}$. Then the graph G_μ is
prime.*

Proof. As in (2) of Lemma 7.8, if M is a nontrivial module of G_μ, then the
largest final segment F of $\mathbb{N}^*$ or of $\mathbb{Z}$ included in M is nonempty. Suppose
for a contradiction that $F \neq \mathbb{N}^*$ and $F \neq \mathbb{Z}$ and let $n := \min(F)$. Then
$n - 1 \notin M$ because otherwise $F \cup \{n - 1\}$ is a final segment included in
M and $F \subseteq F \cup \{n - 1\}$ contradicting the maximality of F. Since M is a
module and $n - 1 \notin M$ we infer that $n - 1$ must be either adjacent to both
n and $n + 1$ or nonadjacent to both n and $n + 1$. Since $n - 1$ and n are
consecutive in $\mathbb{Z}$ we have $\mu(n) \neq \mu(n+1)$. But then every $k < n-1$ separates
n and $n + 1$. It follows from our assumption that M is a module and that
$\{k : k < n - 1\} \subseteq M$. Hence, $M = \mathbb{N}^* \setminus \{n - 1\}$ or $M = \mathbb{Z}^* \setminus \{n - 1\}$. We
get a contradiction since $n - 3 \in M$ and $n - 1$ separates $n - 2$ and $n - 3$.   □

In the next proposition we show that G_μ not being prime forces the se-
quence μ to have a large factor of 0's or of 1's.

Proposition 7.13. *Let $I := \{i_1, \ldots, i_n\}$ be a finite interval of $\mathbb{N}$, $i_0 = i_1 - 1$,
and let μ be a 0-1 sequence on I. Suppose G_μ is not prime and let M be a
nontrivial module of G_μ.*
CASE *1: M has cardinality 2. Then <u>either</u> $M = \{i_0, i_n\}$ and either ($n = 2$
and ($\mu = 00$ or $\mu = 11$)), or $n > 2$ and ($\mu = 1\underbrace{00\ldots0}_{n-3}10$ or*

$\mu = 0\underbrace{11\ldots1}_{n-3}01$), <u>or</u> $M = \{i_1, i_n\}$, and either ($n = 3$ and ($\mu = 100$

or $\mu = 011$)), or $n > 3$ and ($\mu = 11\underbrace{00\ldots0}_{n-4}10$ or $\mu = 00\underbrace{11\ldots1}_{n-4}01$).
CASE *2: M has cardinality n. Then <u>either</u> $M = I$ and ($\mu = 1\underbrace{00\ldots0}_{n-1}$*

or $\mu = 0\underbrace{11\ldots1}_{n-1}$), <u>or</u> $M = \{i_0\} \cup I \setminus \{i_1\}$ and ($\mu = 00\underbrace{11\ldots1}_{n-2}$ or

*$\mu = 11\underbrace{00\ldots0}_{n-2}$). In particular, M induces a path or the complement
of a path in G_μ.*

Proof. Since G_μ and $G_{\overline{\mu}}$ have the same modules (this follows from Lemma 7.1) we may assume without loss of generality that $\mu(i_n) = 0$.

CASE 1: Suppose M has exactly two elements. It follows from item (2) of Proposition 7.9 that either $M = \{i_0, i_n\}$ or $M = \{i_1, i_n\}$. Suppose $M = \{i_0, i_n\}$. It follows from our assumption $\mu(i_n) = 0$ that i_n is not adjacent to i_{n-1} and i_n is adjacent to i_k for all $k < n - 1$. Since $\{i_0, i_n\}$ is a module we infer that i_0 cannot be adjacent to i_{n-1} and i_0 is adjacent to i_k for all $1 \leq k < n - 1$. It follows that $\mu(i_{n-1}) = 1$ if $i_{n-1} \neq i_1$, and $\mu(i_{n-1}) = 0$ if $i_{n-1} = i_1$, that is if $n = 2$. Furthermore, $\mu(i_k) = 0$ for all $1 < k < n - 1$ and $\mu(i_1) = 1$. Thus $\mu = 00$ if $n = 2$ and $\mu = 1\underbrace{00\ldots0}_{n-3}10$ if $n > 2$.

Suppose $M = \{i_1, i_n\}$. It follows from our assumption $\mu(i_n) = 0$ that i_n is not adjacent to i_{n-1} and i_n is adjacent to i_k for all $k < n - 1$. Hence, i_1 cannot be adjacent to i_{n-1} and i_1 is adjacent to i_0 and to i_k for all $2 < k < n-1$. It follows that $(\mu(i_{n-1}) = 0$ if $n = 3)$ and $(\mu(i_{n-1}) = 1$ if $n > 3)$ and $\mu(i_k) = 0$ for all $1 < k < n - 1$ and $\mu(i_1) = 1$. Then $\mu = 100$ if $n = 3$ and $\mu = 11\underbrace{00\ldots0}_{n-4}10$ otherwise.

CASE 1: Suppose M has exactly n elements. It follows from item (2) of Proposition 7.9 that $M = I$ or $M = \{i_0\} \cup I \setminus \{i_1\}$.

Suppose $M = I$. It follows from (3) (a) of Lemma 7.8 that μ is constant on $I \setminus \{i_1\}$ and $\mu(i_1) \neq \mu(i_2)$. It follows from our assumption $\mu(i_n) = 0$ that $\mu(i_1) = 1$ and $\mu(i_k) = 0$ for all $2 \leq k \leq n$, in which case μ induces the complement of a path on M.

Suppose $M = \{i_0\} \cup I \setminus \{i_1\}$. It follows from (3) (a) of Lemma 7.8 that μ is constant on $I \setminus \{i_2\}$ and $\mu(i_2) \neq \mu(i_3)$. It follows from our assumption $\mu(i_n) = 0$ that $\mu(i_2) = 1$, then $\mu(i_k) = 0$ for all $3 \leq k \leq n$ and $\mu(i_1) = 1$, in which case μ induces the complement of a path on M.

The proof of the proposition is now complete. $\square$

Let X be a set of finite 0-1 words. Let $l_0(X)$, respectively $l_1(X)$, be the supremum, over all words μ in X, of the length of factors of $0s$ in μ, respectively of the length of $1s$ in μ. Let $l(X) := \max\{l_0(X), l_1(X)\}$. For a 0-1 sequence μ we let $l(\mu) := l(\mathrm{Fac}(\mu))$. Note that $l(\mu) = l(\overline{\mu})$. We should mention that if μ is uniformly recurrent and nonconstant, then $l(\mu)$ is finite.

Corollary 7.14. *Let X be an infinite set of finite 0-1 words such that $l(X)$ is finite. Then for every $w \in X$ such that $|w| > l(X) + 4$ the graph G_w is prime.*

Proof. Let $w \in X$ be such that $|w| > l(X)+4$ and suppose for a contradiction that G_w is not prime. It follows from Proposition 7.13 that w has a factor of 0's or of 1's of length at least $|w| - 4$. Hence, $|w| - 4 \leq l(X)$. This contradicts our assumption $|w| > l(X) + 4$. $\square$

Corollary 7.15. *If X is an infinite initial segment of $\{0,1\}^*$, then the set X' of $u \in X$ such that G_u is prime is infinite.*

Proof. If X contains factors of 0's or factors of 1's of arbitrary large length, then the corresponding graphs are clearly prime. Otherwise, $l(X)$ is finite and the conclusion follows from Corollary 7.14. □

We denote by 1^k the constant word of length k whose all letters are 1, that is $1^k := \underbrace{11\ldots1}_{k \text{ times}}$. Similarly we define 0^k.

Corollary 7.16. *Let $I := \{i_1, \ldots, i_n\}$ be a finite interval of $\mathbb{N}$, $i_0 := i_1 - 1$, and let μ be a 0-1 sequence on I. Suppose G_μ is prime but at least one of $G_\mu \setminus \{i_0\}$ and $G_\mu \setminus \{i_n\}$ and $G_\mu \setminus \{i_1\}$ is not prime. Then μ has 0^{n-6} or 1^{n-6} as a factor.*

Proof. (1) The graph $G_\mu \setminus \{i_0\}$ is isomorphic to the graph $G_{\mu'}$ where $V(G_{\mu'}) := \{i_1, \ldots, i_n\}$ and $\mu' := \mu_{\restriction \{i_2,\ldots,i_n\}}$. If $G_\mu \setminus \{i_0\}$ is not prime, then the graph $G_{\mu'}$ is not prime and we can apply Proposition 7.13 to this graph with $n' := n - 1$ and deduce that μ' has $0^{n'-4}$ or $1^{n'-4}$ as a factor. Hence, μ has 0^{n-5} or 1^{n-5} as a factor.

The case $G_\mu \setminus \{i_n\}$ not prime can be treated similarly. Apply Proposition 7.13 to the graph $G_{\mu'}$ where $V(G_{\mu'}) := \{i_0, \ldots, i_{n-1}\}$ and $\mu' := \mu_{\restriction \{i_1,\ldots,i_{n-1}\}}$ and $n' := n - 1$.

(2) Suppose $G_\mu \setminus \{i_1\}$ is not prime and let M be a nontrivial module. If $M = \{i_2, \ldots, i_n\}$, then i_0 must be either adjacent to all elements of M or adjacent to none. Thus μ is constant on M, that is $\mu_{\restriction \{i_2,\ldots,i_n\}} = 0^{n-1}$ or $\mu_{\restriction \{i_2,\ldots,i_n\}} = 1^{n-1}$. If $M \neq \{i_2, \ldots, i_n\}$, then M is a nontrivial module of G'_μ where $V(G_{\mu'}) := \{i_2, \ldots, i_n\}$ and $\mu' := \mu_{\restriction \{i_3,\ldots,i_n\}}$. We then apply Proposition 7.13 to $G_{\mu'}$ with $n' = n - 2$ and deduce that μ' has 0^{n-6} or 1^{n-6} as a factor. For the remainder of the proof we may assume that M meets $\{i_2, \ldots, i_n\}$ in a singleton and since M is nontrivial $i_0 \in M$. Let $k \neq 0$ be such that $i_k \in M$. Since i_{k+1} separates i_k from i_0 we infer that $k + 1 > n$. This shows that $k = n$, that is $M = \{i_0, i_n\}$. Suppose $\mu(i_n) = 1$. Then no vertex in $\{i_2, \ldots, i_{n-2}\}$ is adjacent to i_n. Since M is a module no vertex in $\{i_2, \ldots, i_{n-2}\}$ is adjacent to i_0 and therefore μ is constant on $\{i_2, \ldots, i_{n-2}\}$ and takes the value 1. Thus μ has 1^{n-3} as factor. If $\mu(i_n) = 0$, then we obtain that μ has 0^{n-3} as factor. □

8. Embeddings between 0-1 graphs

In this section we study the relation between embeddings of words and embeddings of the corresponding 0-1 graphs, see for example Proposition 8.5. Results obtained in this section will be used in the proof of Theorem 5.11.

Lemma 8.1. *Let μ a 0-1 sequence on an interval I of $\mathbb{N}$. Let $\{i_0, i_1, i_2, i_3\} \subseteq I$ be such that $i_0 < i_1 < i_2 < i_3$. If $G_{\mu \restriction \{i_0, i_1, i_2, i_3\}}$ is isomorphic to a P_4, then $\{i_1, i_2, i_3\}$ is an interval of $\mathbb{N}$ and $\mu_{\restriction \{i_1, i_2, i_3\}}$ can be any 0-1 word of length 3.*

Proof. Since a P_4 is prime it follows from Corollary 7.5 that $\{i_1, i_2, i_3\}$ is an interval of $\mathbb{N}$. Since P_4 is isomorphic to its complement it follows that if $G_{\mu \restriction \{i_0, i_1, i_2, i_3\}}$ is isomorphic to P_4, then so is $G_{\overline{\mu} \restriction \{i_0, i_1, i_2, i_3\}}$. So we may assume without loss of generality that $\mu(i_3) = 1$. If $\mu(i_2) = 0$, then $\{i_1, i_2\}$ is not an edge of $G_{\mu \restriction \{i_0, i_1, i_2, i_3\}}$ and $\{i_0, i_2\}$ is an edge of $G_{\mu \restriction \{i_0, i_1, i_2, i_3\}}$. Hence, $\mu(i_1) = 1$ if i_1 is a successor of i_0 and $\mu(i_1) = 0$ otherwise. If $\mu(i_2) = 1$, then $\{i_1, i_2\}$ is an edge of $G_{\mu \restriction \{i_0, i_1, i_2, i_3\}}$ and $\{i_0, i_2\}$ is not an edge of $G_{\mu \restriction \{i_0, i_1, i_2, i_3\}}$. Hence, $\mu(i_1) = 1$ if i_1 is a successor of i_0 and $\mu(i_1) = 0$ otherwise. $\square$

Lemma 8.2. *Let μ a 0-1 sequence on an interval I of $\mathbb{N}$. Let $\{i_0, i_1, \ldots, i_{k-1}\} \subseteq I$ be such that $k \geq 5$ and $i_0 < i_1 \ldots < i_{k-1}$. If $G_{\mu \restriction \{i_0, \ldots, i_{k-1}\}}$ is isomorphic to P_k, then $\{i_1, \ldots, i_{k-1}\}$ is an interval of $\mathbb{N}$ and $\mu_{\restriction \{i_3, \ldots, i_{k-1}\}} = 1^{k-3}$ and $\mu_{\restriction \{i_1, i_2\}}$ can be any 0-1 word of length 2.*

Proof. Suppose $G_{\mu \restriction \{i_0, \ldots, i_{k-1}\}}$ is isomorphic to P_k. Since P_k is prime for $k \geq 4$ it follows from Corollary 7.5 that $\{i_1, \ldots, i_{k-1}\}$ is an interval of $\mathbb{N}$. Then $\mu(i_{k-1}) = 1$ because otherwise i_{k-1} would be a vertex of degree at least 3 in P_k and this is impossible. Similarly, we have $\mu_{i_{k-2}} = 1$. Since i_{k-1} is a vertex of degree 1 in $G_{\mu \restriction \{i_0, \ldots, i_{k-1}\}}$, which is isomorphic to P_k, we infer that $G_{\mu \restriction \{i_0, \ldots, i_{k-2}\}}$ is isomorphic to P_{k-1}. The required conclusion follows from Lemma 8.1 and an induction on $k \geq 5$. $\square$

Lemma 8.3. *Let μ be a 0-1 sequence on an interval J of $\mathbb{N}$. Let $I := \{i_0, i_1, \ldots, i_n\}$ be a finite interval of $\mathbb{N}$ with $n \geq 2$ and let w be a 0-1 sequence on $I \setminus \{i_0\}$. Suppose G_w embeds into G_μ and let f be such an embedding. If $f(i_n) = \max(f(I))$, then $f(\{i_2, \ldots, i_n\})$ is an interval of J and f is strictly increasing on $\{i_2, \ldots, i_n\}$ and $\mu_{\restriction f(\{i_3, \ldots, i_n\})} = w_3 \ldots w_n$.*

Proof. Let $w = w_1 \ldots w_n$. We notice at once that we can assume without loss of generality that $w_n = 1$. Indeed, if $w_n = 0$, then we consider $\overline{w}$ and $\overline{\mu}$ and recall that $G_{\overline{w}}$ is the complement of G_w. Furthermore, two graphs embed in each other if and only if their corresponding complements embed in each other.

Let f be an embedding of G_w into G_μ such that $f(i_n) = \max(f(I))$. If $n = 2$, there is nothing to prove. Next we suppose $n \geq 3$. It follows from our assumption $w_n = 1$ that i_n has degree 1 in G_w and i_{n-1} is its unique neighbour. Since f is an embedding we infer that $f(i_n)$ has degree 1 in $f(G_w)$. It follows from this and $n \geq 3$ and $f(i_n) = \max(f(I))$ that $\mu(f(i_n)) = 1$. Hence, $f(i_n) - 1$ is the unique neighbour of $f(i_n)$ in G_μ satisfying $f(i_n) - 1 < f(i_n)$, and therefore in $f(G_w)$. Since f is an embedding we must have $f(i_{n-1}) = f(i_n) - 1$. The proof of the lemma follows by induction on $n \geq 3$. $\square$

It should be noted that $f(\{i_1, i_2, \ldots, i_n\})$ is not an interval of J in general. Indeed, $f(i_1) < f(i_0)$ is possible.

Lemma 8.4. *Let $I := \{i_0, i_1, \ldots, i_n\}$ be a finite interval of $\mathbb{N}$ with $n \geq 7$ and let w be a 0-1 sequence on $I \setminus \{i_0\}$ so that G_w is prime. Let μ be a 0-1 sequence on an interval J of $\mathbb{N}$. Suppose G_w embeds into G_μ and let f be such an embedding. Let $f(I) := \{j_0, j_1, \ldots, j_n\}$ so that $j_0 < j_1 < \ldots < j_n$. If $f(i_n) \in \{j_0, j_1\}$, then w and μ have 0^{n-7} or 1^{n-7} as a factor.*

Proof. Let $w := w_1 \ldots w_n$. As in the proof of Lemma 8.3 we may assume without loss of generality that $w_n = 1$. Then i_n has degree 1 in G_w and i_{n-1} is its unique neighbour. Let f be an embedding of G_w into G_μ and suppose $f(i_n) \in \{j_0, j_1\}$. Since f is an embedding we infer that $f(i_n)$ has degree 1 in $f(G_w)$. Furthermore, since G_w is prime, $f(G_w)$ is prime too and therefore $\{j_1, \ldots, j_n\}$ is an interval of $\mathbb{N}$ (Corollary 7.5).

CASE 1: $f(i_n) = j_0$.

Let $k \in \mathbb{N}$ be such that $j_k := f(i_{n-1})$. It follows from Corollary 7.16 that we may assume $k \notin \{1, n\}$. Since f is an embedding and $j_0 = f(i_n)$ it follows that j_k is the unique neighbour of j_0 in $f(G_w)$. It follows from this and $k \notin \{1, n\}$ that $\mu(j_k) = 0$ and μ is constant on $\{j_2, \ldots, j_n\} \setminus \{j_k\}$ and takes the value 1. In particular, j_k has at least j_0 and j_{k+1} as neighbours.

IF $w_{n-1} = 1$, then i_{n-1} has degree 2 in G_w and since f is an embedding $j_k = f(i_{n-1})$ has degree 2 in $f(G_w)$. It follows from $k \notin \{1, n\}$ and $\mu(j_k) = 0$ that $k = 2$. In particular, $\mu_{\restriction \{j_3, \ldots, j_n\}} = 1^{n-2}$ and $f(G_w)$ embeds P_{n-1}. Since f is an embedding we infer that G_w embeds P_{n-1}. It follows from Lemma 8.2 that w has 1^{n-4} as a factor.

ELSE IF $w_{n-1} = 0$, then i_{n-1} has degree $n - 1$. Since f is an embedding we infer that j_k has degree $n - 1$ in $f(G_w)$. This forces $k = n - 1$. It follows from Lemma 8.3 applied to $w' = w_1 \ldots w_{n-1}$ and μ that f is strictly increasing on $\{i_2, \ldots, i_n\}$ and $\mu_{\restriction f(\{i_3, \ldots, i_{n-1}\})} = w_3 \ldots w_{n-1}$. In particular, w has 1^{n-3} as a factor.

CASE 2: $f(i_n) = j_1$.

Let $k \in \mathbb{N}$ be such that $j_k = f(i_{n-1})$. It follows from Corollary 7.16 that we may assume $k \notin \{1, n\}$. Since f is an embedding and $j_1 = f(i_n)$ it follows that j_k is the unique neighbour of j_1 in $f(G_w)$. It follows from this and $k \notin \{1, n\}$ that:

(a) $k = 2$ and μ is constant on $\{j_2, \ldots, j_n\}$ and takes the value 1, or

(b) $k > 2$ and $\mu(j_2) = \mu(j_k) = 0$ and μ is constant on $\{j_3, \ldots, j_n\} \setminus \{j_k\}$ and takes the value 1.

IF $w_{n-1} = 1$, then i_{n-1} has degree 2 in G_w and since f is an embedding $j_k = f(i_{n-1})$ has degree 2 in $f(G_w)$. Then only case (a) holds. Indeed, if not j_k would be adjacent to j_{k+1}, j_1 and j_0 and hence has degree 3 which is impossible. Thus $\mu_{\restriction \{j_2, \ldots, j_n\}} = 1^{n-1}$. In

particular, $f(G_w)$ has an induced P_n and since f is an embedding we infer that G_w has an induced P_n. It follows from Lemma 8.2 that w has 1^{n-3} as a factor.

ELSE IF $w_{n-1} = 0$, then i_{n-1} has degree $n-1$ in G_w. Since f is an embedding we infer that $j_k = f(i_{n-1})$ has degree $n-1$ in $f(G_w)$. Then only case (b) holds. Indeed, if not j_k would be adjacent only to j_1, j_3 and hence has degree 2 which is impossible. This forces $k = n-1$. It follows that $\mu_{\restriction\{j_3,\ldots,j_{n-3}\}} = 1^{n-5}$. In particular, $f(G_w)$ has an induced P_{n-4} and since f is an embedding we infer that G_w has an induced P_{n-4}. It follows from Lemma 8.2 that w has 1^{n-7} as a factor.

The proof of the lemma is now complete. $\qquad\qquad\square$

Proposition 8.5. *Let μ be a recurrent word on $\mathbb{N}$ such that $l(\mu)$ is finite. Let $w := w_0 \ldots w_{n-1}$ be a finite word such that $n > l(\mu) + 7$. If G_w embeds into G_μ and f is such an embedding, then $f(-1), f(0) < f(1) < f(2) < \ldots < f(n-1)$ and either $\{f(-1), f(1), f(2), \ldots, f(n-1)\}$ or $\{f(0), f(1), f(2), \ldots, f(n-1)\}$ is an interval of $\mathbb{N}$ and $w_2 \ldots w_{n-1}$ is a factor of μ.*

Proof. It follows from Corollary 7.14 that G_w is prime. It follows from our assumption $n > l(\mu) + 7$ and Corollary 7.16 that $G_w \setminus \{-1\}$ and $G_w \setminus \{0\}$ and $G_w \setminus \{n-1\}$ are also prime. Since f is an embedding it follows that in $f(G_w)$ removal of one of the vertices $f(-1)$ or $f(0)$ or $f(n-1)$ leaves a prime graph. Since G_w is prime and f is embedding it follows that $f(G_w)$ is also prime. It follows from Corollary 7.5 that $I := f(V(G_w)) \setminus \{\min(f(V(G_w))\}$ is an interval of $\mathbb{N}$. It follows from Corollary 7.7 that $f(n-1) \in \{\min(f(V(G_w)), \min(I), \max(I)\}$. It follows from Lemma 8.4 that $f(n-1) = \max(f(V(G_w)))$. The required conclusion follows then from Lemma 8.3. $\qquad\square$

Corollary 8.6. *Let μ be a recurrent word on $\mathbb{N}$ such that $l(\mu) < 4$. Let u, v be finite words such that $|v| \geq 3$ and $|u| > l(\mu) + 4$ and G_{vu} is prime. If G_{vu} embeds into G_μ, then $u \in \mathrm{Fac}(\mu)$.*

Proof. Follows from Proposition 8.5 applied to $w := vu$. $\qquad\square$

Lemma 8.7. *If μ is recurrent word and $u \in \mathrm{Fac}(\mu)$, then there exists $v \in \{0, 1\}^*$ such that $|v| \geq 4$ and $vu \in \mathrm{Fac}(\mu)$ and G_{vu} is prime.*

Proof. We consider several cases.

CASE 1: μ has 1^4 as a factor.

We can write $\mu = \alpha 1^4 \mu'$ where α is a finite word and μ' is an infinite 0-1 sequence. Since μ is recurrent $\mathrm{Fac}(\mu) = \mathrm{Fac}(1^4 \mu') = \mathrm{Fac}(\mu')$. Hence, we may assume without loss of generality that α is the empty word. Let $u \in \mathrm{Fac}(\mu')$. There exists then $\beta \in \mathrm{Fac}(\mu')$ such that $1^4 \beta u \in \mathrm{Fac}(\mu)$. It follows from Proposition 7.13 that $G_{1^4 \beta u}$ is prime. Choose $v := 1^4 \beta$.

CASE 2: μ has 0^4 as a factor.

We apply Case 1 to $\overline{\mu}$ and $\overline{u}$.

CASE 3: $l(\mu) < 4$. Let $u \in \text{Fac}(\mu)$. Since μ is recurrent there exists $v \in \text{Fac}(\mu)$ such that $vu \in \text{Fac}(\mu)$ and $|v| \geq 4$ and $|vu| > l(\mu) + 4$. It follows from Corollary 7.14 that G_{vu} is prime.

The proof of the lemma is now complete. $\qquad\square$

Lemma 8.8. *Let μ be a 0-1 word on an interval I of $\mathbb{N}$ and let $w := w_1 \ldots w_n$ be any finite word. If G_{1^4w} embeds into G_μ, then $1w$ is a factor of μ.*

Proof. We notice at once that it follows from Proposition 7.13 that G_{1^4w} is prime. Let f be an embedding of G_{1^4w} into G_μ. Then the image of G_{1^4w} under f is prime. We write $f(V(G_{1^4w})) = \{i_0, i_1, i_2, i_3, i_4, j_1, \ldots, j_n\}$ so that $i_0 < \ldots < i_4 < j_1 < \ldots < j_n$. It follows from Corollary 7.5 that $\{i_1, i_2, i_3, i_4, j_1, j_2, \ldots, j_n\}$ is an interval of $\mathbb{N}$.

We use induction on the length $n \geq 1$ of w to prove the following statement: $\mu_{\restriction\{i_4, j_1, \ldots, j_n\}} = 1w_1 \ldots w_n$ and if $w \neq 1^n$, then for all $i \in \{1, \ldots, n\}$, f maps the vertex of G_{1^4w} corresponding to w_i to the vertex j_i.

For the basis case suppose $w \in \{0, 1\}$. If $w = 1$, then $G_{1^4w} = G_{1^5}$ is a path on six vertices. Since f is an embedding we infer that $f(G_{1^4w})$ is a path on six vertices. It follows from Lemma 8.2 that $\mu(i_4) = \mu(j_1) = 1$ and hence $1w = 11$ is a factor of μ as required. Now suppose $w = 0$ and note that G_{1^4w} has exactly one vertex of degree four. We prove that $\mu(j_1) = 0$. Suppose for a contradiction that $\mu(j_1) = 1$. Then $\mu(i_4) = 0$ because otherwise $f(G_{1^4w})$ will not have a vertex of degree four and since f is an embedding neither will G_{1^4w} which is impossible. But then in $f(G_{1^4w})$ the vertex i_4 which has degree four is adjacent to the vertex j_1 which has degree one and hence in G_{1^4w} the vertex of degree four is adjacent to a vertex of degree one and this is not possible. A contradiction. Hence, our supposition that $\mu(j_1) = 1$ is false, that is $\mu(j_1) = 0$ as required. Now since $f(G_{1^4w}) \setminus \{j_1\}$ is a path on five vertices it follows from Lemma 8.2 that $\mu(i_4) = 1$ and hence $1w = 10$ is a factor of μ as required.

Next we consider the inductive case. We first note that if $w = 1^n$, then G_{1^4w} is a path on $n + 5$ vertices. We apply Lemma 8.2 with $k = n + 5$ and deduce that $\mu_{\restriction\{i_3, i_4, j_1, \ldots, j_n\}} = 1^{n+2}$ and hence $1w$ is a factor of μ. We now assume that $w \neq 1^n$. Suppose that $w_1 \ldots w_{n-1} = 1^{n-1}$. Then $G_{1^4w_1 \ldots w_{n-1}}$ is a path on $n + 4$ vertices. It follows from Lemma 8.2 that $\mu_{\restriction\{i_3 i_4, j_1, \ldots, j_{n-1}\}} = 1^{n+1}$. From our assumption that $w \neq 1^n$ we deduce that $w_n = 0$. Hence, G_{1^4w} has a unique vertex of degree $n + 4$ and this vertex is associated to w_n. Since f is an embedding and $\mu_{\restriction\{i_3 i_4, j_1, \ldots, j_{n-1}\}} = 1^{n+1}$ it follows that the image under f of the vertex associated to w_n must be j_n and j_n has degree $n + 4$. This shows that $\mu(j_n) = 0$ and hence $1w$ is a factor of μ.

Next we suppose that $w_1 \ldots w_{n-1} \neq 1^{n-1}$. By the induction hypothesis $\mu_{\restriction\{i_4, j_1, \ldots, j_{n-1}\}} = 1w_1 \ldots w_{n-1}$ and for all $i \in \{1, \ldots, n-1\}$, f maps the vertex of G_{1^4w} corresponding to w_i to the vertex j_i. We note that j_{n-1} is the unique neighbour or the unique nonneighbour of j_n in $f(G_{1^4w})$. Since

f is an embedding it follows that j_n is the image under f of the vertex of G_{1^4w} corresponding to w_n and $\mu(j_n) = w_n$. This completes the proof of the lemma. $\qquad\square$

Corollary 8.9. *Let μ be a 0-1 word on an interval I of $\mathbb{N}$ and let $w := w_1 \ldots w_n$ be any finite word. If G_{0^4w} embeds into G_μ, then $0w$ is a factor of μ.*

Proof. We apply Lemma 8.8 to $\overline{\mu}$ and $\overline{w}$ and recall that $G_{1^4\overline{w}}$ embeds into $G_{\overline{\mu}}$ if and only if the complement of $G_{1^4\overline{w}}$, which is G_{0^4w}, embeds into the complement of $G_{\overline{\mu}}$, which is G_μ. $\qquad\square$

Corollary 8.10. *Let μ be a word 0-1 on $\mathbb{N}$.*
(1) *If w is a bound of $\mathrm{Fac}(\mu)$, then G_{1^4w} and G_{0^4w} do not embed into G_μ.*
(2) *If $(w_i)_{i \in I}$, $I \subseteq \mathbb{N}$, is an antichain (with respect to the factor ordering) of finite words such that no w_i starts with 1, then $(G_{1^4w_i})_{i \in I}$ is an antichain of (permutation) graphs.*

Proof. (1) The fact that G_{1^4w} does not embed into G_μ follows from Lemma 8.8. The fact that G_{0^4w} does not embed into G_μ follows from Corollary 8.9.

(2) Suppose for a contradiction that there exists $i \neq j$ such that $G_{1^4w_i}$ embeds into $G_{1^4w_j}$. It follows from Lemma 8.8 that $1w_i$ is a factor of 1^4w_j. Since w_i does not start with 1 we infer that w_i is a factor of w_j. This is impossible since by assumption the sequence $(w_i)_{i \in I}$ is an antichain of words.

The proof of the corollary is now complete. $\qquad\square$

9. A proof of Theorem 5.11

We prove the following strengthening of Theorem 5.11. For that we introduce first the following notation: if X is a set of finite 0-1 words we set $G_X := \{G_w : w \in X\}$ and

$$\downarrow G_X := \{H : H \text{ embeds into some } G_w \in G_X\}.$$

Theorem 9.1. *Let μ be a recurrent word and X an initial segment of $\{0,1\}^*$ for the factor ordering. If $\mathrm{Age}(G_\mu) \subseteq \downarrow G_X$, then $\mathrm{Fac}(\mu) \subseteq X$.*

Proof. Let $u \in \mathrm{Fac}(\mu)$. We prove that $u \in X$. By Lemma 8.7, if μ is recurrent and $u \in \mathrm{Fac}(\mu)$, then there is some $v \in \{0,1\}^*$ with $|v| \geq 4$ such that $vu \in \mathrm{Fac}(\mu)$ and G_{vu} is prime. Since $G_{vu} \in \mathrm{Age}(G_\mu) \subseteq \downarrow G_X$, G_{vu} embeds in G_w for some $w \in X$. If $l(\mu) < 4$, then it follows from Corollary 8.6 that u is a factor of w. If $l(\mu) \geq 4$, then there is $u' \in \mathrm{Fac}(\mu)$ such that u is a factor of u' and either $0^4u'$ or $1^4u'$ is a factor of μ. It follows from Lemma 8.8 and Corollary 8.9 applied to either $v = 0^4$ or $v = 1^4$ that u' is a factor of w, and so is u. Hence, $u \in X$. $\qquad\square$

Theorem 5.11 now follows by observing that $\mathrm{Age}(G_{\mu'}) = \downarrow G_{\mathrm{Fac}(\mu')}$ and then applying Theorem 9.1 to $X := \mathrm{Fac}(\mu')$.

10. A proof of Theorem 5.13

Proof. $(i) \Rightarrow (ii)$. Let μ be uniformly recurrent. Then trivially, μ is recurrent. Since for two infinite sequence τ and τ', the equality $\mathrm{Fac}(\tau) = \mathrm{Fac}(\tau')$ implies $\mathrm{Age}(G_\tau) = \mathrm{Age}(G_{\tau'})$, it follows from Theorem 3.10 that we may assume that μ is a word on $\mathbb{N}$. It follows from Corollary 7.11 that G_μ is prime. Hence, from Theorem 4.2 it follows that the set of prime graphs in $\mathrm{Age}(G_\mu)$ is cofinal in $\mathrm{Age}(G_\mu)$ hence infinite. Now let $\mathcal{C}$ be a proper age of $\mathrm{Age}(G_\mu)$. We prove that $\mathcal{C}$ contains only a finite number of prime graphs. If $\mathcal{C}$ contains restrictions on intervals of $\mathbb{N}$ of arbitrary large length, then according to Corollary 7.14, $\mathcal{C}$ contains finite prime graphs of arbitrary large length and therefore $\mathcal{C} = \mathrm{Age}(G_\mu)$. Otherwise, $\mathcal{C}$ contains only restrictions to factors of μ of bounded length. It follows from Corollary 7.5 that every prime member of $\mathcal{C}$ of cardinality m induces an interval of $\mathbb{N}$ of cardinality $m - 1$. Therefore prime members of $\mathcal{C}$ have bounded cardinality. That is, there are only finitely many prime members of $\mathcal{C}$.

$(ii) \Rightarrow (i)$. First $\mathrm{Fac}(\mu)$ is infinite since μ is recurrent. Next, let X be an infinite initial segment of $\mathrm{Fac}(\mu)$. We claim that $X = \mathrm{Fac}(\mu)$. Corollary 7.15 asserts that the set $X' := \{u \in X : G_u \text{ is prime}\}$ is infinite. Since $\downarrow G_X$ contains infinitely many primes and $\mathrm{Age}(G_\mu)$ is minimal prime, $\downarrow G_X = \mathrm{Age}(G_\mu)$. Since $\mathrm{Age}(G_\mu) \subseteq \downarrow G_X$, Theorem 9.1 asserts that $\mathrm{Fac}(\mu) \subseteq X$. This proves our claim. $\square$

11. Bounds of 0-1 graphs: a proof of Theorem 5.25.

Let μ be a 0-1 sequence. Then every bound of $\mathrm{Age}(G_\mu)$ is one of the following types:

(1) Finite graphs that are not comparability graphs and that are minimal with this property.
(2) Finite comparability graphs of critical posets of dimension three (see subsection 3.1.2).
(3) Finite comparability graphs of posets of dimension two, that is, finite permutation graphs.

For example if $\mu = 11111\ldots$, then the bounds of $\mathrm{Age}(G_\mu)$ listed according to their type are:

(a) Odd cycles of length at least 5. These are of type (1).
(b) Even cycles of length at least 6. These are of type (2).
(c) The complete bipartite graph $K_{1,3}$ and the complete graph K_3. These are of type (3).

Let μ be a 0-1 sequence. If μ contains factors of 1's of arbitrary length, then it follows from Lemma 8.2 that G_μ embeds P_k for infinitely many k's, hence $\mathrm{Age}(G_\mu)$ contains the age of an infinite path. The cycles C_k are bounds of the infinite path and form an infinite antichain. Since cycles of length at least five are not permutation graphs, these cycles are bounds of $\mathrm{Age}(G_\mu)$. Now suppose that neither P_k nor $\overline{P_k}$ embed in G_μ. In particular,

μ has infinitely many 1's and 0's, that is, G_μ has an infinite independent set and an infinite clique. If we put an upper bound on the length of paths and of complement of paths in members of the lists of Gallai [26] and Kelly [32], there are only finitely many such members, hence $\mathrm{Age}(G_\mu)$ has only finitely many bounds of type (1) and finitely many bounds of type (2). Hence,

Theorem 11.1. *If the age of G_μ does not contain the age of the infinite path nor of its complement, then it has only finitely many bounds which are not permutation graphs.*

It is tempting to think that candidates for bounds of $\mathrm{Age}(G_\mu)$ of type (3) are graphs of the form G_w where w is a bound of $\mathrm{Fac}(\mu)$. This is false.

Lemma 11.2. *Let μ be a recurrent 0-1 sequence on $\mathbb{N}$ and $w := w_1 \ldots w_n$ be a finite word. If $w_2 \ldots w_n$ is a factor of μ, then G_w embeds into G_μ.*

Proof. Suppose $w_2 \ldots w_n$ is a factor of μ. Let $\{j_2, \ldots, j_n\} \subseteq \mathbb{N}$ be such that $\mu(j_k) = w_k$ for all $2 \leq k \leq n$. Since μ is recurrent we may assume that there are at least three elements of $\mathbb{N} \cup \{-1\}$ before j_2. Let $j_1 := j_2 - 1$ and $w'_1 := \mu(j_1)$. If $w'_1 = w_1$, then w is a factor of μ and hence G_w embeds into G_μ. Else if $w'_1 \neq w_1$, then we set $j_0 := j_1 - 2$. It follows that G_w is isomorphic to $G_{\mu \upharpoonright \{j_0, j_1 \ldots, j_n\}}$. $\square$

Corollary 11.3. *Let μ be a recurrent 0-1 sequence on $\mathbb{N}$ and w be a finite word. If w is a bound of $\mathrm{Fac}(\mu)$, then G_w embeds into G_μ.*

11.1. **Proof of** (1) **of Theorem 5.25.** We show first how to construct a bound of $\mathrm{Age}(G_\mu)$ using a bound of $\mathrm{Fac}(\mu)$.

Lemma 11.4. *Let μ be a recurrent 0-1 sequence on $\mathbb{N}$ with $l(\mu)$ finite. Let $w = w_1 \ldots w_n$ be a finite word such that $n > l(\mu) + 7$.*

(1) *If $w := w_1 \ldots w_n$ is a bound of $\mathrm{Fac}(\mu)$ and $w_0 \in \{0, 1\}$ is such that $w_0 \ldots w_{n-1}$ is a factor of μ and $w' := w_0 w_1 \ldots w_n$, then $G_{w'}$ is a bound of $\mathrm{Age}(G_\mu)$.*
(2) *If G_w is a bound of $\mathrm{Age}(G_\mu)$, then $w_2 \ldots w_n$ is a bound of $\mathrm{Fac}(\mu)$.*

Proof. (1) We need to prove that $G_{w'}$ does not embed into G_μ and that deleting any vertex from $G_{w'}$ yields a graph that embeds into G_μ. We notice at once that it follows from our assumption $n > l(\mu) + 7$ that $G_{w'}$ is prime. We first prove that $G_{w'}$ does not embed into G_μ. Suppose not and let f be an embedding of $G_{w'}$ into G_μ. Then $f(i_n) = \max(f(V(G_{w'})))$ because otherwise it follows from Corollary 7.16 that w' has 0^{n-6} or 1^{n-6}. Hence, w has 0^{n-7} or 1^{n-7} as a factor. Since $w_0 \ldots w_{n-1}$ is a factor of μ we infer that $n - 7 < l(\mu)$ contradicting our assumption that $n > l(\mu) + 7$. This proves that $f(i_n) = \max(f(V(G_{w'})))$. It follows then for Lemma 8.3 that w is a factor of μ contradicting our assumption that w is a bound of $\mathrm{Fac}(\mu)$. This proves that $G_{w'}$ does not embed into G_μ.
Next we prove that deleting any vertex from $G_{w'}$ yields a graph that embeds into G_μ. Set $V(G_{w'}) = \{-1, 0, \ldots, n\}$. First we consider the graph

$G_{w'} \setminus \{-1\}$ and observe that it is isomorphic to G_w. It follows from Corollary 11.3 that G_w embeds into G_μ. We now consider the graph $G_{w'} \setminus \{n\}$ and observe that it is isomorphic to $G_{w_0 \ldots w_{n-1}}$. Since $w_0 \ldots w_{n-1}$ is a factor of μ we infer that $G_{w_0 \ldots w_{n-1}}$ is an induced subgraph of G_μ. Let $k \notin \{-1, n\}$ and consider the graph $G_{w'} \setminus \{k\}$. Then $G_{w' \restriction \{-1, \ldots, k-1\}}$ is the graph $G_{w_0 \ldots w_{k-1}}$ and $G_{w' \restriction \{k+1, \ldots, n\}}$ is the graph $G_{w_{k+2} \ldots w_n}$. Since $w_0 \ldots w_{k-1}$ and $w_{k+2} \ldots w_n$ are factors of μ the graphs $G_{w_0 \ldots w_{k-1}}$ and $G_{w_{k+2} \ldots w_n}$ are induced subgraphs of G_μ, and hence, so is $G_{w'} \setminus \{k\}$. This completes the proof of (1).

(2) Suppose G_w is a bound of $\mathrm{Age}(G_\mu)$. Then w cannot be a factor of μ and it follows from Lemma 11.2 that w is not a bound of $\mathrm{Fac}(\mu)$. Hence, w has a factor which is a bound of $\mathrm{Fac}(\mu)$. We prove that $w_2 \ldots w_n$ is a bound of $\mathrm{Fac}(\mu)$, that is, $w_2 \ldots w_n$ is a not a factor of μ and both words $w_3 \ldots w_n$ and $w_2 \ldots w_{n-1}$ are factors of μ. The fact that $w_2 \ldots w_n$ is not a factor of μ follows from Lemma 11.2 and the fact that G_w does not embed in G_μ. Next we prove that $w_3 \ldots w_n$ and $w_2 \ldots w_{n-1}$ are factors of μ. It follows from our assumption $n > l(\mu) + 7$ and Corollary 7.14 that G_w is prime. Next we set $V(G_w) := \{i_0, i_1, \ldots, i_n\}$ so that w is a word on $\{i_1, \ldots, i_n\}$. It follows from Corollary 7.16 that $G_w \setminus \{i_0\}$ and $G_w \setminus \{i_n\}$ are prime. It follows from our assumption that G_w is a bound of $\mathrm{Age}(G_\mu)$ that $G_w \setminus \{i_0\}$ and $G_w \setminus \{i_n\}$ embed in G_μ. It follows from Lemma 8.4 and our assumption $n > l(\mu) + 7$ that if f and g are such embeddings, then $f(i_n) = \max(f(G_w \setminus \{i_0\}))$ and $g(i_{n-1}) = \max(g(G_w \setminus \{i_n\}))$. Lemma 8.3 yields that $\mu_{\restriction f(\{i_3, \ldots, i_n\})} = w_3 \ldots w_n$ and $\mu_{\restriction g(\{i_2, \ldots, i_{n-1}\})} = w_2 \ldots w_{n-1}$. This proves that $w_3 \ldots w_n$ and $w_2 \ldots w_{n-1}$ are factors of μ as required. $\qquad \square$

The proof of (1) of Theorem 5.25 follows from Theorem 3.14 and (1) of Lemma 11.4.

11.2. **Proof of (2) of Theorem 5.25.** Clearly if μ is periodic and u is a period, then $\overline{\mu}$ is periodic and $\overline{u}$ is a period.

Lemma 11.5. *Let μ be a 0-1 word on $\mathbb{N}$, let $I := \{i_0, i_1, \ldots, i_{n-1}\} \subseteq \mathbb{N} \cup \{-1\}$ so that $i_0 < i_1 < \ldots < i_{n-1}$ and $H := G_{\mu \restriction I}$ be an induced subgraph of G_μ. Let $j < k < n-1$. If i_j is adjacent to all vertices in $\{i_k, \ldots, i_{n-1}\}$, then μ is constant on $\{i_{k+1}, \ldots, i_{n-1}\}$ and takes the value 0. In particular, if $l(\mu)$ is finite and $\{i_{k+1}, \ldots, i_{n-1}\}$ is an interval of $\mathbb{N}$, then $n - l(\mu) - 1 \leq k$.*

Proof. Straightforward. $\qquad \square$

Lemma 11.6. *Let μ be a 0-1 word on $\mathbb{N}$ such $l(\mu)$ is finite. Let $J := \{j_0, j_1, \ldots, j_k\} \subseteq \mathbb{N}$ be such that $j_0 < j_1 < \ldots < i_k$ and $\{j_1, \ldots, j_k\}$ is an interval of $\mathbb{N}$ and $k > l(\mu) + 5$. Then $G := G_\mu \restriction J$ is prime.*

Proof. Suppose for a contradiction that G is not prime. Let M be a nontrivial module of G. Then $M \cap \{j_1, \ldots, j_k\}$ is a module of $G \setminus \{j_0\}$. It follows from our assumption that $k > l(\mu) + 5$ and Corollary 7.14 that $G \setminus \{i_0\}$ is prime. Hence, $M \cap \{j_1, \ldots, j_k\}$ is either empty, reduced to a singleton or is

equal to $\{j_1, \ldots, j_k\}$. Since M is nontrivial we infer that $M = \{j_1, \ldots, j_k\}$ or $M \cap \{j_1, \ldots, j_k\}$ is a singleton. If $M = \{j_1, \ldots, j_k\}$, then j_0 must be either adjacent to all elements of M or adjacent to none. Thus μ is constant on M, that is, $k \leq l(\mu) < k - 5$. A contradiction. Else if $M \cap \{j_1, \ldots, j_k\}$ is a singleton, then $M = \{j_0, j_m\}$ for some $1 \leq m \leq k$. Necessarily $m = k$, because otherwise j_{m+1} separates j_m from j_0. That is, $M = \{j_0, j_k\}$. Suppose $\mu(j_k) = 1$. Then no vertex in $\{j_2, \ldots, j_{k-2}\}$ is adjacent to j_k. Since M is a module, no vertex in $\{j_2, \ldots, j_{k-2}\}$ is adjacent to j_0 and therefore μ is constant on $\{j_2, \ldots, j_{k-2}\}$ and takes the value 1. Thus μ has 1^{k-3} as factor. If $\mu(j_k) = 0$, then we obtain that μ has 0^{k-3} as factor. Therefore, $k - 3 \leq l(\mu)$ and from our assumption $k > l(\mu) + 5$ we get $k - 3 < k - 5$ which is impossible. $\square$

Corollary 11.7. *Let μ be a 0-1 word on $\mathbb{N}$ such $l(\mu)$ is finite. Let $J := \{j_0, j_1, \ldots, j_k\} \subseteq \mathbb{N}$ be such that $j_0 < j_1 < \ldots < i_k$ and $\{j_1, \ldots, j_k\}$ is an interval of $\mathbb{N}$ and $k > l(\mu) + 6$. Let $G := G_\mu \restriction J$ and $x \in J$. Then $G \setminus \{x\}$ is prime if and only if $x \in \{j_0, j_1, j_k\}$.*

Lemma 11.8. *Let μ be a 0-1 word on $\mathbb{N}$ such that $l(\mu)$ is finite. Let $\{i_0, i_1, \ldots, i_{n-1}\} \subseteq \mathbb{N}$ be such that $i_0 < i_1 < \ldots < i_{n-1}$ and $\{i_1, \ldots, i_{n-1}\}$ is an interval of $\mathbb{N}$ and $n > l(\mu) + 8$. Let $x \notin \mathbb{N}$ and H be the graph whose vertex set is $\{i_0, i_1, \ldots, i_{n-1}\} \cup \{x\}$ and edge set $E := E(G_{\mu \restriction \{i_0, i_1, \ldots, i_{n-1}\}}) \cup \{\{i_1, x\}, \{i_2, x\}, \ldots, \{i_{n-1}, x\}\}$. Then H does not embed into G_μ.*

Proof. Suppose for a contradiction that H embeds into G_μ and let f be such an embedding. Then f induces an embedding of $H \setminus \{x\}$ into G_μ. It follows from Lemma 11.6 that $H \setminus \{x\}$ is prime. According to Corollary 7.5 the image of $H \setminus \{x\}$ under f decomposes into a point y and an interval J to its right. It follows from Corollary 11.7 that $f(\{i_0, i_1, i_n - 1\}) = \{y, \min(J), \max(J)\}$. It follows from Lemma 8.4 that $f(i_{n-1}) = \max(J)$. Hence, $f(\{i_0, i_1\}) = \{y, \min(J)\}$. Now, we argue on the possible position of $f(x)$. Suppose that $f(x)$ is to the left of $f(i_{n-1})$. Since $\{f(i_2), \ldots, f(i_{n-1})\}$ is an interval of $\mathbb{N}$ we infer that $f(x)$ is to the left of $f(i_2)$. Since $f(x)$ is adjacent to all vertices in $\{f(i_1), f(i_2), \ldots, f(i_{n-1})\}$ it follows from Lemma 11.5 that μ is constant on $\{f(i_2), \ldots, f(i_{n-1})\}$. Hence, $n - 2 \leq l(\mu)$. From our assumption that $n > l(\mu) + 8$ we get $n < n - 8$ which is impossible. Now suppose that $f(x)$ is to the right of $f(i_{n-1})$. Since $f(x)$ is adjacent to all vertices in $\{f(1), \ldots, f(n - 1)\}$ it is adjacent to $f(i_{n-2})$ and $f(i_{n-1})$. It follows that $\mu(f(x)) = 0$. Thus $f(x)$ is adjacent to $f(i_0)$, hence x is adjacent to i_0 in H. A contradiction. This proves that our supposition H embeds into G_μ is false. $\square$

A vertex x of a graph G is -1-*extremal* if either x is not adjacent to at most one vertex of $V(G) \setminus \{x\}$ or if x is adjacent to at most one vertex of $V(G) \setminus \{x\}$. Note that if x is -1-extremal in G, then x is also -1-extremal in $\overline{G}$.

Lemma 11.9. *Let $\mathcal{C}$ be a hereditary class of finite graphs which is 1^--well-quasi-ordered. Then $\mathcal{C}$ has only finitely many bounds having a -1-extremal vertex.*

Proof. Since $\mathcal{C}$ is w.q.o. there are only finitely many bounds of $\mathcal{C}$ having a vertex adjacent to all other vertices. Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of bounds of $\mathcal{C}$ such that each G_n has a -1-extremal vertex x_n. We may suppose that there is a unique vertex y_n distinct from x_n and not adjacent to x_n. Let $H_n := G_{n\restriction V(G_n)\setminus\{x_n\}}$. Since $\mathcal{C}$ is 1^--well-quasi-ordered from the sequence (H_n, y_n) we can extract an increasing subsequence. Clearly, if (H_n, y_n) embeds into (H_m, y_m), then G_n embeds into G_m. This contradicts the fact that $\{G_n : n \in \mathbb{N}\}$ forms an antichain. $\square$

Corollary 11.10. *Let μ be a periodic 0-1 sequence on $\mathbb{N}$. Then there are only finitely many bounds of $\mathrm{Age}(G_\mu)$ having a -1-extremal vertex.*

Proof. Follows from Lemma 11.9 and the fact that $\mathrm{Age}(G_\mu)$ is 1^--well-quasi-ordered. $\square$

Lemma 11.11. *Let μ be a periodic 0-1 sequence on $\mathbb{N}$. Then the number of non prime bounds of $\mathrm{Age}(G_\mu)$ is finite.*

Proof. Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of bounds of $\mathrm{Age}(G_\mu)$. Suppose G_n is not prime. Let M_n be a nontrivial module of G_n and x_n any vertex of M_n. Let $H_n := G_{n\restriction(V(G_n)\setminus M_n)\cup\{x_n\}}$. Since M_n is nontrivial and G_n is a bound we infer that H_n and M_n are elements of $\mathrm{Age}(G_\mu)$. Since $\mathrm{Age}(G_\mu)$ is w.q.o there exists an infinite subset I of $\mathbb{N}$ so that the sequence $(G_{n\restriction M_n})_{n\in I}$ is increasing with respect to embeddability. Since $\mathrm{Age}(G_\mu)$ is 1^--well-quasi-ordered we infer that we can extract from the sequence $(H_n, x_n)_{n\in I}$ an increasing subsequence. Then note that if (H_n, x_n) embeds into (H_m, x_m) and M_n embeds into M_m, then G_n embeds into G_m. $\square$

We now prove (2) of Theorem 5.25. Let μ be a nonconstant and periodic 0-1 sequence on $\mathbb{N}$ and let H be a bound of G_μ. It follows from Lemma 11.11 that we may assume that H is prime. Since the examples of critically prime graphs of Schmerl and Trotter [63] split into two totally ordered sets with respect to embeddability we may assume that H is not critically prime. There exists then $x \in V(H)$ such that $H \setminus \{x\}$ is prime. Since H is a bound of G_μ we infer that $H \setminus \{x\}$ embeds into G_μ. Let f_x be such an embedding. We write $f_x(V(H \setminus \{x\}) := \{i_0, i_1, \ldots, i_n\}$ so that $i_0 < i_1 < \ldots < i_n$. Since $H \setminus \{x\}$ is prime it follows from Corollary 7.5 that $\{i_1, \ldots, i_n\}$ is an interval of $\mathbb{N}$. Since μ is periodic $l(\mu)$ is finite. For $n > l(\mu) + 5$, $G_\mu \restriction \{i_1 < \ldots < i_n\}$ is prime, hence $H \setminus \{x, f_x^{-1}(i_0)\}$ is prime. We may assume that $\mu(i_n) = 0$ (if not consider $\overline{G_\mu} = G_{\overline{\mu}}$ and $\overline{H}$ and note that $\overline{\mu}$ is also periodic). By Lemma 11.9 we may assume that H has no -1-extremal vertices. It follows that $\{x, f_x^{-1}(i_n)\}$ is not an edge of H (otherwise $f_x^{-1}(i_n)$ is -1-extremal in H). We now consider the graph $H \setminus \{f_x^{-1}(i_0)\}$. Let g_{i_0} be an embedding of $H \setminus \{f_x^{-1}(i_0)\}$ into G_μ. For $1 \le k \le n$, we define $i'_k := g_{i_0}(f_x^{-1}(i_k))$.

Suppose $n > l(\mu) + 7$. It then follows from Proposition 8.5 that every embedding of $H \setminus \{f_x^{-1}(i_0)\}$ in G_μ maps $\{i'_2, \ldots, i'_n\}$ into an interval and in that order. Hence, such an embedding agree with f_x and g_{i_0}. From our assumption that $\mu(i_n) = 0$ and $\{x, f_x^{-1}(i_n)\}$ is not an edge of H we deduce that $g_{i_0}(x)$ is to the right of i'_n. Indeed, if $g_{i_0}(x)$ is to the left of i'_n, then since $\mu(i'_n) = \mu(i_n) = 0$ we infer that $g_{i_0}(x)$ is on the left of i'_3. But then $\{g_{i_0}(x), i'_n\}$ is an edge, therefore $\{x, f_x^{-1}(i_0)\}$ is an edge of H hence x is -1-extremal, which is not possible . Thus, $g_{i_0}(x)$ is to the right of i'_n. It follows then that $g_{i_0}(x)$ is either adjacent to all vertices in $\{i'_1, \ldots, i'_{n-1}\}$ or adjacent to none. This last case is not possible, otherwise x would be -1-extremal. So we are left with the case that $g_{i_0}(x)$ is adjacent to all vertices in $\{i'_1, \ldots, i'_{n-1}\}$. Since x is not -1-extremal, x is adjacent to all vertices of $H \setminus \{f_x^{-1}(i_0), f_x^{-1}(i_n)\}$ and not adjacent to either $f^{-1}(i_0)$ or $f^{-1}(i_n)$. It follows from Lemma 11.8 that $H \setminus \{i_n\}$ does not embed into G_μ. This contradicts our assumption that H is a bound of $\mathrm{Age}(G_\mu)$.

12. Conclusion

This work on hereditary classes of finite graphs containing relatively few primes puts a light on hereditary classes which are well-quasi-ordered and also on those made of permutation graphs. The result of [19] was crucial in proving that our list of hereditary classes of graphs which are minimal prime was complete. Kim [34] obtained for tournaments a result similar to Chudnovski et al. [19]. It remains to see if results similar to ours can be obtained in the case of tournaments; and also, if they shed light on the case of binary relations and binary relational structures and allow to solve the problems mentioned in the text about minimal prime hereditary classes. Among questions which interest us are first the rank of minimal prime classes of permutation graphs; in this respect note that it is unknown if there are hereditary well-quasi-ordered classes of graphs with arbitrary countable rank (see [59]). Next, the question to know whether well-quasi-ordered hereditary classes of finite graphs are better-quasi-ordered.

A consequence of our study is the existence of an uncountable antichain of well-quasi-ordered ages of permutation graphs. The existence of uncountably many well-quasi-ordered ages of binary structures was obtained in 1978 [51]. This was obtained by means of an encoding via uniformly recurrent sequences. The same existence for graphs, permutation graphs or posets, is a nontrivial fact which requires work. The same encoding than the one we use in this paper was used first in 1992 [64] and in 2002 [65]. In chapter 5 of [47] the first author proved with a simpler encoding the existence of uncountably many hereditary classes of oriented graphs which are minimal prime. We conclude by mentioning the existence of uncountably many well-quasi-ordered ages of permutation graphs with distinct enumeration functions (alias profile) due to Brignall and Vatter [15].

Acknowledgements

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