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CONGRUENCE PROPERTIES MODULO POWERS OF 2 FOR PARTITION PAIRS INTO DISTINCT PARTS

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Abstract. Let $Q(n)$ denote the number of partitions of n into distinct parts. In 1997, Gordon and Ono proved that almost all values of $Q(n)$ are divisible by 2^m with any fixed positive integer m. Let $Q_2(n)$ denote the number of partition pairs of n into distinct parts. A result derived by Ray and Barman reveals that almost all values of $Q_2(n)$ are also divisible by 2^m with any fixed positive integer m. Quite recently, the author derived several internal congruences and congruences modulo powers of 2 satisfied by $Q(n)$. In this paper, we prove some internal congruences and congruences modulo powers of 2 for $Q_2(n)$. Moreover, we prove an infinite family of congruence relations modulo 4 and dozens of congruence relations modulo powers of 2 enjoyed by $Q_2(n)$. Finally, we pose two conjectures on congruence properties modulo powers of 2 for $Q_2(n)$.

1. Introduction

The purpose of this paper is to establish some internal congruences and congruences modulo powers of 2 for $Q_2(n)$ by utilizing some q-series identities and iterative computations, where $Q_2(n)$ denotes the number of partition pairs of n into distinct parts. Moreover, we also derive dozens of congruence relations modulo powers of 2 enjoyed by $Q_2(n)$ by using the theory of modular forms. These results reveal that there is an inseparable relation between arithmetic density property on powers of 2 and congruence properties modulo powers of 2 for $Q_2(n)$.

A partition π of a positive integer n is a finite weakly decreasing sequence of positive integers $\pi_1 \geq \pi_2 \geq \cdots \geq \pi_r$ such that $\sum_{i=1}^{r} \pi_i = n$. The numbers

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π_i are called the parts of the partition π. Let $p(n)$ denote the number of partitions of n with the convention that $p(0) = 1$. The generating function of $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty},$$

where here and throughout the paper, we always assume that q is a complex number such that $|q| < 1$ and adopt the following standard notation:

$$(a;q)_\infty := \prod_{j=0}^{\infty}(1 - aq^j).$$

In 1919, Ramanujan [19] discovered, and later confirmed, the following three remarkable congruences

$$p(5n + 4) \equiv 0 \pmod 5, \tag{1.1}$$

$$p(7n + 5) \equiv 0 \pmod 7, \tag{1.2}$$

$$p(11n + 6) \equiv 0 \pmod{11}. \tag{1.3}$$

In the theory of partitions, the most famous partition theorem is Euler's partition theorem, which states that there are as many partitions of n into distinct parts as into odd parts. This partition theorem can be written via the generating function, namely,

$$\sum_{n=0}^{\infty} Q(n)q^n = (-q;q)_\infty = \frac{1}{(q;q^2)_\infty} = \frac{(q^2;q^2)_\infty}{(q;q)_\infty},$$

where $Q(n)$ denotes the number of partitions of n into distinct parts. Ramanujan's congruences (1.1)–(1.3) have been a continuing source of inspiration and have motivated a tremendous amount of research for over a century. Therefore, congruence properties of $Q(n)$ were also considered by some scholars; see, for example, [4, 5, 13, 14, 21]. In 1997, Gordon and Ono [11] derived the following arithmetic density property on powers of 2 for $Q(n)$:

Theorem 1.1 (Gordon–Ono)**.** *Let m be a positive integer. Then $Q(n)$ is almost always divisible by 2^m, namely,*

$$\lim_{X\to\infty} \frac{\#\{0 \le n < X : Q(n) \equiv 0 \pmod{2^m}\}}{X} = 1. \tag{1.4}$$

In 2018, Andrews [1] investigated the partition function $\mathcal{EO}(n)$ which denotes the number of partitions of n where every even part is less than each odd part. Uncu [25] further considered a certain subset of the partitions enumerated by $\mathcal{EO}(n)$. We denote by $\mathcal{EO}_u(n)$ the partition function defined by Uncu, and its generating function is given by

$$\sum_{n=0}^{\infty} \mathcal{EO}_u(n)q^n = \frac{1}{(q^2;q^4)_\infty^2} = \frac{(q^4;q^4)_\infty^2}{(q^2;q^2)_\infty^2}. \tag{1.5}$$

Let $Q_2(n)$ denote the number of partition pairs of n into distinct parts, and the generating function of $Q_2(n)$ is given by

$$\sum_{n=0}^{\infty} Q_2(n)q^n = (-q;q)_\infty^2 = \frac{1}{(q;q^2)_\infty^2} = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty^2}. \tag{1.6}$$

In 2019, Andrews and Newman [3] defined the minimal excludant of an integer partition π as the least positive integer missing from the partition, denoted by $\mathrm{mex}(\pi)$. For instance, there are five partitions of 4: 4 with $\mathrm{mex}(\pi) = 1$; $3+1$ with $\mathrm{mex}(\pi) = 2$; $2+2$ with $\mathrm{mex}(\pi) = 1$; $2+1+1$ with $\mathrm{mex}(\pi) = 3$; $1+1+1+1$ with $\mathrm{mex}(\pi) = 2$. Andrews and Newman [3, Theorem 1.1] derived an elegant identity for the generating function of $\sigma\mathrm{mex}(n)$, which denotes the sum of minimal excludants over all the partitions of n. More precisely, they proved that

$$\sum_{n=0}^{\infty} \sigma\mathrm{mex}(n)q^n = (-q;q)_\infty^2 = \sum_{n=0}^{\infty} Q_2(n)q^n. \tag{1.7}$$

It follows immediately from (1.5)–(1.7) that $\mathcal{EO}_u(2n) = Q_2(n) = \sigma\mathrm{mex}(n)$ holds for any $n \geq 0$. In 2020, Ray and Barman [20] proved the following arithmetic density property on powers of 2 for $Q_2(n)$:

Theorem 1.2 (Ray–Barman)**.** *Let m be a positive integer. Then $Q_2(n)$ is almost always divisible by 2^m, namely,*

$$\lim_{X\to\infty} \frac{\#\{0 \leq n < X : Q_2(n) \equiv 0 \pmod{2^m}\}}{X} = 1. \tag{1.8}$$

Although (1.4) is a powerful result for $Q(n)$, the theory of modular forms used to derive the arithmetic density property is not constructive and it does not give an explicit congruence satisfied by $Q(n)$. Therefore, it is still desirable to derive explicit congruence modulo powers of 2 for $Q(n)$. In a recent paper, Merca [15] established some congruences modulo 4 and 8 for $Q(n)$ by utilizing Smoot's `Mathematica` implementation of Radu's algorithm (see [22]) for proving partition congruences. Motivated by this work, the author [24] further derived several internal congruences and congruences modulo powers of 2 enjoyed by $Q(n)$. For instance, he proved that for any $n \geq 0$ and $1 \leq i \leq 4$,

$$Q\left(5^{256}n + \frac{5^{256}-1}{24}\right) \equiv 257Q(n) \pmod{512} \tag{1.9}$$

and

$$Q\left(5^{256}n + \frac{(24i+5)\times 5^{255}-1}{24}\right) \equiv 0 \pmod{512}. \tag{1.10}$$

The identities (1.4) and (1.8) reveal that the two partition functions $Q(n)$ and $Q_2(n)$ have the same arithmetic density property on powers of 2. Therefore, for (1.9) and (1.10), a natural question is whether there exist some

similar internal congruences and congruences modulo powers of 2 for $Q_2(n)$. The following theorem says that the answer is positive.

Theorem 1.3. *For any $n \geq 0$,*

$$Q_2(25n+2) \equiv 3Q_2(n) \pmod{16}, \tag{1.11}$$

and for any $n \geq 0$ and $2 \leq \alpha \leq 8$,

$$Q_2\left(5^{2^\alpha}n + \frac{5^{2^\alpha}-1}{12}\right) \equiv \left(3 \times 2^{\alpha+1}+1\right)Q_2(n) \pmod{2^{\alpha+3}}. \tag{1.12}$$

Moreover, for any $n \geq 0$, $1 \leq \alpha \leq 8$ and $1 \leq i \leq 4$,

$$Q_2\left(5^{2^\alpha}n + \frac{(12i+5) \times 5^{2^\alpha-1}-1}{12}\right) \equiv 0 \pmod{2^{\alpha+3}}. \tag{1.13}$$

Corollary 1.4. *For any $n \geq 0$, $1 \leq \alpha \leq 8$, $1 \leq i \leq 4$ and $\beta \geq 1$,*

$$Q_2\left(5^{2^\alpha\beta}n + \frac{(12i+5) \times 5^{2^\alpha\beta-1}-1}{12}\right) \equiv 0 \pmod{2^{\alpha+3}} \tag{1.14}$$

The following theorem gives an infinite family of congruence relations modulo 4 enjoyed by $Q_2(n)$.

Theorem 1.5. *Let $p \geq 5$ be a prime number such that $p \equiv 3 \pmod 4$. Then*

$$\sum_{n=0}^{\infty} Q_2\left(pn + \frac{p^2-1}{12}\right)q^n \equiv \sum_{n=0}^{\infty} Q_2(n)q^{pn} \pmod 4. \tag{1.15}$$

In particular, for any $n \geq 0$ and $1 \leq i \leq p-1$,

$$Q_2\left(p^2n + \frac{p^2-1}{12}\right) \equiv Q_2(n) \pmod 4. \tag{1.16}$$

Moreover, for $1 \leq i \leq p-1$,

$$Q_2\left(p^2n + \frac{(12i+p) \times p-1}{12}\right) \equiv 0 \pmod 4. \tag{1.17}$$

Although (1.15) provides an infinite family of congruence relations modulo 4 satisfied by $Q_2(n)$, the modulus in (1.15) may not be the best possible for a given prime $p \geq 5$ and $p \equiv 3 \pmod 4$. Utilizing the theory of modular forms, we prove the following congruence relations modulo powers of 2 enjoyed by $Q_2(n)$.

Theorem 1.6. *Let*

$$\begin{aligned} S := \big\{&(5,3,4),(7,9,6),(11,93,10),(13,1,1),(17,15,5),\\ &(19,101,7),(23,89,11),(29,11,4),(31,33,6),(37,1,1),\\ &(41,7,5),(43,125,7),(47,305,11),(53,3,4),(59,1805,11),\\ &(61,1,1),(67,149,8),(71,5769,13),(73,1,1),(79,17,6),\\ &(83,37,10),(89,23,6),(97,1,1),(101,3,4),(103,41,6)\big\}. \end{aligned}$$

Then for any $(p, c_p, m) \in S$,

$$\sum_{n=0}^{\infty} Q_2\left(pn + \frac{p^2-1}{12}\right) q^n \equiv c_p \sum_{n=0}^{\infty} Q_2(n) q^{pn} \pmod{2^m}. \tag{1.18}$$

In particular, for any $(p, c_p, k) \in S$ *and any* $n \geq 0$,

$$Q_2\left(p^2 n + \frac{p^2-1}{12}\right) \equiv c_p Q_2(n) \pmod{2^m}. \tag{1.19}$$

Moreover, for $1 \leq i \leq 4$,

$$Q_2\left(p^2 n + \frac{(12i+p) \times p - 1}{12}\right) \equiv 0 \pmod{2^m}. \tag{1.20}$$

A natural question worth studying is whether there exists a formula for c_p in (1.18).

The rest of this paper is organized as follows. In Section 2, we establish some important initial and general relations which will be applied to prove (1.11)–(1.14). The proofs of Theorem 1.3 and Corollary 1.4 are presented in Section 3. In Section 4, we give the proofs of Theorems 1.5 and 1.6. Finally, we pose two related conjectures and some follow-up questions to motivate further investigation.

2. Some initial and general relations

In this section, we collect some necessary identities which will be applied to prove the main results.

For the sake of convenience, we denote

$$E(q) := (q;q)_\infty.$$

We first introduce the following three auxiliary functions given by

$$\gamma = \frac{E(q^2)^2 E(q^5)^2}{E(q)^4}, \qquad \delta = \frac{E(q^2)^4 E(q^5)^4}{E(q)^6 E(q^{10})^2}, \qquad \zeta = q\frac{E(q^2)E(q^{10})^3}{E(q)^3 E(q^5)}. \tag{2.1}$$

Define the U-operator by

$$U\left(\sum_{n=n_0}^{\infty} a(n) q^n\right) = \sum_{n=\lceil n_0/5 \rceil}^{\infty} a(5n) q^n.$$

Next, we sketch the rough steps in the proofs of (1.12) and (1.13). Using some q-series identities and recurrences involving the Rogers–Ramanujan continued fraction (see (2.17)–(2.26)), we establish some q-series identities involving γ, δ and ζ (see (2.4)–(2.13)). Based on these identities, we find

that for any $\alpha \geq 1$,

$$\sum_{n=0}^{\infty} Q_2\Big(5^{2\alpha-1}n + \frac{5^{2\alpha-1}-1}{12}\Big)q^n = \sum_{i=1}^{(5^{2\alpha}-1)/12} g_{2\alpha-1,i}\,\gamma\zeta^{i-1}, \tag{2.2}$$

$$\sum_{n=0}^{\infty} Q_2\Big(5^{2\alpha}n + \frac{5^{2\alpha}-1}{12}\Big)q^n = \sum_{i=1}^{(5^{2\alpha+1}-5)/12} g_{2\alpha,i}\,\delta\zeta^{i-1}, \tag{2.3}$$

where $\{g_{\alpha,i}\}_{\alpha\geq 1, i\geq 1}$ are some positive integers. Utilizing another identity (see (3.3)) and analyzing the 2-adic orders of the sequences $\{g_{\alpha,i}\}_{\alpha\geq 1,i\geq 1}$, we obtain the desired internal congruences and congruences involving $Q_2(n)$. However, we also need to utilize two recurrences (i.e., replacing ζ^i by $\gamma\zeta^i$ and $q^{-2}\delta\zeta^i$ in (2.31)). From this perspective, the computations in proofs of (1.12) and (1.13) are lengthier and more complicated than those of (1.9) and (1.10) in [24].

2.1. **Initial relations.** The following lemma plays a vital role in the proofs of (1.11)–(1.14).

Lemma 2.1. *Let γ, δ and ζ be defined by* (2.1). *Then*

(i)

$$U(\gamma) = \delta\big(1 + 16\times 10\zeta + 28\times 10^2\zeta^2 + 16\times 10^3\zeta^3 + 32\times 10^3\zeta^4\big), \tag{2.4}$$

$$\begin{aligned} U(\gamma\zeta) = \delta\big(&385\zeta + 401\times 10^2\zeta^2 + 13128\times 10^2\zeta^3 \\ &+ 20912\times 10^3\zeta^4 + 18992\times 10^4\zeta^5 + 10432\times 10^5\zeta^6 \\ &+ 3456\times 10^6\zeta^7 + 64\times 10^8\zeta^8 + 512\times 10^7\zeta^9\big), \end{aligned} \tag{2.5}$$

$$\begin{aligned} U(\gamma\zeta^2) = \delta\big(&29\times 10\zeta + 119015\zeta^2 + 112356\times 10^2\zeta^3 \\ &+ 476348\times 10^3\zeta^4 + 1153776\times 10^4\zeta^5 \\ &+ 1794344\times 10^5\zeta^6 + 1908992\times 10^6\zeta^7 \\ &+ 14377472\times 10^6\zeta^8 + 7778304\times 10^7\zeta^9 \\ &+ 3016448\times 10^8\zeta^{10} + 821248\times 10^9\zeta^{11} \\ &+ 149504\times 10^{10}\zeta^{12} + 16384\times 10^{11}\zeta^{13} + 8192\times 10^{11}\zeta^{14}\big), \end{aligned} \tag{2.6}$$

$$\begin{aligned} U(\gamma\zeta^3) = \delta\big(&99\zeta + 157795\zeta^2 + 36522125\zeta^3 \\ &+ 33085695\times 10^2\zeta^4 + 16194315\times 10^4\zeta^5 \\ &+ 49956038\times 10^5\zeta^6 + 1059335888\times 10^5\zeta^7 \\ &+ 1628976896\times 10^6\zeta^8 + 1879743552\times 10^7\zeta^9 \end{aligned} \tag{2.7}$$

$$
\begin{aligned}
&+ 1663609088 \times 10^8\zeta^{10} + 1143762304 \times 10^9\zeta^{11}\\
&+ 614230016 \times 10^{10}\zeta^{12} + 2572978176 \times 10^{10}\zeta^{13}\\
&+ 833304576 \times 10^{11}\zeta^{14} + 204857344 \times 10^{12}\zeta^{15}\\
&+ 37003264 \times 10^{13}\zeta^{16} + 4636672 \times 10^{14}\zeta^{17}\\
&+ 360448 \times 10^{15}\zeta^{18} + 131072 \times 10^{15}\zeta^{19}\big),
\end{aligned}
$$

$$
\begin{aligned}
U(\gamma\zeta^4) = \delta\big(&16\zeta + 11809 \times 10\zeta^2 + 638351 \times 10^2\zeta^3 + 11315760375\zeta^4\\
&+ 1002222145 \times 10^3\zeta^5 + 537784392 \times 10^5\zeta^6\\
&+ 19463929732 \times 10^5\zeta^7 + 50789296612 \times 10^6\zeta^8\\
&+ 99869648352 \times 10^7\zeta^9 + 152569328944 \times 10^8\zeta^{10}\\
&+ 185007570368 \times 10^9\zeta^{11} + 180767148928 \times 10^{10}\zeta^{12}\\
&+ 1437629353984 \times 10^{10}\zeta^{13} + 936303455232 \times 10^{11}\zeta^{14}\\
&+ 500636522496 \times 10^{12}\zeta^{15} + 219558209536 \times 10^{13}\zeta^{16}\\
&+ 78607884288 \times 10^{14}\zeta^{17} + 22768123904 \times 10^{15}\zeta^{18}\\
&+ 52564656128 \times 10^{15}\zeta^{19} + 945225728 \times 10^{17}\zeta^{20}\\
&+ 127664128 \times 10^{18}\zeta^{21} + 12189696 \times 10^{19}\zeta^{22}\\
&+ 7340032 \times 10^{19}\zeta^{23} + 2097152 \times 10^{19}\zeta^{24}\big).
\end{aligned}
\tag{2.8}
$$

(ii)

$$
\begin{aligned}
U(q^{-2}\delta) = \gamma\big(&23 + 276 \times 10\zeta + 708 \times 10^2\zeta^2 + 768 \times 10^3\zeta^3\\
&+ 4192 \times 10^3\zeta^4 + 1152 \times 10^4\zeta^5 + 128 \times 10^5\zeta^6\big),
\end{aligned}
\tag{2.9}
$$

$$
\begin{aligned}
U(q^{-2}\delta\zeta) = \gamma\big(&9 + 7525\zeta + 7386 \times 10^2\zeta^2 + 279588 \times 10^2\zeta^3\\
&+ 562776 \times 10^3\zeta^4 + 691552 \times 10^4\zeta^5 + 554496 \times 10^5\zeta^6\\
&+ 298176 \times 10^6\zeta^7 + 107264 \times 10^7\zeta^8 + 248832 \times 10^7\zeta^9\\
&+ 33792 \times 10^8\zeta^{10} + 2048 \times 10^9\zeta^{11}\big),
\end{aligned}
\tag{2.10}
$$

$$
\begin{aligned}
U(q^{-2}\delta\zeta^2) = \gamma\big(&1 + 789 \times 10\zeta + 2306665\zeta^2\\
&+ 2127634 \times 10^2\zeta^3 + 9747472 \times 10^3\zeta^4\\
&+ 26876968 \times 10^4\zeta^5 + 49352536 \times 10^5\zeta^6\\
&+ 64017568 \times 10^6\zeta^7 + 607788352 \times 10^6\zeta^8\\
&+ 430976512 \times 10^7\zeta^9 + 230242816 \times 10^8\zeta^{10}\\
&+ 92517376 \times 10^9\zeta^{11} + 27597824 \times 10^{10}\zeta^{12}\\
&+ 59392 \times 10^{13}\zeta^{13} + 8732672 \times 10^{11}\zeta^{14}\\
&+ 786432 \times 10^{12}\zeta^{15} + 32768 \times 10^{13}\zeta^{16}\big),
\end{aligned}
\tag{2.11}
$$

(2.12)
$$\begin{aligned}U(q^{-2}\delta\zeta^3) = \gamma\big(&4289\zeta + 355983\times 10\zeta^2 + 710828225\zeta^3\\ &+ 636122475\times 10^2\zeta^4 + 326815927\times 10^4\zeta^5\\ &+ 1096212234\times 10^5\zeta^6 + 25909641648\times 10^5\zeta^7\\ &+ 4531493912\times 10^7\zeta^8 + 6059231232\times 10^8\zeta^9\\ &+ 63324664896\times 10^8\zeta^{10} + 52483174784\times 10^9\zeta^{11}\\ &+ 34796112384\times 10^{10}\zeta^{12} + 185224641536\times 10^{10}\zeta^{13}\\ &+ 79074091008\times 10^{11}\zeta^{14} + 2690289664\times 10^{13}\zeta^{15}\\ &+ 7203004416\times 10^{13}\zeta^{16} + 1485668352\times 10^{14}\zeta^{17}\\ &+ 227966976\times 10^{15}\zeta^{18} + 245235712\times 10^{15}\zeta^{19}\\ &+ 16515072\times 10^{16}\zeta^{20} + 524288\times 10^{17}\zeta^{21}\big),\end{aligned}$$

(2.13)
$$\begin{aligned}U(q^{-2}\delta\zeta^4) = \gamma\big(&1338\zeta + 3284195\zeta^2 + 13575158\times 10^2\zeta^3\\ &+ 221326582375\zeta^4 + 19455727335\times 10^3\zeta^5\\ &+ 107922559155\times 10^4\zeta^6 + 414312241256\times 10^5\zeta^7\\ &+ 1168106573468\times 10^6\zeta^8 + 2518614201968\times 10^7\zeta^9\\ &+ 4273799987216\times 10^8\zeta^{10} + 5826933222912\times 10^9\zeta^{11}\\ &+ 6480065706112\times 10^{10}\zeta^{12} + 59417717491712\times 10^{10}\zeta^{13}\\ &+ 45253146312704\times 10^{11}\zeta^{14} + 28755008360448\times 10^{12}\zeta^{15}\\ &+ 15271931029504\times 10^{13}\zeta^{16} + 6773946425344\times 10^{14}\zeta^{17}\\ &+ 2500362952704\times 10^{15}\zeta^{18} + 7628372312064\times 10^{15}\zeta^{19}\\ &+ 1903003828224\times 10^{16}\zeta^{20} + 381921263616\times 10^{17}\zeta^{21}\\ &+ 6018564096\times 10^{19}\zeta^{22} + 7174356992\times 10^{19}\zeta^{23}\\ &+ 6083837952\times 10^{19}\zeta^{24} + 327155712\times 10^{20}\zeta^{25}\\ &+ 8388608\times 10^{21}\zeta^{26}\big).\end{aligned}$$

Proof. By the definition of the U-operator, one easily obtains that

(2.14)
$$U\left(\left(\sum_{m=0}^{\infty} a(m)q^{5m}\right)\left(\sum_{n=0}^{\infty} b(n)q^n\right)\right) = \left(\sum_{m=0}^{\infty} a(m)q^m\right)\cdot U\left(\sum_{n=0}^{\infty} b(n)q^n\right).$$

The identities (2.4)–(2.8) follow from (5.6)–(5.10) in [5] and (2.14) immediately.

For (2.9)–(2.13), we only present the proof of (2.9) here, the remaining cases can be demonstrated similarly but with lengthier calculations, and thus, we omit the details. For this purpose, we require the following two

identities (see [12, Eqs. (8.1.1) and (8.4.4)]):

$$E(q)=E(q^{25})\big(R(q^5)^{-1}-q-q^2R(q^5)\big), \tag{2.15}$$

$$\begin{aligned}\frac{1}{E(q)}=\frac{E(q^{25})^5}{E(q^5)^6}\big(&R(q^5)^{-4}+qR(q^5)^{-3}+2q^2R(q^5)^{-2}+3q^3R(q^5)^{-1}\\&+5q^4-3q^5R(q^5)+2q^6R(q^5)^2-q^7R(q^5)^3+q^8R(q^5)^4\big),\end{aligned} \tag{2.16}$$

where

$$R(q)=\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}.$$

Moreover, we also need the following two identities, namely,

$$K+1=q^{-1}\frac{E(q^2)^4E(q^5)^2}{E(q)^2E(q^{10})^4}, \tag{2.17}$$

$$K-4=q^{-1}\frac{E(q)^3E(q^5)}{E(q^2)E(q^{10})^3}, \tag{2.18}$$

where

$$K=q^{-1}\frac{E(q^2)E(q^5)^5}{E(q)E(q^{10})^5}. \tag{2.19}$$

Actually, (2.17) and (2.18) are equivalent to (9.10) and (9.11) in [5], respectively.

Further, for any $m\in\mathbb{Z}_{\geq 0}$ and $n\in\mathbb{Z}$, we define

$$P(m,n)=\frac{1}{q^mR(q)^{m+2n}R(q^2)^{2m-n}}+(-1)^{m+n}q^nR(q)^{m+2n}R(q^2)^{2m-n}. \tag{2.20}$$

Chern and the author [6, Theorem 1.1] proved that

$$P(m,n+1)=4K^{-1}P(m,n)+P(m,n-1), \tag{2.21}$$

$$P(m+2,n)=KP(m+1,n)+P(m,n), \tag{2.22}$$

where

$$P(0,0)=2, \tag{2.23}$$

$$P(0,1)=4K^{-1}, \tag{2.24}$$

$$P(1,0)=K, \tag{2.25}$$

$$P(1,-1)=4K^{-1}-2+K. \tag{2.26}$$

Now we proceed with the proof of (2.9).

Replacing q by q^2 in (2.15) gives

$$E(q^2)=E(q^{50})\big(R(q^{10})^{-1}-q^2-q^4R(q^{10})\big). \tag{2.27}$$

According to (2.14), (2.16) and (2.27), we find that

$$U(\delta)=U\left(\frac{E(q^2)^4E(q^5)^4}{E(q)^6E(q^{10})^2}\right)=\frac{E(q)^4}{E(q^2)^2}U(\Pi), \tag{2.28}$$

where

$$\begin{aligned}\Pi = \bigg(& E(q^{50})^4\big(R(q^{10})^{-1} - q^2 - q^4R(q^{10})\big)^4 \\ & \times \left(\frac{E(q^{25})^5}{E(q^5)^6}\right)^6 \big(R(q^5)^{-4} + qR(q^5)^{-3} + 2q^2R(q^5)^{-2} + 3q^3R(q^5)^{-1} \\ & + 5q^4 - 3q^5R(q^5) + 2q^6R(q^5)^2 - q^7R(q^5)^3 + q^8R(q^5)^4\big)^6\bigg).\end{aligned}$$

Now, we extract all the terms of the form q^{5n} in the right-hand side of (2.28), and replace q by $q^{1/5}$, then express them in terms of $P(\cdot,\cdot)$ by (2.20), we further deduce that

$$\begin{aligned}U(\delta) = q^6\frac{E(q^5)^{30}E(q^{10})^4}{E(q)^{32}E(q^2)^2}\big(&-4P(6,9) + 27P(6,8) + 48P(5,9) \\ &+ 196P(5,8) - 3432P(5,7) + 4806P(5,6) + 2P(4,10) \\ &- 216P(4,9) - 1575P(4,8) + 17104P(4,7) + 19878P(4,6) \\ &- 131136P(4,5) + 78758P(4,4) + 6P(3,10) + 392P(3,9) \\ &+ 1716P(3,8) - 38448P(3,7) - 93200P(3,6) \\ &+ 421152P(3,5) + 215424P(3,4) - 642208P(3,3) \\ &+ 171984P(3,2) + 2138P(2,8) + 39756P(2,7) \\ &+ 65568P(2,6) - 630064P(2,5) - 696525P(2,4) \\ &+ 1412736P(2,3) + 334062P(2,2) - 508044P(2,1) \\ &+ 98667P(2,0) + 52644P(1,6) + 430848P(1,5) \\ &+ 321104P(1,4) - 1375872P(1,3) - 635100P(1,2) \\ &+ 631888P(1,1) + 55836P(1,0) + 111672P(1,-1) \\ &- 78986P(1,-2) - 78986P(0,4) + 668124P(0,3) \\ &+ 254022P(0,2) - 789336P(0,1) - 532125\big).\end{aligned} \tag{2.29}$$

Substituting (2.21)–(2.26) into (2.29) and utilizing (2.17)–(2.19), after simplification, we obtain that

$$\begin{aligned}U(\delta) = q^6\frac{E(q^5)^{30}E(q^{10})^4}{E(q)^{32}E(q^2)^2}\frac{(K-4)^4(K+1)^6}{K^{10}} \\ \times \big(23K^6 + 2208K^5 + 21120K^4 + 47360K^3 \\ + 94720K^2 + 114688K + 32768\big)\end{aligned}$$

$$
\begin{aligned}
&= q^6 \frac{E(q^5)^{30}E(q^{10})^4}{E(q)^{32}E(q^2)^2}\frac{(K-4)^{10}(K+1)^6}{K^{10}}\\
&\quad\times \big(23 + 2760(K-4)^{-1} + 70800(K-4)^{-2}\\
&\qquad + 768000(K-4)^{-3} + 4192000(K-4)^{-4}\\
&\qquad + 11520000(K-4)^{-5} + 12800000(K-4)^{-6}\big)\\
&= \gamma\big(23 + 2760\zeta + 70800\zeta^2 + 768000\zeta^3\\
&\quad + 4192000\zeta^4 + 11520000\zeta^5 + 12800000\zeta^6\big),
\end{aligned}
$$

which is nothing but (2.9).

We therefore complete the proof of Lemma 2.1. □

2.2. **General relations.** Chern and Hirschhorn [5, Theorem 4.1] established the following modular equation involving ζ and $Z = \zeta(q^5)$:

$$
\begin{aligned}
(2.30)\qquad \zeta^5 &- \big(205Z + 4300Z^2 + 34000Z^3 + 120000Z^4 + 160000Z^5\big)\zeta^4\\
&- \big(215Z + 4475Z^2 + 35000Z^3 + 122000Z^4 + 160000Z^5)\zeta^3\\
&- \big(85Z + 1750Z^2 + 13525Z^3 + 46500Z^4 + 60000Z^5\big)\zeta^2\\
&- \big(15Z + 305Z^2 + 2325Z^3 + 7875Z^4 + 10000Z^5\big)\zeta\\
&- \big(Z + 20Z^2 + 150Z^3 + 500Z^4 + 625Z^5\big) = 0.
\end{aligned}
$$

Based on (2.30), Chern and Hirschhorn [5, Eq. (5.1)] proved that for any $i \geq 5$, $U(\zeta^i)$ satisfies the following recurrence:

(2.31)

$$
\begin{aligned}
U(\zeta^i) &= \big(205\zeta + 4300\zeta^2 + 34000\zeta^3 + 120000\zeta^4 + 160000\zeta^5\big)U(\zeta^{i-1})\\
&\quad + \big(215\zeta + 4475\zeta^2 + 35000\zeta^3 + 122000\zeta^4 + 160000\zeta^5)U(\zeta^{i-2})\\
&\quad + \big(85\zeta + 1750\zeta^2 + 13525\zeta^3 + 46500\zeta^4 + 60000\zeta^5\big)U(\zeta^{i-3})\\
&\quad + \big(15\zeta + 305\zeta^2 + 2325\zeta^3 + 7875\zeta^4 + 10000\zeta^5\big)U(\zeta^{i-4})\\
&\quad + \big(\zeta + 20\zeta^2 + 150\zeta^3 + 500\zeta^4 + 625\zeta^5\big)U(\zeta^{i-5}).
\end{aligned}
$$

Interestingly, multiplying (2.30) by γ and $q^{-2}\delta$ and applying the U-operator, we readily find that for any $i \geq 5$, $U(\gamma\zeta^i)$ and $U(q^{-2}\delta\zeta^i)$ also satisfy the recurrence (2.31).

3. Proofs of Theorem 1.3 and Corollary 1.4

This section is devoted to the proofs of Theorem 1.3 and Corollary 1.4.

In this section, all the following congruences are modulo 4096 unless otherwise specified.

We first prove Theorem 1.3.

Proof of Theorem 1.3. Applying the operator U to the generating function of $Q_2(n)$ and utilizing (2.15), (2.16) and (2.20), we find that

(3.1)
$$\begin{aligned}
&\sum_{n=0}^{\infty} Q_2(5n+2)q^n \\
&\quad = U\bigg(\sum_{n=0}^{\infty} Q_2(n)q^{n-2}\bigg) \\
&\quad = U\bigg(q^{-2}\frac{E(q^2)^2}{E(q)^2}\bigg) \\
&\quad = U\bigg(q^{-2}E(q^{50})^2\big(R(q^{10})^{-1} - q^2 - q^4R(q^{10})\big)^2 \\
&\qquad \times \bigg(\frac{E(q^{25})^5}{E(q^5)^6}\bigg)^2\big(R(q^5)^{-4} + qR(q^5)^{-3} + 2q^2R(q^5)^{-2} + 3q^3R(q^5)^{-1} \\
&\qquad\quad + 5q^4 - 3q^5R(q^5) + 2q^6R(q^5)^2 - q^7R(q^5)^3 + q^8R(q^5)^4\big)^2\bigg) \\
&\quad = q^2\frac{E(q^5)^{10}E(q^{10})^2}{E(q)^{12}}\big(-2P(2,3) + 5P(2,2) + 4P(1,3) - 10P(1,2) \\
&\qquad - 32P(1,1) + 20P(1,0) + 20P(0,2) + 54P(0,1) - 15\big).
\end{aligned}$$

Plugging (2.21)–(2.26) into (3.1), upon simplification, we deduce that

$$\begin{aligned}
\sum_{n=0}^{\infty} Q_2(5n+2)q^n &= q^2\frac{E(q^5)^{10}E(q^{10})^2}{E(q)^{12}}\frac{(K-4)^3(K+1)^2}{K^3} \\
&\quad \times \big(3 + 20(K-4)^{-1}\big) \\
&= \gamma(3+20\zeta).
\end{aligned} \tag{3.2}$$

Multiplying the factor $qE(q^2)E(q^{10})^3/(E(q)^3E(q^5))$ in both sides of (2.18) gives that

$$\frac{E(q^2)^2E(q^5)^4}{E(q)^4E(q^{10})^2} = 1 + 4q\frac{E(q^2)E(q^{10})^3}{E(q)^3E(q^5)},$$

from which we obtain that

$$\begin{aligned}
\gamma = \frac{E(q^2)^2E(q^5)^2}{E(q)^4} &= \frac{E(q^{10})^2}{E(q^5)^2}\cdot\frac{E(q^2)^2E(q^5)^4}{E(q)^4E(q^{10})^2} \\
&= \frac{E(q^{10})^2}{E(q^5)^2}\bigg(1 + 4q\frac{E(q^2)E(q^{10})^3}{E(q)^3E(q^5)}\bigg) = \frac{E(q^{10})^2}{E(q^5)^2}(1+4\zeta).
\end{aligned} \tag{3.3}$$

Substituting (3.3) into (3.2) yields that

$$\sum_{n=0}^{\infty} Q_2(5n+2)q^n = \frac{E(q^{10})^2}{E(q^5)^2}\big(3+32\zeta+80\zeta^2\big) \equiv 3\frac{E(q^{10})^2}{E(q^5)^2} \pmod{16}. \tag{3.4}$$

The congruence (1.11) follows from (3.4) immediately.

Moreover, it is easy to see that there are all the terms of the form q^{5n} in the series expansion of the right-hand side of (3.4). Thus we get that for $1 \le i \le 4$,

$$Q_2\big(5(5n+i)+2\big) \equiv 0 \pmod{16}.$$

This proves that (1.13) is true for $\alpha = 1$.

Applying the operator U to (3.2) and using (2.4) and (2.5), after simplification, we find that

$$\sum_{n=0}^{\infty} Q_2(25n+2)q^n \equiv \delta\big(3+4084\zeta+3488\zeta^2+3584\zeta^3+3328\zeta^4+3072\zeta^5\big),$$

or, equivalently,

$$\sum_{n=0}^{\infty} Q_2(25n+2)q^{n-2} \equiv q^{-2}\delta\big(3+4084\zeta+3488\zeta^2+3584\zeta^3+3328\zeta^4+3072\zeta^5\big). \tag{3.5}$$

Applying the operator U to (3.5) and utilizing (2.9)–(2.13) and (2.31), upon simplification, we find that

$$\sum_{n=0}^{\infty} Q_2\bigg(5^3n+\frac{5^4-1}{12}\bigg)q^n \equiv \gamma\big(3449+220\zeta+3440\zeta^2+2624\zeta^3+1792\zeta^4+3072\zeta^5\big). \tag{3.6}$$

Next, we apply the operator U and employ (2.4)–(2.13) and (2.31) to simplify the corresponding expressions repeatedly. Through some tedious but straightforward calculation, we further conclude that

$$\sum_{n=0}^{\infty} Q_2\bigg(5^7n+\frac{5^8-1}{12}\bigg)q^n \equiv \gamma\big(753+3516\zeta+2256\zeta^2 + 3264\zeta^3+3328\zeta^4+3072\zeta^5\big), \tag{3.7}$$

$$\sum_{n=0}^{\infty} Q_2\bigg(5^{15}n+\frac{5^{16}-1}{12}\bigg)q^n \equiv \gamma\big(1505+2940\zeta+2448\zeta^2 + 2496\zeta^3+2304\zeta^4+3072\zeta^5\big), \tag{3.8}$$

$$\sum_{n=0}^{\infty} Q_2\Big(5^{31}n+\frac{5^{32}-1}{12}\Big)q^n \equiv \gamma\big(3009+1788\zeta+784\zeta^2 + 960\zeta^3+256\zeta^4+3072\zeta^5\big), \tag{3.9}$$

$$\sum_{n=0}^{\infty} Q_2\Big(5^{63}n+\frac{5^{64}-1}{12}\Big)q^n \equiv \gamma\big(1921+3580\zeta+1552\zeta^2 + 1984\zeta^3+256\zeta^4+3072\zeta^5\big), \tag{3.10}$$

$$\sum_{n=0}^{\infty} Q_2\Big(5^{127}n+\frac{5^{128}-1}{12}\Big)q^n \equiv \gamma\big(3841+3068\zeta+3088\zeta^2 + 4032\zeta^3+256\zeta^4+3072\zeta^5\big), \tag{3.11}$$

$$\sum_{n=0}^{\infty} Q_2\Big(5^{255}n+\frac{5^{128}-1}{12}\Big)q^n \equiv \gamma\big(3585+2044\zeta+2064\zeta^2 + 4032\zeta^3+256\zeta^4+3072\zeta^5\big). \tag{3.12}$$

Plugging (3.3) into (3.6)–(3.12), after simplification, we obtain that

$$\begin{aligned}
&\sum_{n=0}^{\infty} Q_2\Big(5^{3}n+\frac{5^{4}-1}{12}\Big)q^n \\
&\quad\equiv \frac{E(q^{10})^2}{E(q^5)^2}\big(3449+14016\zeta+4320\zeta^2 \\
&\qquad +16384\zeta^3+12288\zeta^4+8192\zeta^5+4096\zeta^6\big), \\
&\sum_{n=0}^{\infty} Q_2\Big(5^{7}n+\frac{5^{8}-1}{12}\Big)q^n \\
&\quad\equiv \frac{E(q^{10})^2}{E(q^5)^2}\big(753+6528\zeta+16320\zeta^2 \\
&\qquad +12288\zeta^3+16384\zeta^4+16384\zeta^5+12288\zeta^6\big), \\
&\sum_{n=0}^{\infty} Q_2\Big(5^{15}n+\frac{5^{16}-1}{12}\Big)q^n \\
&\quad\equiv \frac{E(q^{10})^2}{E(q^5)^2}\big(1505+8960\zeta+14208\zeta^2 \\
&\qquad +12288\zeta^3+12288\zeta^4+12288\zeta^5+12288\zeta^6\big), \\
&\sum_{n=0}^{\infty} Q_2\Big(5^{31}n+\frac{5^{32}-1}{12}\Big)q^n \\
&\quad\equiv \frac{E(q^{10})^2}{E(q^5)^2}\big(3009+13824\zeta+7936\zeta^2 \\
&\qquad +4096\zeta^3+4096\zeta^4+4096\zeta^5+12288\zeta^6\big),
\end{aligned}$$

$$\begin{aligned}
&\sum_{n=0}^{\infty} Q_2\Big(5^{63}n+\frac{5^{64}-1}{12}\Big)q^n\\
&\quad\equiv \frac{E(q^{10})^2}{E(q^5)^2}\big(1921+11264\zeta+15872\zeta^2\\
&\qquad+8192\zeta^3+8192\zeta^4+4096\zeta^5+12288\zeta^6\big),\\
&\sum_{n=0}^{\infty} Q_2\Big(5^{127}n+\frac{5^{128}-1}{12}\Big)q^n\\
&\quad\equiv \frac{E(q^{10})^2}{E(q^5)^2}\big(3841+18432\zeta+15360\zeta^2\\
&\qquad+16384\zeta^3+16384\zeta^4+4096\zeta^5+12288\zeta^6\big),\\
&\sum_{n=0}^{\infty} Q_2\Big(5^{255}n+\frac{5^{128}-1}{12}\Big)q^n\\
&\quad\equiv \frac{E(q^{10})^2}{E(q^5)^2}\big(3585+16384\zeta+10240\zeta^2\\
&\qquad+12288\zeta^3+16384\zeta^4+4096\zeta^5+12288\zeta^6\big),
\end{aligned}$$

from which we obtain that

$$\sum_{n=0}^{\infty} Q_2\Big(5^3n+\frac{5^4-1}{12}\Big)q^n \equiv 25\frac{E(q^{10})^2}{E(q^5)^2} \pmod{32}, \tag{3.13}$$

$$\sum_{n=0}^{\infty} Q_2\Big(5^7n+\frac{5^8-1}{12}\Big)q^n \equiv 49\frac{E(q^{10})^2}{E(q^5)^2} \pmod{64}, \tag{3.14}$$

$$\sum_{n=0}^{\infty} Q_2\Big(5^{15}n+\frac{5^{16}-1}{12}\Big)q^n \equiv 97\frac{E(q^{10})^2}{E(q^5)^2} \pmod{128}, \tag{3.15}$$

$$\sum_{n=0}^{\infty} Q_2\Big(5^{31}n+\frac{5^{32}-1}{12}\Big)q^n \equiv 193\frac{E(q^{10})^2}{E(q^5)^2} \pmod{256}, \tag{3.16}$$

$$\sum_{n=0}^{\infty} Q_2\Big(5^{63}n+\frac{5^{64}-1}{12}\Big)q^n \equiv 385\frac{E(q^{10})^2}{E(q^5)^2} \pmod{512}, \tag{3.17}$$

$$\sum_{n=0}^{\infty} Q_2\Big(5^{127}n+\frac{5^{128}-1}{12}\Big)q^n \equiv 769\frac{E(q^{10})^2}{E(q^5)^2} \pmod{1024}, \tag{3.18}$$

$$\sum_{n=0}^{\infty} Q_2\Big(5^{255}n+\frac{5^{256}-1}{12}\Big)q^n \equiv 1537\frac{E(q^{10})^2}{E(q^5)^2} \pmod{2048}. \tag{3.19}$$

The congruence (1.12) follows from (3.13)–(3.19) immediately. Moreover, one readily sees that there are all the terms of the form q^{5n} in the right-hand sides of (3.13)–(3.19). The cases $2 \le \alpha \le 8$ of (1.13) thus follow.

This completes the proof of Theorem 1.3. □

Now we turn to prove Corollary 1.4.

Proof of Corollary 1.4. We only prove the case $\alpha = 1$ in (1.14), and the remaining cases can be proved similarly.

Replacing n by $25n + 2$ in (1.11) gives that

$$Q_2\left(5^4 n + \frac{5^4 - 1}{12}\right) \equiv 3Q_2(25n + 2) \equiv 3^2 Q_2(n) \pmod{16}.$$

By induction, we find that for any $\beta \geq 1$,

$$Q_2\left(5^{2\beta} n + \frac{5^{2\beta} - 1}{12}\right) \equiv 3^\beta Q_2(n) \pmod{16}. \tag{3.20}$$

Taking $\alpha = 1$ in (1.13), we get that

$$Q_2\left(5^2 n + \frac{(12i + 5) \times 5 - 1}{12}\right) \equiv 0 \pmod{16}. \tag{3.21}$$

Replacing n by $5(5n + i) + (5^2 - 1)/12$ in (3.20), we find that for any $\beta \geq 1$,

$$Q_2\left(5^{2\beta+2} n + \frac{(12i + 5) \times 5^{2\beta+1} - 1}{12}\right) \equiv 0 \pmod{16}. \tag{3.22}$$

Combining (3.21) and (3.22), we conclude that for any $\beta \geq 1$,

$$Q_2\left(5^{2\beta} n + \frac{(12i + 5) \times 5^{2\beta-1} - 1}{12}\right) \equiv 0 \pmod{16}.$$

This proves that (1.14) is true for $\alpha = 1$.

We therefore complete the proof of Corollary 1.4. □

4. Proofs of Theorems 1.5 and 1.6

In this section, we give the proofs of Theorems 1.5 and 1.6.

For this purpose, we first recall that Ramnujan's theta function is given by

$$\begin{aligned} f(a, b) &:= \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} \\ &= (-a, ab)_\infty (-b; ab)_\infty (ab; ab)_\infty, \qquad |ab| < 1. \end{aligned} \tag{4.1}$$

where the last equality in (4.1) is the celebrated Jacobi triple product identity [2, p. 17, Eq. (1.4.8)]. One of three special cases of $f(a, b)$ is the Euler product $f(-q)$, given by

$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} q^{n(3n+1)/2} = E(q).$$

The following p-dissection for $f(-q)$ due to Cui and Gu [8] is the main ingredient in proof of (1.15).

Lemma 4.1. [8, Theorem 2.2] *Let $p \geq 5$ be a prime number. Then*

(4.2)

$$f(-q) = \sum_{\substack{k=-(p-1)/2 \\ k\neq(\pm p-1)/6}}^{(p-1)/2} (-1)^k q^{k(3k+1)/2} f\big(-q^{(3p^2+(6k+1)p)/2}, -q^{(3p^2-(6k+1)p)/2}\big) + (-1)^{(\pm p-1)/6} q^{(p^2-1)/24} f\big(-q^{p^2}\big),$$

where

$$(\pm p-1)/6 = \begin{cases} (p-1)/6 & \text{if } p \equiv 1 \pmod 6, \\ (-p-1)/6 & \text{if } p \equiv 5 \pmod 6. \end{cases}$$

Next, we collect some necessary terminology and lemmas in theory of modular forms. The full modular group is given by

$$\Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, \text{ and } ad - bc = 1 \right\},$$

and for a positive integer N, the congruence subgroup $\Gamma_0(N)$ is defined by

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \colon c \equiv 0 \pmod N \right\}.$$

Let γ be the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ hereinafter. Define γ act on $\tau \in \mathbb{C}$ by the linear fractional transformation

$$\gamma\tau = \frac{a\tau + b}{c\tau + d} \qquad \text{and} \qquad \gamma\infty = \lim_{\tau\to\infty} \gamma\tau.$$

Let N, k be positive integers and $\mathbb{H} = \{\tau \in \mathbb{C} \colon \mathrm{Im}(\tau) > 0\}$. A holomorphic function $f \colon \mathbb{H} \to \mathbb{C}$ is called a modular function of weight k for $\Gamma_0(N)$, if it satisfies the following two conditions:

(1) for all $\gamma \in \Gamma_0(N)$, $f(\gamma\tau) = (c\tau + d)^k f(\tau)$;
(2) for any $\gamma \in \Gamma$, $(c\tau + d)^{-k} f(\gamma\tau)$ has a Fourier expansion of the form

$$(c\tau + d)^{-k} f(\gamma\tau) = \sum_{n=n_\gamma}^{\infty} a(n) q_{w_\gamma}^n,$$

where $a(n_\gamma) \neq 0$, $q_{w_\gamma} = e^{2\pi i\tau/w_\gamma}$ and $w_\gamma = N/\gcd(c^2, N)$.

In particular, if $n_\gamma \geq 0$ for all $\gamma \in \Gamma$, then we call that f is a modular form of weight k for $\Gamma_0(N)$. A modular function with weight 0 for $\Gamma_0(N)$ is referred to as a modular function for $\Gamma_0(N)$. For a modular function $f(\tau)$ of weight k with respect to $\Gamma_0(N)$, the order of $f(\tau)$ at the cusp $a/c \in \mathbb{Q} \cup \{\infty\}$ is defined by

$$\mathrm{ord}_{a/c}(f) = n_\gamma$$

for some $\gamma \in \Gamma$ such that $\gamma\infty = a/c$. It is known that $\mathrm{ord}_{a/c}(f)$ is well-defined, see [9, p. 72]. Radu [18] developed the Ramanujan–Kolberg algorithm to establish the Ramanujan–Kolberg identities on a class of partition functions defined in terms of eta-quotients using modular functions for $\Gamma_0(N)$. A description of the Ramanujan–Kolberg algorithm can be found in Paule and Radu [17]. Smoot [22] developed a Mathematica package `RaduRK` to implement Radu's algorithm.

Let the partition function $a(n)$ be defined by

$$\sum_{n=0}^{\infty} a(n)q^n = \prod_{\delta|M} (q^\delta, q^\delta)_\infty^{r_\delta}, \tag{4.3}$$

where M, δ are positive integers, and r_δ are integers. For any $m \geq 1$ and $0 \leq t \leq m-1$, Radu [18] defined

$$g_{m,t}(\tau) = q^{(t+\ell)/m} \sum_{n=0}^{\infty} a(mn+t)q^n, \tag{4.4}$$

where

$$\ell = \frac{1}{24} \sum_{\delta|M} \delta r_\delta,$$

and gave a criterion for a function involving $g_{m,t}(\tau)$ to be a modular function with respect to $\Gamma_0(N)$, where N satisfies the following: let $\kappa = \gcd(1 - m^2, 24)$,

(1) for every prime p, $p|m$ implies $p|N$;

(2) for every $\delta|M$ with $r_\delta \neq 0$, $\delta|M$ implies $\delta|mN$;

(3) $\kappa m N^2 \sum_{\delta|M} r_\delta/\delta \equiv 0 \pmod{24}$;

(4) $\kappa N \sum_{\delta|M} r_\delta \equiv 0 \pmod{8}$;

(5) $24m/\gcd\left(\kappa(-24t - \sum_{\delta|M} \delta r_\delta), 24m\right) | N$;

(6) if $2|m$, then $\kappa N \equiv 0 \pmod{4}$ and $8|Ns$, or $2|s$ and $8|N(1-j)$, where $\prod_{\delta|M} \delta^{|r_\delta|} = 2^s j$, and $j, s \in \mathbb{Z}$, j is odd.

Given a positive integer n and an integer x, we denote by $[x]_n$ the residue class of x modulo n. Let

$$\mathbb{Z}_n^* = \{[x]_n \in \mathbb{Z}_n \colon \gcd(x, n) = 1\} \qquad \text{and} \qquad \mathbb{S}_n = \{y^2 \colon y \in \mathbb{Z}_n^*\}.$$

Let the set

$$P_m(t) = \left\{ \left[ts + \frac{s-1}{24} \sum_{\delta|M} \delta r_\delta \right]_m \colon s \in \mathbb{S}_{24m} \right\}.$$

Recall that the Dedekind eta-function $\eta(\tau)$ is defined by

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n),$$

where $q = e^{2\pi i\tau}$ and $\tau \in \mathbb{H}$.

Theorem 4.2. [18, Theorem 45] *For a partition function $a(n)$ defined as (4.3), and integers $m \geq 1, 0 \leq t \leq m-1$, suppose that N is a positive integer satisfying the conditions (1)–(6). Let*

$$F(\tau) = \prod_{\delta|N}\eta^{s_\delta}(\delta\tau)\prod_{t'\in P_m(t)}g_{m,t'}(\tau),$$

where s_δ are integers. Then $F(\tau)$ is a modular function for $\Gamma_0(N)$ if and only if s_δ satisfy the following

[(1)]

(1) $|P_m(t)|\sum_{\delta|M} r_\delta + \sum_{\delta|N} s_\delta = 0$;

(2) $\sum_{t'\in P_m(t)}(1-m^2)\big(24t' + \sum_{\delta|M}\delta r_\delta\big)/m$
$+|P_m(t)|m\sum_{\delta|M}\delta r_\delta + \sum_{\delta|N}\delta s_\delta \equiv 0 \pmod{24}$;

(3) $|P_m(t)|mN\sum_{\delta|M} r_\delta/\delta + \sum_{\delta|N}(N/\delta)s_\delta \equiv 0 \pmod{24}$;

(4) $\big(\prod_{\delta|M}(m\delta)^{|r_\delta|}\big)^{|P_m(t)|}\prod_{\delta|N}\delta^{|s_\delta|}$ *is a square.*

Radu [18, Theorem 47] also gave the lower bounds of the orders of $F(\tau)$ at cusps of $\Gamma_0(N)$.

Theorem 4.3. *For a partition function $a(n)$ defined as (4.3), and integers $m \geq 1$, $0 \leq t \leq m-1$, let*

$$F(\tau) = \prod_{\delta|N}\eta^{s_\delta}(\delta\tau)\prod_{t'\in P_m(t)}g_{m,t'}(\tau),$$

be a modular function for $\Gamma_0(N)$, where s_δ are integers and N satisfies the conditions (1)–(6). Let $\{s_1, s_2, \ldots, s_\epsilon\}$ be a complete set of inequivalent cusps of $\Gamma_0(N)$, and for each $1 \leq i \leq \epsilon$, let $\gamma_i \in \Gamma$ be such that $\gamma_i\infty = s_i$. Then

$$\mathrm{ord}_{s_i}(F(\tau)) \geq \frac{N}{\gcd(c^2,N)}\big(|P_m(t)|p(\gamma_i) + p^*(\gamma_i)\big),$$

where

$$p(\gamma_i) = \min_{\lambda\in\{0,1,\ldots,m-1\}}\frac{1}{24}\sum_{\delta|M} r_\delta\frac{\gcd^2(\delta(a+\kappa\lambda c), mc)}{\delta m},$$

and

$$p^*(\gamma_i)=\frac{1}{24}\sum_{\delta|N}s_\delta\frac{\gcd^2(\delta,c)}{\delta}.$$

The following theorem of Sturm [23, Theorem 1] plays a crucial role in proving congruences using the theory of modular forms.

Theorem 4.4. *Let k be an integer and $g(\tau)=\sum_{n=0}^{\infty}c(n)q^n$ a modular form of weight k for $\Gamma_0(N)$. For any given positive integer u, if $c(n)\equiv 0 \pmod u$ holds for all $n\le (kN/12)\prod_{p|N,\ p\text{ prime}}(1+1/p)$, then $c(n)\equiv 0 \pmod u$ holds for any $n\ge 0$.*

Now we proceed with the proof of Theorem 1.5.

Proof of Theorem 1.5. From (1.6) we find that

$$\sum_{n=0}^{\infty}Q_2(n)q^n=\frac{E(q^2)^2}{E(q)^2}=\frac{E(q^2)^2}{E(q)^4}\cdot E(q)^2\equiv f(-q)^2 \pmod 4. \tag{4.5}$$

For a prime $p\ge 5$ and two integers $-(p-1)/2\le j,\ k\le (p-1)/2$, assume that

$$\frac{3j^2+j}{2}+\frac{3k^2+k}{2}\equiv\frac{p^2-1}{12} \pmod p,$$

which implies that

$$(6j+1)^2+(6k+1)^2\equiv 0 \pmod p. \tag{4.6}$$

Since $p\equiv 3 \pmod 4$, one obtains $\left(\frac{-1}{p}\right)=(-1)^{(p-1)/2}=-1$ by the quadratic reciprocal law, where $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol. Therefore, it follows from (4.6) that $j=k=(\pm p-1)/6$. Combining (4.2) and (4.5), we conclude that

$$\sum_{n=0}^{\infty}Q_2\left(pn+\frac{p^2-1}{12}\right)q^n\equiv f(-q^p)^2\equiv\sum_{n=0}^{\infty}Q_2(n)q^{pn} \pmod 4,$$

which is nothing but (1.15).

The proof of Theorem 1.5 is complete. □

The following lemma plays an important role in the proof of Theorem 1.6.

Lemma 4.5. *For any prime $p\ge 5$, let $k_1=\lceil (p^2-1)/(24p)\rceil$ and $k_2=\lceil (p^2-1)/(12p^2)\rceil$. Then for any constant c,*

$$\frac{\eta^{24k_1}(\tau)\eta^{16k_2}(2p\tau)}{\eta^{8k_2}(p\tau)}\left(q^{p/12}\frac{\eta^2(p\tau)}{\eta^2(2p\tau)}\sum_{n=0}^{\infty}Q_2\left(pn+\frac{p^2-1}{12}\right)q^n-c\right)$$

is a modular form of weight $12k_1+4k_2$ for $\Gamma_0(2p)$.

Proof. Recall that the generating function of $Q_2(n)$ is

$$\sum_{n=0}^{\infty} Q_2(n)q^n = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty^2}.$$

Taking $M = 2$, $(r_1, r_2) = (-2, 2)$, $m = p$, $t = (p^2-1)/12$ in Theorem 4.2, one easily find that $N = 2p$ satisfies the conditions (1)–(6), and for $(s_1, s_2, s_p, s_{2p}) = (0, 0, 2, -2)$,

$$F(\tau) = q^{p/12}\frac{\eta^2(p\tau)}{\eta^2(2p\tau)}\sum_{n=0}^{\infty} Q_2\left(pn + \frac{p^2-1}{12}\right)q^n$$

is a modular function for $\Gamma_0(2p)$.

The set of inequivalent cusps of $\Gamma_0(2p)$ is given by

$$\left\{0, \frac{1}{2}, \frac{1}{p}, \infty\right\}.$$

According to Theorem 4.3, we derive the following lower bounds of $F(\tau)$ at the cusps of $\Gamma_0(2p)$:

$$\begin{aligned} \mathrm{ord}_0(F(\tau)) &\geq -\frac{p^2-1}{12}, & \mathrm{ord}_{1/2}(F(\tau)) &\geq 0, \\ \mathrm{ord}_{1/p}(F(\tau)) &\geq \frac{p^2-1}{12p}, & \mathrm{ord}_\infty(F(\tau)) &\geq -\frac{p^2-1}{12p}, \end{aligned}$$

which implies that

$$\begin{aligned} \mathrm{ord}_0(F(\tau)-c) &\geq -\frac{p^2-1}{12}, & \mathrm{ord}_{1/2}(F(\tau)-c) &\geq 0, \\ \mathrm{ord}_{1/p}(F(\tau)-c) &\geq 0, & \mathrm{ord}_\infty(F(\tau)-c) &\geq -\frac{p^2-1}{12p}. \end{aligned}$$

With the help of Theorem 1.64 and Theorem 1.65 in [16], one easily obtains that

$$F_1(\tau) = \eta^{24}(\tau) \qquad \text{and} \qquad F_2(\tau) = \frac{\eta^{16}(2p\tau)}{\eta^8(p\tau)}$$

are modular forms with weight 12 and 4 for $\Gamma_0(2p)$, respectively, and the orders at the cusps of $\Gamma_0(2p)$ are

$$\begin{aligned} &\mathrm{ord}_0(F_1(\tau)) = 2p, && \mathrm{ord}_{1/2}(F_1(\tau)) = p, && \mathrm{ord}_{1/p}(F_1(\tau)) = 2, \\ &\mathrm{ord}_\infty(F_1(\tau)) = 1, && \mathrm{ord}_0(F_2(\tau)) = 0, && \mathrm{ord}_{1/2}(F_2(\tau)) = 1, \\ &\mathrm{ord}_{1/p}(F_2(\tau)) = 0, && \mathrm{ord}_\infty(F_2(\tau)) = p. \end{aligned}$$

Therefore, we obtain that all orders of $F_1^{k_1}(\tau)F_2^{k_2}(\tau)F(\tau)$ at all cusps of $\Gamma_0(2p)$ are nonnegative, and then $F_1^{k_1}(\tau)F_2^{k_2}(\tau)F(\tau)$ is a modular form with weight $12k_2 + 4k_2$ for $\Gamma_0(2p)$.

The proof is therefore complete. □

Finally, we prove Theorem 1.6.

Proof of Theorem 1.6. For a given integer $m \geq 1$. By Lemma 4.5 and Sturm's theorem, to prove

$$\frac{(q^p;q^p)_\infty^2}{(q^{2p};q^{2p})_\infty^2}\sum_{n=0}^{\infty}Q_2\bigg(pn+\frac{p^2-1}{12}\bigg)q^n-c_p\equiv 0 \pmod{2^m},$$

we only need to check that the coefficients of the first $l_p=(3k_1+k_2)(p+1)$ terms of the expansion

$$\frac{\eta^{24k_1}(\tau)\eta^{16k_2}(2p\tau)}{\eta^{8k_2}(p\tau)}\bigg(q^{p/12}\frac{\eta^2(p\tau)}{\eta^2(2p\tau)}\sum_{n=0}^{\infty}Q_2\bigg(pn+\frac{p^2-1}{12}\bigg)q^n-c_p\bigg)$$

are congruent to 0 modulo 2^m. Here, k_1 and k_2 are defined in Lemma 4.5 and the corresponding l_p are displayed in Table 1. This information allows us to do the computations to complete the proof of Theorem 1.6. □

TABLE 1. A table of values of l_p.

p	5	7	11	13	17	19	23	29	31	37	41	43
l_p	24	32	48	56	72	80	96	210	224	266	294	308
p	47	53	59	61	67	71	73	79	83	89	101	103
l_p	336	540	600	620	680	720	962	1040	1092	1170	1632	1664

5. Final remarks

We conclude this paper with several remarks.

First, it appears that (1.12) and (1.13) are just some initial cases in the following corresponding internal congruence family and congruence family.

Conjecture 5.1. *For any $\alpha\geq 2$ and $n\geq 0$,*

$$Q_2\bigg(5^{2^\alpha}n+\frac{5^{2^\alpha}-1}{12}\bigg)\equiv\big(3\times 2^{\alpha+1}+1\big)Q_2(n)\ \big(\mathrm{mod}\ 2^{\alpha+3}\big). \tag{5.1}$$

Moreover, for any $\alpha\geq 1$ and $1\leq i\leq 4$,

$$Q_2\bigg(5^{2^\alpha}n+\frac{(12i+5)\times 5^{2^\alpha-1}-1}{12}\bigg)\equiv 0\ \big(\mathrm{mod}\ 2^{\alpha+3}\big). \tag{5.2}$$

Second, (1.18) together with numerical evidence suggests the following conjecture.

Conjecture 5.2. *Let $p\geq 5$ be a prime number such that $p\not\equiv 1 \pmod{12}$. Then*

$$\sum_{n=0}^{\infty}Q_2\bigg(pn+\frac{p^2-1}{12}\bigg)q^n\equiv c_p\sum_{n=0}^{\infty}Q_2(n)q^{pn} \pmod{16}, \tag{5.3}$$

where c_p is an odd positive integer depending on p.

The appearance of (1.18) suggests that there are a large number of internal congruences and congruences modulo powers of 2 enjoyed by $Q_2(n)$. Motivated by (5.1) and (5.2), a natural question worth further investigation is whether there exist some similar internal congruences and congruences modulo powers of 2 satisfied by $Q_2(n)$ for another prime p. From (1.8) and (1.18), we find that there is an inseparable relation between congruence properties modulo powers of 2 and arithmetic density property on powers of 2. Therefore, the other natural question is whether there exist internal congruences and congruences modulo powers of 2 similar to (1.11)–(1.13) for another partition function, which has the same arithmetic density property on powers of 2 as $Q(n)$ and $Q_2(n)$. We would like to address this question in a forthcoming paper [10].

Third, if we consider the partition function $Q_k(n)$, which denotes the number of k-colored partitions of n into distinct parts, and its generating function is given by

$$\sum_{n=0}^{\infty} Q_k(n)q^n = (-q;q)_\infty^k = \frac{(q^2;q^2)_\infty^k}{(q;q)_\infty^k} = \frac{E(q^2)^k}{E(q)^k}.$$

Obviously, $Q_1(n) = Q(n)$. Following a similar strategy of proving (1.11)–(1.13), one can prove that

$$\sum_{n=0}^{\infty} Q_3(5n+3)q^n \equiv 13\frac{E(q^{10})^3}{E(q^5)^3} \pmod{64},$$
$$\sum_{n=0}^{\infty} Q_3(125n+78)q^n \equiv 105\frac{E(q^{10})^3}{E(q^5)^3} \pmod{128},$$
$$\sum_{n=0}^{\infty} Q_4(5n+4)q^n \equiv 51\frac{E(q^{10})^3}{E(q^5)^3} \pmod{256},$$
$$\sum_{n=0}^{\infty} Q_4(125n+104)q^n \equiv 297\frac{E(q^{10})^3}{E(q^5)^3} \pmod{512}.$$

Unfortunately, for $k = 5$, we have not found similar results yet. Based on these congruences, it is natural to ask whether there exists a criterion which can be applied to search for an internal congruence family and congruence family similar to (5.1), (5.2) and (5.3) for $Q_k(n)$ with a given positive integer k.

Finally, the results involved in $Q(n)$ and $Q_2(n)$ imply that there is an inseparable connection between congruences properties modulo powers of 2 and arithmetic density property on powers of 2 for these two objects. In 2020, Cotron et al. [7, Theorem 1.1] proved a powerful theorem on the arithmetic density property for a class of infinite products by utilizing the theory of modular forms. As an immediate consequence, one easily gets that

for a given positive integer k and any positive integer m,

$$\lim_{X\to\infty}\frac{\#\{0\le n<X\colon Q_k(n)\equiv 0 \pmod{2^m}\}}{X}=1. \tag{5.4}$$

Therefore, another natural question is whether there exist other types of internal congruences families and congruence families modulo any powers of 2 for $Q_k(n)$, where k is a positive integer.

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