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IDENTITIES INVOLVING TRIGONOMETRIC, ZETA AND PARTITION FUNCTIONS

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Abstract. Using well-known trigonometric functions, we establish identities involving the multiple zeta function and the multiple lambda function. Furthermore, we derive new identities by applying classical trigonometric relations. Some of these results are expressed in terms of colored partitions, highlighting connections between partition theory and special functions.

1. Introduction

We recall that the multiple zeta function (MZV) is given by

$$\zeta(s_1, s_2, \dots, s_k) = \sum_{n_1 > n_2 > \dots > n_k \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}},$$

where $\zeta(s_1, s_2, \dots, s_k)$ converges absolutely in the domain $\Re(s_1 + \cdots + s_j) > j$ for every $j = 1, \dots, k$. The k-tuple having all entries equal to s is denoted by $\{s\}^k$. Thus,

$$\zeta(\{s\}^k) = \sum_{n_1 > n_2 > \dots > n_k \geq 1} \frac{1}{(n_1 n_2 \cdots n_k)^s}.$$

For convenience we let $\zeta(\{n\}^0) = 1$. A multi-index $s = (s_1, s_2, \dots, s_k)$ is said to be admissible if the series $\zeta(s)$ converges (see Hoffman [16] and Spanier [28]).

Working with the infinite product representation for $\sin x$, namely

$$\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2 \pi^2}\right)$$

(see Eberlein [10]) and its representation as a Maclaurin series naturally yields an identity involving the multivariate zeta function as we can see in

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the first theorem of this paper. In addition, using the same approach with the function $\cos x$ a different function appears, namely multiple lambda function, which is defined as follows.

Definition 1.1. *The multiple lambda function is defined by*

$$\lambda(s_1, s_2, \ldots, s_k) = \sum_{n_1>n_2>\cdots>n_k\geq 1} \frac{1}{(2n_1-1)^{s_1}(2n_2-1)^{s_2}\ldots(2n_k-1)^{s_k}}.$$

In particular,

$$\lambda(\{s\}^k) = \sum_{n_1>n_2>\cdots>n_k\geq 1} \frac{1}{((2n_1-1)(2n_2-1)\ldots(2n_k-1))^s}.$$

This paper is devoted to establishing identities involving the multiple zeta function and the multiple lambda function using some well-known trigonometric identities and elementary tools. Some of the identities we present here involve colored partitions, including

$$\sum_{\substack{w_1+\cdots+w_m\in C(n)\\ w_i\geq 2}} \frac{(-1)^m p_2(w_1-2)\cdots p_2(w_m-2)}{(2m+1)!} = \sum_{k=1}^{\lfloor n/2\rfloor} \frac{(-1)^k p_{2k}(n-2k)}{(2k+1)!},$$

where $C(n)$ is the set of integer compositions of n and $p_k(n)$ denotes the number of partitions of n whose parts have one of k colors. In Section 2, we present a number of identities involving the multiple zeta and the multiple lambda functions. Two new series, namely $Z(k,\ell,m)$ and $\Lambda(k,\ell,m)$, are introduced in Section 3 and some identities involving them are derived. Finally, in Section 4, we present some results involving k-colored partitions.

2. Results involving basic trigonometric functions

In order to prove some results in this section, we need the three lemmas below.

Lemma 2.1. *Let f be a polynomial. The coefficient of z^m in the product $\prod_{n=1}^{\infty}\left(1+\frac{az}{f(n)^s}\right)$ is given by*

$$\sum_{i_1>\cdots>i_m\geq 1} \frac{a^m}{f(i_1)^s\ldots f(i_m)^s}. \tag{2.1}$$

Proof. In order to obtain z^m from $\prod_{n=1}^{\infty}\left(1+\frac{az}{f(n)^s}\right)$ we take m factors of the type $\frac{az}{f(i_j)^s}$, with different i_j. So the coefficient of z^m is

$$\sum_{i_1>\cdots>i_m\geq 1} \frac{a^m}{f(i_1)^s\ldots f(i_m)^s}.$$

□

Lemma 2.2. *Let f and g be polynomials. The coefficient of z^m in*

$$\prod_{n=1}^{\infty}\left(1+\frac{az}{f(n)^s}\right)\prod_{n=1}^{\infty}\left(1+\frac{bz}{g(n)^t}\right)$$

is given by

$$\sum_{k=0}^{m}\left[\left(\sum_{i_1>\cdots>i_k\geq 1}\frac{1}{f(i_1)^s\ldots f(i_k)^s}\right)\right.$$
$$\left.\times\left(\sum_{j_1>\cdots>j_{m-k}\geq 1}\frac{1}{g(j_1)^t\ldots g(j_{m-k})^t}\right)a^k b^{m-k}\right]. \tag{2.2}$$

Proof. The term z^m is obtained by combining a z^k from the first product with z^{m-k} from the second one, where k varies from 0 to m. By Lemma 2.1, the coefficient of z^k in the first product is

$$\left(\sum_{i_1>\cdots>i_k\geq 1}\frac{1}{f(i_1)^s\ldots f(i_k)^s}\right)a^k$$

and the coefficient of z^{m-k} in the second product is

$$\left(\sum_{j_1>\cdots>j_{m-k}\geq 1}\frac{1}{g(j_1)^t\ldots g(j_{m-k})^t}\right)b^{m-k}.$$

Therefore, the coefficient of z^m is given by (2.2). ☐

We recall below the notion of the composition of a positive integer.

Definition 2.3. *A composition of a positive integer n is an ordered collection of positive integers whose sum is n. The set of compositions of n is denoted by $C(n)$.*

Example 2.4. *The eight compositions of 4 are*

$$C(4)=\{(4),(3,1),(1,3),(2,2),(2,1,1),(1,2,1),(1,1,2),(1,1,1,1)\}.$$

More about integer compositions can be found in Heubach and Mansour [15] and Sills [27].

Lemma 2.5. *Let f be a polynomial and $a,s\in\mathbb{R}$. If $|z|<\left|\frac{f(n)^s}{a}\right|$, for all $n\geq 1$, then the coefficient of z^m in $\prod_{n=1}^{\infty}\left(1+\frac{az}{f(n)^s}\right)^{-1}$ is given by*

$$\sum_{(w_1,\ldots,w_j)\in C(m)}\sum_{i_1>\cdots>i_j\geq 1}\frac{(-a)^m}{f(i_1)^{w_1 s}\ldots f(i_j)^{w_j s}},$$

where $C(m)$ is the set of integer compositions of m.

Proof. Note that the factor $\left(1+\frac{az}{f(n)^s}\right)^{-1}$ can be seen as the sum of the geometric series having ratio $-\frac{az}{f(n)^s}$, since the absolute value of this ratio is less than 1, or equivalently, $|z| < \left|\frac{f(n)^s}{a}\right|$ for all $n \geq 1$. Rewriting the infinite product in the form

$$\prod_{n=1}^{\infty}\left(1-\frac{az}{f(n)^s}+\frac{a^2z^2}{f(n)^{2s}}-\frac{a^3z^3}{f(n)^{3s}}+\dots\right)$$

we see that the coefficient of z^m is obtained by taking a factor of the type $\frac{(-az)^{w_k}}{f(i_k)^{w_k s}}$ in j different parenthesis, with $(w_1,\dots,w_j) \in C(m)$. So, $w_1+\cdots+w_j = m$ and the coefficient of z^m is given by

$$\sum_{(w_1,\dots,w_j)\in C(m)}\ \sum_{i_1>\dots>i_j\geq 1}\frac{(-a)^m}{f(i_1)^{w_1 s}\dots f(i_j)^{w_j s}}.$$

□

Throughout this paper, we make use of the expansions of the trigonometric functions below:

$$\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1} = x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right) \tag{2.3}$$

$$\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} = \prod_{n=1}^{\infty}\left(1-\frac{4x^2}{(2n-1)^2\pi^2}\right) \tag{2.4}$$

Analytic proofs of these identities can be found in Eberlein [10].

For $|x| < \frac{\pi}{2}$, we have

$$\sec x = \sum_{n=0}^{\infty}\frac{(-1)^n E_{2n}}{(2n)!}x^{2n} = \prod_{n=1}^{\infty}\frac{(2n-1)^2\pi^2}{((2n-1)^2)\pi^2-4x^2}. \tag{2.5}$$

For $|x| < \pi$, we have

$$\begin{aligned}\csc x &= \frac{1}{x}+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2^{2n}-2)B_{2n}}{(2n)!}x^{2n-1} = \\ &= \frac{1}{x}\prod_{n=1}^{\infty}\frac{n^2\pi^2}{n^2\pi^2-x^2}.\end{aligned} \tag{2.6}$$

The tangent and cotangent functions satisfy

$$\begin{aligned}\tan x &= \sum_{n=0}^{\infty}\frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}}{(2n)!}x^{2n-1} \\ &= x\prod_{n=1}^{\infty}\left(\frac{n^2\pi^2-x^2}{n^2}\right)\left(\frac{(2n-1)^2}{(2n-1)^2\pi^2-4x^2}\right),\end{aligned} \tag{2.7}$$

for $|x| < \pi/2$, and

$$\cot x = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}2^{2n}B_{2n}}{(2n)!}x^{2n-1} = \frac{1}{x}\prod_{n=1}^{\infty}\left(\frac{(2n-1)^2\pi^2-4x^2}{(2n-1)^2}\right)\left(\frac{n^2}{n^2\pi^2-x^2}\right), \tag{2.8}$$

for $|x| < \pi$. The numbers B_n and E_n, for $n \geq 0$, are Bernoulli and Euler numbers, respectively (see Abramowitz [1] and Arfken [7]). We recall that the nth Euler number E_n is defined by the Taylor series expansion

$$\frac{1}{\cosh x} = \sum_{n=0}^{\infty} \frac{E_n}{n!}x^n,$$

and the nth Bernoulli numbers B_n is defined by the Taylor series expansion

$$\frac{x}{e^x-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!}x^n.$$

The next theorem is a well-known result due to Euler (see Hoffman [16], Terasoma [29] and Zagier [31]). We present an alternative proof of this result.

Theorem 2.6 (Euler). *For all $m \geq 1$, we have*

$$\zeta(\{2\}^m) = \frac{\pi^{2m}}{(2m+1)!} \tag{2.9}$$

Proof. Taking $a = -\frac{1}{\pi^2}$, $p(n) = n$, and $s = 2$ in Lemma 2.1, it follows that the coefficient of z^m in (2.3) is equal to

$$\frac{(-1)^m}{\pi^{2m}} \sum_{i_1 > \cdots > i_m} \frac{1}{i_1^2 \ldots i_m^2}.$$

Let $z = x^2$ in (2.3). Since there is an x multiplying the product in (2.3), by Lemma 2.1 the coefficient of x^{2m+1} in the expansion of the sine function is equal to $\frac{(-1)^m}{\pi^{2m}}\zeta(2^{\{m\}})$. On the other hand, considering its Taylor series we recall that this coefficient is equal to $\frac{(-1)^m}{(2m+1)!}$. Hence,

$$\zeta(\{2\}^m) = \frac{\pi^{2m}}{(2m+1)!}.$$

$\square$

Example 2.7. *In particular, for $m = 1$ we have $\zeta(2) = \frac{\pi^2}{6}$, while for $m = 2$ we know that $\zeta(2,2) = \frac{\pi^4}{120}$.*

We prove below a theorem for the multiple lambda function, which is analogous to Theorem 2.6.

Theorem 2.8. *For all $m \geq 1$, we have*

$$\lambda(\{2\}^m) = \frac{\pi^{2m}}{2^{2m}(2m)!}. \tag{2.10}$$

Proof. Taking $a = -\frac{4}{\pi^2}$, $f(n) = 2n-1$ and $s = 2$ in Lemma 2.1, we see that the coefficient of z^m on the right-hand side of (2.4) is equal to

$$\frac{(-4)^m}{\pi^{2m}} \sum_{i_1 > \cdots > i_m} \frac{1}{(2i_1-1)^2 \dots (2i_m-1)^2}.$$

Taking $z = x^2$, it follows by Theorem 2.1 that the coefficient of x^{2m} in the expansion of the cosine function is equal to $\frac{(-4)^m}{\pi^{2m}}\lambda(\{2\}^m)$. On the other hand its Taylor series yields $\frac{(-1)^m}{(2m)!}$ as the coefficient of x^{2m}. Therefore,

$$\lambda(\{2\}^m) = \frac{\pi^{2m}}{4^m(2m)!}.$$

□

Example 2.9. *For $m = 1$ we have $\lambda(2) = \frac{\pi^2}{8}$, while $m = 2$ yields $\lambda(2,2) = \frac{\pi^4}{384}$.*

The next result follows directly from Theorems 2.6 and 2.8.

Corollary 2.10. *For all $m \geq 1$, we have*

$$\lambda(\{2\}^m) = \left(\frac{2m+1}{2^{2m}}\right)\zeta(\{2\}^m). \tag{2.11}$$

In the next theorem, if $w = (w_1, w_2, \dots, w_j) \in C(m)$, we let $\zeta(2w)$ denote $\zeta(2w_1, 2w_2, \dots, 2w_j)$.

Theorem 2.11. *For all $m \geq 1$, we have*

$$\sum_{w \in C(m)} \zeta(2w) = \frac{(-1)^{m-1}(2^{2m}-2)\pi^{2m}B_{2m}}{(2m)!}, \tag{2.12}$$

where $C(m)$ is the set of integer compositions of m.

Proof. Thanks to (2.7) we see that the cosecant function can be written as the infinite product

$$\csc(x) = \frac{1}{x}\prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2\pi^2}\right)^{-1}.$$

For $|x| < \pi$, we have $\frac{x^2}{n^2\pi^2} < 1$ and then $\frac{1}{1-\frac{x^2}{n^2\pi^2}}$ is the limit of the geometric series with ratio $\frac{x^2}{n^2\pi^2}$. Taking $a = -\frac{1}{\pi^2}$, $f(n) = n$, and $s = 2$ in Lemma 2.5, we see that the coefficient of z^m in the infinite product above is equal to

$$\sum_{(w_1,\dots,w_j) \in C(m)} \sum_{i_1 > \cdots > i_j \geq 1} \frac{\frac{1}{\pi^{2m}}}{i_1^{2w_1} \dots i_j^{2w_j}}.$$

This expression can be rewritten as $\frac{1}{\pi^{2m}}\sum_{(w_1,\ldots,w_j)\in C(m)}\zeta(2w_1,\ldots,2w_j)$ or simply

$$\frac{1}{\pi^{2m}}\sum_{w\in C(m)}\zeta(2w).$$

Since $z = x^2$ and there is a factor $1/x$ multiplying the infinite product, we see that the coefficient of x^{2m-1} in the cosecant function is equal to $\frac{1}{\pi^{2m}}\sum_{w\in C(m)}\zeta(2w)$. On the other hand, considering the Taylor expansion of the cosecant function, we know that such a coefficient is $\frac{(-1)^{m-1}(2^{2m}-2)B_{2m}}{(2m)!}$. Therefore, $\sum_{w\in C(m)}\zeta(2w) = \frac{(-1)^{m-1}(2^{2m}-2)\pi^{2m}B_{2m}}{(2m)!}$. □

Example 2.12. *Since* $C(2) = \{(2),(1,1)\}$ *and* $C(3) = \{(3),(2,1),(1,2),(1,1,1)\}$*, we have*

$$\zeta(4)+\zeta(2,2) = \frac{(2-2^4)\pi^4 B_4}{4!},$$
$$\zeta(6)+\zeta(4,2)+\zeta(2,4)+\zeta(2,2,2) = \frac{(2^6-2)\pi^6 B_6}{6!}.$$

The corollary below is an immediate consequence of combining Theorem 2.11 and the following identity (see [1, Identity 23.2.6])

$$\zeta(2m) = \frac{(2\pi)^{2m}(-1)^{m-1}B_{2m}}{2(2m)!}.$$

Corollary 2.13. *We have*

$$\sum_{w\in C(m)}\zeta(2w) = \left(\frac{2^{2m-1}-1}{2^{2(m-1)}}\right)\zeta(2m).$$

The next result can be proven similarly to Theorem 2.11, using the expression (2.7) for the cosecant function instead of the secant function. So its proof will be omitted.

Theorem 2.14. *For all* $m \geq 1$*, we have*

$$\sum_{w\in C(m)}\lambda(2w) = \frac{(-1)^m\pi^{2m}E_{2m}}{2^{2m}(2m)!}. \tag{2.13}$$

Example 2.15. *We have*

$$\lambda(4)+\lambda(2,2) = \frac{\pi^4 E_4}{2^4 4!},$$
$$\lambda(6)+\lambda(4,2)+\lambda(2,4)+\lambda(2,2,2) = -\frac{\pi^6 E_6}{2^6 6!}.$$

The next theorem is obtained by using the infinite products expansion of tangent function (2.8).

Theorem 2.16. *For all $m \geq 1$, we have*

$$\sum_{k=0}^{m-1}\left((-1)^k 2^{2(m-1-k)}\zeta(\{2\}^k)\sum_{w\in C(m-1-k)}\lambda(2w)\right)$$
$$=\frac{(-1)^{m-1}2^{2m}(2^{2m}-1)\pi^{2m-2}B_{2m}}{(2m)!}.$$

Proof. We rewrite (2.8) as

$$\tan(x)=x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right)\prod_{n=1}^{\infty}\left(1-\frac{2^2x^2}{(2n-1)^2\pi^2}\right)^{-1}.$$

Since we have a factor x multiplying the product above, in order to obtain x^{2m-1} we have to get x^{2m-2} from the two products: x^{2k} from the first one and $x^{2(m-1-k)}$ from the second one, with $0 \leq k \leq m-1$. By Lemma 2.1 we have that the coefficient of x^{2k} in the first product is $\frac{(-1)^k}{\pi^{2k}}\zeta(\{2\}^k)$. On the other hand, it follows by Lemma 2.5 that the coefficient of $x^{2(m-1-k)}$ in second one is $\frac{2^{2(m-1-k)}}{\pi^{2(m-1-k)}}\sum_{w\in C(m-1-k)}\lambda(2w)$. Thus, the coefficient of x^{2m-1} is equal to

$$\sum_{k=0}^{m-1}\left(\frac{(-1)^k}{\pi^{2k}}\zeta(\{2\}^k\cdot\frac{2^{2(m-1-k)}}{\pi^{2(m-1-k)}}\sum_{w\in C(m-1-k)}\lambda(2w)\right).$$

The previous sum can be rewritten as

$$\frac{1}{\pi^{2m-2}}\sum_{k=0}^{m-1}\left((-1)^k\zeta(2^{\{k\}})2^{2(m-1-k)}\sum_{w\in C(m-1-k)}\lambda(2w)\right).$$

Comparison of this and the coefficient of x^{2m-1} obtained from the Taylor series expansion of the tangent function yields

$$\sum_{k=0}^{m-1}\left((-1)^k 2^{2(m-1-k)}\zeta(\{2\}^k)\sum_{w\in C(m-1-k)}\lambda(2w)\right)$$
$$=\frac{(-1)^{m-1}2^{2m}(2^{2m}-1)\pi^{2m-2}B_{2m}}{(2m)!},$$

which completes the proof. □

Example 2.17. *For $m = 3$ we have*

$$16(\lambda(4)+\lambda(2,2))-4\zeta(2)\lambda(2)+\zeta(2,2)=\frac{2^6(2^6-1)\pi^4B_6}{6!}.$$

Similarly, using (2.9) we can prove the following theorem.

Theorem 2.18. *For all $m \geq 1$, we have*

$$\sum_{k=0}^{m}\left((-1)^{m-k}2^{2k}\lambda(\{2\}^k)\sum_{w\in C(m-k)}\zeta(2w)\right)=\frac{(-1)^{m-1}2^{2m}\pi^{2m}B_{2m}}{(2m)!}.$$

Proof. Thanks to (2.9), we know that

$$\cot(x)=\frac{1}{x}\prod_{n=1}^{\infty}\left(1-\frac{2^2x^2}{(2n-1)^2\pi^2}\right)\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right)^{-1}.$$

Considering the factor $1/x$ in the expression above, we see that x^{2m-1} is obtained by taking x^{2k} from first product and $x^{2(m-k)}$ from the second one, with $0 \leq k \leq m$. Then the coefficient of x^{2m-1} is

$$\sum_{k=0}^{m}\left(\frac{2^{2k}}{\pi^{2k}}\lambda(\{2\}^k)\cdot\frac{(-1)^{m-k}}{\pi^{2(m-k)}}\sum_{w\in C(m-k)}\zeta(2w)\right),$$

which can be rewritten as

$$\frac{1}{\pi^{2m}}\sum_{k=0}^{m}(-1)^{m-k}\lambda(\{2\}^k)2^{2k}\sum_{w\in C(m-k)}\zeta(2w).$$

Comparing this with the coefficient of x^{2m-1} that we obtain from the Taylor series of the cotangent function, we have

$$\sum_{k=0}^{m}(-1)^{m-k}2^{2k}\lambda(\{2\}^k)\sum_{w\in C(m-k)}\zeta(2w)=\frac{(-1)^{m-1}2^{2m}\pi^{2m}B_{2m}}{(2m)!}.$$

□

Example 2.19. *For $m = 2$ we have*

$$(\zeta(4)+\zeta(2,2))-4\lambda(2)\zeta(2)+16\lambda(2,2)=-\frac{2^4\pi^4B_4}{4!}$$

Combining Theorems 2.14 and 2.16, we get

$$\sum_{k=0}^{m-1}(-1)^k2^{2m-2-2k}\frac{\pi^{2k}}{(2k+1)!}\left(\frac{(-1)^{m-k-1}\pi^{2m-2k-2}E_{2m-2k-2}}{2^{2m-2k-2}(2m-2k-2)!}\right)$$
$$=\frac{(-1)^{m-1}2^{2m}(2^{2m}-1)\pi^{2m-2}B_{2m}}{(2m)!}.$$

With a few algebraic manipulations, we can deduce the identity relating Bernoulli numbers and Euler numbers below.

Corollary 2.20. *For all $m \geq 1$, we have*

$$\sum_{k=0}^{m-1}\frac{E_{2m-2k-2}}{(2k+1)!(2m-2k-2)!}=\frac{2^{2m}(2^{2m}-1)B_{2m}}{(2m)!}.$$

Combining Theorems 2.11 and 2.18, we obtain

$$\sum_{k=0}^{m} \frac{(-1)^{m-k}4^k\pi^{2k}}{4^k(2k)!} \sum_{w\in C(m-k)} \zeta(2w) = \frac{(-1)^{m-1}2^{2m}\pi^{2m}B_{2m}}{(2m)!}.$$

After some manipulations, we can deduce the recurrence for the Bernoulli numbers below.

Corollary 2.21. *For all $m \geq 1$, we have*

$$\sum_{k=0}^{m} \frac{(2^{2m-2k}-2)B_{2m-2k}}{(2k)!(2m-2k)!} = \frac{(-1)^m 2^{2m} B_{2m}}{(2m)!}.$$

We now prove a result that follows from the well-known trigonometric identity $\sin(2x) = 2\sin(x)\cos(x)$.

Theorem 2.22. *For all $m \geq 1$, we have*

$$\zeta(\{2\}^m) = \sum_{k=0}^{m} \left[\frac{1}{4^k}\zeta(\{2\}^k)\lambda(\{2\}^{m-k})\right].$$

Proof. For $x \neq 0$, we use (2.3) and (2.4) to write

$$\prod_{n=1}^{\infty}\left(1-\frac{(2x)^2}{n^2\pi^2}\right) = \prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right)\prod_{n=1}^{\infty}\left(1-\frac{4x^2}{(2n-1)^2\pi^2}\right).$$

By Theorem 2.1 we know that the coefficient of x^{2m} on the left-hand side of the identity above is

$$\frac{(-1)^m 4^m}{\pi^{2m}}\zeta(\{2\}^m).$$

By Theorem 2.2 we see that the coefficient of x^{2m} on the right-hand side is

$$\frac{(-1)^m}{\pi^{2m}}\sum_{k=0}^{m} 4^{m-k}\zeta(\{2\}^k)\lambda(\{2\}^{m-k}).$$

Then we have

$$\zeta(\{2\}^m) = \sum_{k=0}^{m} \left[\frac{1}{4^k}\zeta(\{2\}^k)\lambda(\{2\}^{m-k})\right].$$

□

Example 2.23. *For $m = 1$ the identity gives $\zeta(2) = \lambda(2) + \frac{1}{4}\zeta(2)$ while for $m = 2$ we have*

$$\zeta(2,2) = \lambda(2,2) + \frac{1}{4}\zeta(2)\lambda(2) + \frac{1}{16}\zeta(2,2).$$

Combining the above result with Theorems 2.6 and 2.8, we obtain the identity below.

Corollary 2.24. *For all $m \geq 1$, we have*

$$\sum_{k=0}^{m} \frac{1}{(2k+1)!(2m-2k)!} = \frac{4^m}{(2m+1)!}.$$

Our next result is a consequence of the well-known elementary fact $\sin^2(x) + \cos^2(x) = 1$.

Theorem 2.25. *For all $m \geq 1$, we have*

$$\sum_{k=0}^{m} \zeta(\{2\}^k)\zeta(\{2\}^{m-k}) = \frac{4^{m+1}}{\pi^2} \sum_{k=0}^{m+1} \lambda(\{2\}^k)\lambda(\{2\}^{m+1-k}).$$

Proof. Using (2.3) and (2.4) we can write the trigonometric identity $\sin^2(x)+\cos^2(x) = 1$ as

$$x^2 \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right) \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right)$$
$$+ \prod_{n=1}^{\infty} \left(1 - \frac{4x^2}{(2n-1)^2\pi^2}\right) \prod_{n=1}^{\infty} \left(1 - \frac{4x^2}{(2n-1)^2\pi^2}\right) = 1.$$

Since the first term is multiplied by x^2, the coefficient of x^{2m+2} on the left side of the identity above is given by a combination of the coefficient of x^{2m} from the first factor and the coefficient of x^{2m+2} from the second one. By Theorem 2.2, the coefficient from first factor is

$$\frac{(-1)^m}{\pi^{2m}} \sum_{k=0}^{m} \zeta(\{2\}^k)\zeta(\{2\}^{m-k})$$

and the coefficient from the second factor is

$$\frac{(-1)^{m+1}4^{m+1}}{\pi^{2m+2}} \sum_{k=0}^{m+1} \lambda(\{2\}^k)\lambda(\{2\}^{m+1-k}).$$

Therefore,

$$\sum_{k=0}^{m} \zeta(\{2\}^k)\zeta(\{2\}^{m-k}) = \frac{4^{m+1}}{\pi^2} \sum_{k=0}^{m+1} \lambda(\{2\}^k)\lambda(\{2\}^{m+1-k}).$$

□

The following two identities were proved by Olver [23] and Schmidt [25]:

$$\frac{(-1)^n E_{2n-1}(x)\pi^{2n}}{4(2n-1)!} = \sum_{k=0}^{\infty} \frac{\cos((2k+1)\pi x)}{(2k+1)^{2n}}, \tag{2.14}$$

and

$$\frac{(-1)^n E_{2n}(x)\pi^{2n+1}}{4(2n)!} = \sum_{k=0}^{\infty} \frac{\sin((2k+1)\pi x)}{(2k+1)^{2n+1}}, \tag{2.15}$$

where $E_n(x)$ denotes the Euler's polynomial (see [11]), which is given by

$$E_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{E_k}{2^k} \left(x - \frac{1}{2}\right)^{n-k}.$$

Using (2.14) and treating the right-hand side as we did in Theorems 2.6 and 2.8, we have following result.

Theorem 2.26. *Given a positive integer l, we have*

$$\lambda(2l) = \frac{(-1)^l \pi^{2l}}{2^{2l+1}} \sum_{k=0}^{2l-1} \frac{(-1)^{k+1} E_k}{k!(2l-k-1)!}.$$

Next, using our techniques, which is different from the other known proofs, we prove a well-known result involving Dirichlet's beta function (see Koch [18]), given by

$$\beta(s) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^s}.$$

Theorem 2.27. *For all $l \geq 0$, we have*

$$\beta(2l-1) = \frac{(-1)^l \pi^{2l} E_{2l-2}}{2^{2l}(2l-2)!}.$$

Proof. We begin by replacing x by $x + \frac{1}{2}$ in (2.14) to obtain

$$\frac{(-1)^n E_{2n-1}(x+\frac{1}{2})\pi^{2n}}{4(2n-1)!} = \sum_{k=0}^{\infty} \frac{\cos((2k+1)\pi(x+\frac{1}{2}))}{(2k+1)^{2n}}. \tag{2.16}$$

Thus, for $n > j$, the coefficient of x^{2j+1} in $E_{2n-1}(x + \frac{1}{2})$ is equal to

$$\binom{2n-1}{2n-2j-2} \frac{E_{2n-2j-2}}{2^{2n-2j-2}}.$$

On the other hand, the coefficient of x^{2j+1} in $\cos((2k+1)\pi(x+\frac{1}{2}))$ is equal to

$$(-1)^{k+1}(2k+1)^{2j+1}(-1)^j \zeta(\{2\}^j) = \frac{(-1)^{k+1+j}(2k+1)^{2j+1}\pi^{2j}}{(2j+1)!},$$

which can be derived using the identity:

$$\begin{aligned}
&\cos\left((2k+1)\pi(x+\frac{1}{2})\right) \\
=&\cos((2k+1)\pi x)\cos\left((2k+1)\frac{\pi}{2}\right) - \sin((2k+1)\pi x)\sin\left((2k+1)\frac{\pi}{2}\right) \\
=&(-1)^{k+1}\sin((2k+1)\pi x).
\end{aligned}$$

Hence,

$$\sum_{k=0}^{\infty} \frac{(-1)^{k+1+j}(2k+1)^{2j+1}\pi^{2j}}{(2j+1)!} = \frac{(-1)^{j+1}\pi^(2j)}{(2j+1)!}\beta(2n-2j-1).$$

Thus, extracting the coefficients of x^{2j+1} in

$$\frac{(-1)^n E_{2n-1}(x+\frac{1}{2})\pi^{2n}}{4(2n-1)!}$$

and

$$\sum_{k=0}^{\infty} \frac{\cos((2k+1)\pi(x+\frac{1}{2}))}{(2k+1)^{2n}},$$

we have the desired result. $\square$

In the next theorem we make use of the sum

$$\sum_{\substack{a_1>\cdots>a_k\geq 1\\ b_1>\cdots>b_\ell\geq 1\\ c_1>\cdots>c_m\geq 1\\ a_i\neq b_i\neq c_i}} \frac{1}{\prod_{i=1}^{k}(2a_i-1)^2\prod_{i=1}^{\ell}(2b_i-1)^2\prod_{i=1}^{m}(2c_i-1)^2},$$

which is equal to

$$\binom{k+l+m}{k}\binom{l+m}{m}\lambda(\{2\}^{k+l+m}).$$

Theorem 2.28. *Let r and s be nonnegative integers with $r \equiv s \pmod 2$ and $r \leq s$. Then*

$$\sum_{k=0}^{\lfloor r/2\rfloor} 2^{r-2k}\binom{\frac{s+r}{2}}{k}\binom{\frac{s+r}{2}-k}{r-2k}\lambda(\{2\}^{\frac{s+r}{2}}) = \frac{1}{r!s!}\left(\frac{\pi}{2}\right)^{r+s}.$$

Proof. It follows from (2.3) that

(2.17)

$$\cos(x+y) = \prod_{n=1}^{\infty}\left(1-\frac{4(x+y)^2}{(2n+1)^2\pi^2}\right)$$

$$\left(1-\frac{4}{1^2\pi^2}x^2-\frac{4}{1^2\pi^2}y^2-\frac{8}{1^2\pi^2}xy\right)\left(1-\frac{4}{3^2\pi^2}x^2-\frac{4}{3^2\pi^2}y^2-\frac{8}{3^2\pi^2}xy\right)\cdots$$

The expansion of this expression only has powers of the type $x^r y^s$, with r and s having the same parity. Let $r = 2k+m$ and $s = 2\ell+m$, with k,ℓ,m nonnegative integers and $k \leq \ell$. The coefficient obtained by multiplying k terms $-\frac{4}{(2a_i-1)^2\pi^2}x^2$, ℓ terms $-\frac{4}{(2b_i-1)^2\pi^2}y^2$, and m mixed terms $-\frac{8}{(2c_i-1)^2\pi^2}xy$ is equal to

$$\left(-\frac{4}{\pi^2}\right)^k\left(-\frac{4}{\pi^2}\right)^\ell\left(-\frac{8}{\pi^2}\right)^m\binom{k+l+m}{k}\binom{l+m}{m}\lambda(\{2\}^{k+l+m}).$$

Then the coefficient is equal to

$$a(k,\ell,m) = 2^m\left(\frac{-2^2}{\pi^2}\right)^{k+\ell+m}\binom{k+l+m}{k}\binom{l+m}{m}\lambda(\{2\}^{k+l+m}).$$

However, there are many ways to obtain $x^r y^s$. Let $c(r,s)$ be the coefficient of $x^r y^s$ in the product (2.17). We have $r < s$, $r \equiv s \pmod 2$, $(r,s) = (2k+m, 2\ell+m)$, where $\ell = \frac{s-r}{2} + k$, $m = r-2k$, and $0 \le k \le \left\lfloor \frac{r}{2} \right\rfloor$. Thus,

$$c(r,s) = \sum_{k=0}^{\lfloor r/2 \rfloor} a\left(k, \frac{s-r}{2} + k, r-2k\right),$$

which can be rewritten as

$$c(r,s) = \left(\frac{2\sqrt{-1}}{\pi}\right)^{r+s} \sum_{k=0}^{\lfloor r/2 \rfloor} 2^{r-2k} \binom{\frac{s+r}{2}}{k} \binom{\frac{s+r}{2} - k}{r-2k} \lambda(\{2\}^{\frac{s+r}{2}}).$$

On the other hand, using the sum angle identity and Taylor series, we know that the coefficient of $x^r y^s$ is $\frac{(\sqrt{-1})^{r+s}}{r!s!}$. Therefore, it follows that

$$\sum_{k=0}^{\lfloor r/2 \rfloor} 2^{r-2k} \binom{\frac{s+r}{2}}{k} \binom{\frac{s+r}{2} - k}{r-2k} \lambda(\{2\}^{\frac{s+r}{2}}) = \frac{1}{r!s!} \left(\frac{\pi}{2}\right)^{r+s}.$$

□

Combining the previous result with Theorem 2.8 we can state the well-known combinatorial identity below.

Corollary 2.29. *For r and s nonnegative integers with $r \equiv s \pmod 2$ and $r \le s$, then*

$$\sum_{k=0}^{\lfloor r/2 \rfloor} 2^{r-2k} \binom{\frac{s+r}{2}}{k} \binom{\frac{s+r}{2} - k}{r-2k} = \binom{r+s}{r}.$$

This identity appears as equation (3.22) in Gould [14].

In the next theorem, we make use of the sum

$$\sum_{\substack{a_1 > \cdots > a_k \ge 1 \\ b_1 > \cdots > b_\ell \ge 1 \\ c_1 > \cdots > c_m \ge 1 \\ a_i \ne b_i \ne c_i}} \left(\frac{1}{a_1^2 \ldots a_k^2}\right) \left(\frac{1}{b_1^2 \ldots b_\ell^2}\right) \left(\frac{1}{c_1^2 \ldots c_m^2}\right),$$

is equal to

$$\binom{k+l+m}{k} \binom{l+m}{m} \zeta(\{2\}^{k+l+m}).$$

Theorem 2.30. *Let r and s be positive integers with $r \not\equiv s \pmod 2$ and $r < s$. Then*

$$\sum_{k=0}^{\lfloor (r-1)/2 \rfloor} 2^{(r-1)-2k} \binom{\frac{s+r-1}{2}}{k} \binom{\frac{s+r-1}{2} - k}{r-2k-1} \zeta(\{2\}^{\frac{s+r-1}{2}})$$

$$+ \sum_{k=0}^{\lfloor r/2 \rfloor} 2^{r-2k} \binom{\frac{s+r-1}{2}}{k} \binom{\frac{s+r-1}{2} - k}{r-2k-1} \zeta(\{2\}^{\frac{s+r}{2}})) = \frac{\pi^{r+s-1}}{r!s!}.$$

Proof. Using (2.3), we see that

$$\begin{aligned}(2.18)\ \sin(x+y) &= (x+y)\prod_{n=1}^{\infty}\left(1-\frac{(x+y)^2}{n^2\pi^2}\right)\\ &= (x+y)\prod_{n=1}^{\infty}\left(1-\frac{1}{n^2\pi^2}x^2-\frac{1}{n^2\pi^2}y^2-\frac{2}{n^2\pi^2}xy\right)\end{aligned}$$

The coefficient obtained by multiplying k terms $-\frac{1}{a_i^2\pi^2}x^2$, ℓ terms $-\frac{1}{b_i^2\pi^2}y^2$, and m mixed terms $-\frac{2}{c_i^2\pi^2}xy$ is equal to

$$\left(-\frac{1}{\pi^2}\right)^k\left(-\frac{1}{\pi^2}\right)^\ell\left(-\frac{2}{\pi^2}\right)^m\binom{k+l+m}{k}\binom{l+m}{m}\zeta(\{2\}^{k+l+m}).$$

Then this coefficient is equal to

$$a(k,\ell,m)=2^m\left(\frac{-1}{\pi^2}\right)^{k+\ell+m}\binom{k+l+m}{k}\binom{l+m}{m}\zeta(\{2\}^{k+l+m}).$$

Let $c(r,s)$ be the coefficient of x^ry^s in the expansion of $\sin(x+y)$ and $\bar{c}(r,s)$ be the coefficient of x^ry^s in the infinity product (2.19) divided by $(x+y)$. Then, $c(r,s)=\bar{c}(r-1,s)+\bar{c}(r,s-1)$. If $r\not\equiv s \pmod 2$, then $(r-1)\equiv s \pmod 2$ and $r\equiv(s-1)\pmod 2$. Both cases are similar to Theorem 2.28. Thus, the coefficient $c(r,s)$ is equal to

$$\left(\frac{\sqrt{-1}}{\pi}\right)^{r+s-1}\left(\sum_{k=0}^{\lfloor (r-1)/2\rfloor}2^{(r-1)-2k}\binom{\frac{s+r-1}{2}}{k}\binom{\frac{s+r-1}{2}-k}{r-2k-1}\zeta(\{2\}^{\frac{s+r-1}{2}})\right.$$
$$\left.+\sum_{k=0}^{\lfloor r/2\rfloor}2^{r-2k}\binom{\frac{s+r-1}{2}}{k}\binom{\frac{s+r-1}{2}-k}{r-2k-1}\zeta(\{2\}^{\frac{s+r}{2}})\right).$$

On the other hand, using the sum angle identity and Taylor series, we see that the coefficient of x^ry^s in $\sin(s+y)$ is $\frac{(\sqrt{-1})^{r+s-1}}{r!s!}$, for $r\not\equiv s \pmod 2$. Thus, for $0<r<s$, we have

$$\sum_{k=0}^{\lfloor (r-1)/2\rfloor}2^{(r-1)-2k}\binom{\frac{s+r-1}{2}}{k}\binom{\frac{s+r-1}{2}-k}{r-2k-1}\zeta(\{2\}^{\frac{s+r-1}{2}})$$
$$+\sum_{k=0}^{\lfloor r/2\rfloor}2^{r-2k}\binom{\frac{s+r-1}{2}}{k}\binom{\frac{s+r-1}{2}-k}{r-2k-1}\zeta(\{2\}^{\frac{s+r}{2}})=\frac{\pi^{r+s-1}}{r!s!}.$$

□

Combining the previous Theorem with Theorem 2, we deduce the following combinatorial identity.

Corollary 2.31. *For r and s positive integers with $r \not\equiv s \pmod 2$ and $r < s$, then*

$$\sum_{k=0}^{\lfloor (r-1)/2 \rfloor} 2^{r-2k-1} \binom{\frac{s+r-1}{2}}{k} \binom{\frac{s+r-1}{2} - k}{r-2k-1} + \sum_{k=0}^{\lfloor r/2 \rfloor} 2^{r-2k} \binom{\frac{s+r-1}{2}}{k} \binom{\frac{s+r-1}{2} - k}{r-2k} = \binom{r+s}{r}.$$

3. Integer partitions

An integer partition of n is a nonincreasing sequence of natural numbers whose sum is n. The five partitions of $n = 4$ are: 4, $3+1$, $2+2$, $2+1+1$, and $1+1+1+1$. More information about integer partitions can be found in [3, 4, 5, 6]. A partition is called a k-colored partition if each part can appear in k different colors. We denote each part of a k-colored partition with a subscript that represents its color. For instance, if $k = 2$, and the colors are black and red, the twenty 2-colored partitions of $n = 4$ are: 4_b, 4_r, $3_b + 1_b$, $3_b + 1_r$, $3_r + 1_b$, $3_r + 1_r$, $2_b + 2_b$, $2_r + 2_b$, $2_r + 2_r$, $2_b + 1_b + 1_b$, $2_b + 1_b + 1_b$, $2_b + 1_r + 1_b$, $2_r + 1_r + 1_b$, $2_b + 1_r + 1_r$, $2_r + 1_r + 1_r$, $1_b + 1_b + 1_b + 1_b$, $1_r + 1_b + 1_b + 1_b$, $1_r + 1_r + 1_b + 1_b$, $1_r + 1_r + 1_r + 1_b$, and $1_r + 1_r + 1_r + 1_r$.

Let $p_k(n)$ denote the number of k-colored partitions of n. The generating function for $p_k(n)$ is given by

$$\sum_{n=0}^{\infty} p_k(n) q^n = \frac{1}{(q;q)_\infty^k},$$

where we are using the standard q-Pochhammer symbol defined by

$$(a,q)_n = \begin{cases} (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}), & \text{if } n > 0; \\ 1, & \text{if } n = 0. \end{cases}$$

If $n \to \infty$, we have

$$(a,q)_\infty = \lim_{n\to\infty} (a,q)_n.$$

In the next theorem we analyze the following equation, for $|q| < 1$:

$$\frac{q}{(q;q)_\infty} \prod_{n=1}^{\infty} \left(1 - \frac{q^2}{\pi^2 n^2 (q;q)_\infty^2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{2n+1}}{(2n+1)!(q;q)_\infty^{2n+1}}, \tag{3.1}$$

which is obtained by using the product expansion and Taylor expansion of

$$\sin\left(\frac{q}{(q;q)_\infty}\right).$$

We note that this composition of generating functions is well-defined. Indeed, the independent term in $\frac{q}{(q;q)_\infty}$ is equal to 0, which is a sufficient condition to ensure the convergence of the composition of generating functions (see Wilf [30]).

We can rewrite (3.1) in a more simply form as

$$\prod_{n=1}^{\infty}\left(1-\frac{q^2}{\pi^2 n^2 (q;q)_\infty^2}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n q^{2n}}{(2n+1)!(q;q)_\infty^{2n}}. \tag{3.2}$$

The next result is a consequence of extracting the coefficient of q^n from (3.2).

Theorem 3.1. *For all $n > 2$,*

$$\sum_{\substack{(w_1,\dots,w_m)\in C(n)\\ w_i\geq 2}}\frac{(-1)^m p_2(w_1-2)\cdots p_2(w_m-2)}{(2m+1)!}=\sum_{k=1}^{\lfloor n/2\rfloor}\frac{(-1)^k p_{2k}(n-2k)}{(2k+1)!}. \tag{3.3}$$

Proof. Firstly we will show that

$$\sum_{\substack{(w_1,\dots,w_m)\in C(n)\\ w_i\geq 2}}\sum_{1\leq l_1<\dots<l_m}\frac{(-1)^m p_2(w_1-2)\cdots p_2(w_m-2)}{l_1^2\cdots l_m^2\pi^{2m}}$$

$$=\sum_{k=1}^{\lfloor n/2\rfloor}\frac{(-1)^k p_{2k}(n-2k)}{(2k+1)!}.$$

The left side of equation (3.2) can be rewritten as

$$\prod_{n=1}^{\infty}\left(1-\frac{q^2\sum_{k=0}^{\infty}p_2(k)q^k}{\pi^2 n^2}\right).$$

In order to find the coefficient of q^n for $n > 2$ in the previous infinite product, we consider an integer composition $(w_1, w_2, \dots, w_m) \in C(n)$, with $w_i \geq 2$. In the term

$$-\frac{1}{\pi^2 l_1^2}\sum_{k=0}^{\infty}p_2(k)q^{k+2},$$

for $l_1 \geq 1$, the coefficient of q^{w_1} is equal to

$$-\frac{p_2(w_1-2)}{\pi^2 l_1^2}.$$

Also, for $l_2 > l_1$, the contribution to the coefficient of q^{w_2} is

$$-\frac{p_2(w_2-2)}{\pi^2 l_2^2}.$$

In general, given a sequence $l_1, l_2, \dots, l_m$, with $1 \leq l_1 < l_2 < \dots < l_m$, the contribution to the coefficient of $q^{w_1+w_2+\cdots+w_m} = q^n$ is

$$\frac{(-1)^m p_2(w_1-2)p_2(w_2-2)\cdots p_2(w_m-2)}{\pi^{2m}l_1^2 l_2^2\dots l_m^2}.$$

Considering all sequences $l_1, l_2, \ldots, l_m$, with the aforementioned restrictions and all integer compositions in $C(n)$, for each part greater than 1, the coefficient of q^n in the left side of (3.2) is

$$\sum_{\substack{(w_1,\ldots,w_m)\in C(n)\\ w_i\geq 2}} \sum_{1\leq l_1<l_2<\ldots<l_m} (-1)^m \frac{p_2(w_1-2)\cdots p_2(w_m-2)}{\pi^{2m} l_1^2\cdots l_m^2}$$

$$= \sum_{\substack{(w_1,\ldots,w_m)\in C(n)\\ w_i\geq 2}} (-1)^m \frac{p_2(w_1-2)\cdots p_2(w_m-2)}{\pi^{2m}} \sum_{1\leq l_1<l_2<\ldots<l_m} \frac{1}{l_1^2\cdots l_m^2}.$$

Since $\zeta(\{2\}^m) = \pi^{2m}/(2m+1)!$, we have

$$\sum_{\substack{(w_1,\ldots,w_m)\in C(n)\\ w_i\geq 2}} (-1)^m \frac{p_2(w_1-2)\cdots p_2(w_m-2)}{\pi^{2m}} \sum_{1\leq l_1<l_2<\ldots<l_m} \frac{1}{l_1^2\cdots l_m^2}$$

$$= \sum_{\substack{(w_1,\ldots,w_m)\in C(n)\\ w_i\geq 2}} (-1)^m \frac{p_2(w_1-2)\cdots p_2(w_m-2)}{(2m+1)!}.$$

To finish the proof, we extract the coefficient of q^n in the right side of (3.2):

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{2n}}{(2n+1)!(q;q)_\infty^{2n}} = 1 - \frac{q^2}{3!}\sum_{k=0}^{\infty} p_2(k)q^k + \frac{q^4}{5!}\sum_{k=0}^{\infty} p_4(k)q^k$$
$$- \frac{q^6}{7!}\sum_{k=0}^{\infty} p_6(k)q^k + \ldots$$

It is easy to see that the coefficient of q^n in this sum is given by

$$-\frac{1}{3!}p_2(n-2) + \frac{1}{5!}p_4(n-4) - \frac{1}{7!}p_6(n-6) + \ldots + \frac{(-1)^n}{(2n+1)!}p_{2m}(n-2m),$$

where $m = \lfloor n/2 \rfloor$. Equating the coefficients of q^n on both sides of (3.2), we obtain the desired result. □

Example 3.2. *For $n = 6$, we have 5 compositions of 6 with parts greater than 1, namely*

$$\{(6), (4,2), (2,4), (3,3), (2,2,2)\}.$$

Applying Theorem 3.1 we obtain

$$-\frac{1}{3!}p_2(4) + \frac{1}{5!}(p_2(2)p_2(0) + p_2(0)p_2(2) + p_2(1)p_2(1)) - \frac{1}{7!}p_2(0)p_2(0)p_2(0)$$

as the left side of (3.3). *On the other hand, the right side of* (3.3) *is equal to*

$$-\frac{1}{3!}p_2(4) + \frac{1}{5!}p_4(2) - \frac{1}{7!}p_6(0)$$

Simplifying the resulting identity, we are left with

$$2p_2(2) + [p_2(1)]^2 = p_4(2),$$

which can also be verified by using $p_2(2) = 5, p_2(1) = 2$ *and* $p_4(2) = 14$.

Analogously to the previous theorem, the next one is obtained by analyzing the expansion of the cosine function (2.4) with

$$x = \frac{q}{(q;q)_\infty}.$$

Theorem 3.3. *For all* $n \geq 0$,

$$\sum_{\substack{(w_1,\ldots,w_m)\in C(n)\\ w_i\geq 2}} \frac{(-1)^m p_2(w_1-2)\cdots p_2(w_m-2)}{(2m)!} = \sum_{k=1}^{\lfloor n/2\rfloor} \frac{(-1)^k p_{2k}(n-2k)}{(2k)!}.$$

Now we present generalizations of the two theorems above.

Theorem 3.4. *For any generating function* $q\sum_{k=0}^{\infty} a_k q^k$, *taking*

$$b_{2n}(i) = \sum_{k_1+\cdots+k_{2n}\in C(i)} a_{k_1}\ldots a_{k_{2n}},$$

we have

$$\sum_{n=1}^{\lfloor m/2\rfloor} \frac{(-1)^n}{(2n)!} b_{2n}(m-2n) = \sum_{\substack{(w_1,\ldots,w_s)\in C(m)\\ w_i\geq 2}} \frac{(-1)^s b_2(w_1-2)\ldots b_2(w_s-2)}{(2s)!}.$$

Proof. By the Taylor expansion of $\cos x$ for $x = q\sum_{k=0}^{\infty} a_k q^k$, we have

$$\cos\left(q\sum_{k=0}^{\infty} a_k q^k\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(q\sum_{k=0}^{\infty} a_k q^k\right)^{2n}.$$

Since $\left(q\sum_{k=0}^{\infty} a_k q^k\right)^{2n} = q^{2n}\sum_{i=0}^{\infty} b_{2n}(i) q^i$, with $b_{2n}(i) = \sum_{\substack{k_1+\cdots+k_{2n}=i\\ k_j\geq 0}} a_{k_1}\ldots a_{k_{2n}}$, the coefficient of q^m is equal to

$$\sum_{n=1}^{\lfloor m/2\rfloor} \frac{(-1)^n}{(2n)!} b_{2n}(m-2n).$$

On the other hand, by the product expansion of $\cos\left(q\sum_{k=0}^{\infty} a_k q^k\right)$ we known that

$$\prod_{n=1}^{\infty}\left(1 - \frac{4}{(2n+1)^2\pi^2} q^2 \left(\sum_{i=0}^{\infty} b_2(i) q^i\right)\right),$$

which can be written as

$$1 - \frac{4}{\pi^2}\lambda(2) b_2(0) q^2 - \frac{4}{\pi^2}\lambda(2) b_2(1) q^3$$
$$+\left(-\frac{4}{\pi^2}\lambda(2) b_2(2) + \left(-\frac{4}{\pi^2}\right)^2 \lambda(2,2) b_2(0) b_2(0)\right) q^4 + \cdots.$$

Thus, the coefficient of q^m, for $m > 1$, is given by

$$\sum_{\substack{(w_1,\dots,w_s)\in C(m)\\ w_i\geq 2}} \left(-\frac{4}{\pi^2}\right)^s \lambda(\{2\}^s) b_2(w_1-2)\dots b_2(w_s-2).$$

Substituting $\lambda(\{2\}^s)$ by the result of Theorem 2.8, the last expression becomes

$$\sum_{\substack{(w_1,\dots,w_s)\in C(m)\\ w_i\geq 2}} \frac{(-1)^s b_2(w_1-2)\dots b_2(w_s-2)}{(2s)!}.$$

Then, for all $m > 1$, we have

$$\sum_{n=1}^{\lfloor m/2\rfloor} \frac{(-1)^n}{(2n)!} b_{2n}(m-2n) = \sum_{\substack{(w_1,\dots,w_s)\in C(m)\\ w_i\geq 2}} \frac{(-1)^s b_2(w_1-2)\dots b_2(w_s-2)}{(2s)!},$$

which completes the proof. □

Example 3.5. *The five compositions of 6 with parts greater than* 1 *are* $\{(6),(4,2),(2,4),(3,3),(2,2,2)\}$. *Thus, by Theorem 3.4 we have*

$$\begin{aligned}-\frac{1}{2!}b_2(4)+\frac{1}{4!}b_4(2)-\frac{1}{6!}b_6(0) = -\frac{1}{2!}b_2(4) &+ \frac{1}{4!}(b_2(2)b_2(0)+b_2(0)b_2(2)\\ &+ b_2(1)b_2(1)) - \frac{1}{6!}b_2(0)b_2(0)b_2(0),\end{aligned}$$

which, after some cancellations, gives $b_4(2) = 2b_2(2)b_2(0) + b_2^2(1)$. *In terms of the coefficients* a_k *of the generating function, this equation can be rewritten as* $4a_0^3a_2 + 6a_0^2a_1^2 = 2(2a_0a_2 + a_1^2)a_0^2 + (2a_0a_1)^2$. *Hence,* $4a_0^3a_2 + 6a_0^2a_1^2 = (4a_0^3a_2 + 2a_0^2a_1^2) + 4a_0^2a_1^2$.

Example 3.6. *The thirteen compositions of 8 with parts greater than 1 are*

(8)	*(6,2)*	*(2,6)*	*(5,3)*	*(3,5)*	*(4,4)*	*(4,2,2)*
(2,4,2)	*(2,2,4)*	*(3,3,2)*	*(3,2,3)*	*(2,3,3)*	*(2,2,2,2)*	

Then by Theorem 3.4 we have

$$\begin{aligned}30b_4(4) - b_6(2) = 60b_2(4)b_2(0) &+ 60b_2(3)b_2(1) + 30b_2^2(2)\\ &- 3b_2(2)b_2^2(0) - 3b_2^2(1)b_2(0).\end{aligned}$$

Theorem 3.7. *For any generating function* $q\sum_{k=0}^{\infty} a_kq^k$, *taking*

$$b_{2n}(i) = \sum_{(k_1,\dots,k_{2n})\in C(i)} a_{k_1}\dots a_{k_{2n}},$$

we have

$$\sum_{n=1}^{\lfloor m/2\rfloor} \frac{(-1)^n}{(2n+1)!} b_{2n}(m-2n) = \sum_{\substack{(w_1,\dots,w_s)\in C(m)\\ w_i\geq 2}} \frac{(-1)^s b_2(w_1-2)\dots b_2(w_s-2)}{(2s+1)!}.$$

Example 3.8. *Let $p^1_m(n)$ denote the number of partitions of n having only parts equal to 1 in m different colors and having at least one part 1 of each color. For instance: $p^1_m(n) = 0$ if $n < m$ and $p^1_m(m) = 1$. The generating function for $p^1_m(n)$ is given by*

$$\sum_{n=0}^{\infty} p^1_m(n) q^n = \frac{q^m}{(1-q)^m}.$$

Thus, by Theorem 3.7, we know that, for all $n \geq 0$,

$$\sum_{m=0}^{n} \sum_{(w_1,\ldots,w_m)\in C(n)} \frac{(-1)^m p^1_2(w_1)\cdots p^1_2(w_m)}{(2m+1)!} = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^m}{(2m+1)!} p^1_{2m}(n).$$

4. Concluding Remarks

There are many trigonometric identities not explored here. Considering compositions with convergent q-series, gamma function, and other special functions, we believe that there are more identities like the ones presented here to be discovered. We leave such an investigation to the interested reader. Another direction to be taken is to look for combinatorial proofs for the theorems presented in this paper.

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