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MATRIX DECOMPOSITIONS OF STIRLING-LIKE MATRICES AND THEIR DETERMINANTS

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In memory of Professor Robert Woodrow

Abstract. In this article, we study certain matrices whose entries are Stirling numbers of the second kind; these are called Stirling-like matrices of the second kind. We obtain, among other results, several matrix decompositions of these matrices and evaluate their determinants. In particular, we find some identities for Stirling numbers.

1. Introduction

Let n and k be nonnegative integers. We define $c(n,k)$ as the number of permutations of n (distinct) objects into exactly k cycles. The numbers $c(n,k)$ are known as the *unsigned Stirling number of the first kind*, and the numbers $s(n,k) = (-1)^{n-k}c(n,k)$ are called the *Stirling numbers of the first kind*. We also define $S(n,k)$ to be the number of partitions of a set with n elements into exactly k nonempty subsets. The integers $S(n,k)$ are called the *Stirling numbers of the second kind*. It should be clear that $S(n,n) = 1$ and $s(n,k) = S(n,k) = 0$ for $k > n \geq 0$, and it is easy to check that for all $n \geq 1$, $s(n,0) = S(n,0) = 0$, $s(n,1) = (-1)^{n-1}(n-1)!$ and $S(n,1) = 1$. By convention, we put $s(0,0) = S(0,0) = 1$. It is well-known that the Stirling numbers of the second kind $S(n,k)$ satisfy the following basic recurrence ([8, Eq. (1.93)]):

$$S(n,k) = S(n-1,k-1) + kS(n-1,k), \quad (n,k \geq 1). \tag{1.1}$$

The Stirling numbers of the first kind $s(n,k)$ and of the second kind $S(n,k)$ can also be defined as the coefficients in the following expansion of a variable x:

$$[x]_n = \sum_{k=0}^{n} s(n,k)x^k \qquad \text{and} \qquad x^n = \sum_{k=0}^{n} S(n,k)[x]_k, \tag{1.2}$$

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where

$$[x]_n = \begin{cases} x(x-1)\cdots(x-n+1), & \text{if } n \geq 1; \\ 1, & \text{if } n = 0, \end{cases}$$

is the falling factorial. Let $\mathbb{F}[x]$ denote the set of all polynomials in the variable x with coefficients in the field $\mathbb{F}$, which forms a vector space over $\mathbb{F}$. Clearly, the sets $\mathscr{B} = \{1, x, x^2, \ldots\}$ and $\mathscr{C} = \{1, [x]_1, [x]_2, \ldots\}$ form two bases of $\mathbb{F}[x]$. Then, Equation (1.2) indicates that the (infinite) matrix $s = [s(n,k)]_{k,n\geq 1}$ is the transition matrix from $\mathscr{B}$ to $\mathscr{C}$, while the (infinite) matrix $S = [S(n,k)]_{n,k\geq 1}$ is the transition matrix from $\mathscr{C}$ to $\mathscr{B}$. Therefore, the matrices s and S are inverses of each other. We call the matrices s and S *Stirling matrix of the first kind* and *of the second kind*, respectively.

As usual, we denote by $s(n)$ the $n \times n$ *Stirling matrix of the first kind*, that is, the $n \times n$ submatrix $s(n) = [s(i,j)]_{1\leq i,j\leq n}$ of s. Similarly, we denote by $S(n)$ the $n \times n$ *Stirling matrix of the second kind*, that is, $S(n) = [S(i,j)]_{1\leq i,j\leq n}$. Thus, for instance, the following matrices correspond to the Stirling matrix of the first kind $s(n)$ and of the second kind $S(n)$, with $n = 5$:

$$s(5) = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ -1 & 1 & \cdot & \cdot & \cdot \\ 2 & -3 & 1 & \cdot & \cdot \\ -6 & 11 & -6 & 1 & \cdot \\ 24 & -50 & 35 & -10 & 1 \end{pmatrix}, \qquad S(5) = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot \\ 1 & 3 & 1 & \cdot & \cdot \\ 1 & 7 & 6 & 1 & \cdot \\ 1 & 15 & 25 & 10 & 1 \end{pmatrix}.$$

We now define a matrix related to S, which is obtained by moving the column entries of S upwards to eliminate all the zeros. More precisely, we define the *Stirling-like matrix of the second kind* $M = [M_{i,j}]_{i,j\geq 1}$ to be the infinite matrix whose entries are given by $M_{i,j} = S(i+j-1, j)$. As before, the $n \times n$ Stirling-like matrix of the second kind, denoted by $M(n)$, is the $n \times n$ submatrix of M with entries $M_{i,j}$ for which $1 \leq i, j \leq n$. For instance, we have

$$M(5) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 7 & 25 & 65 & 140 \\ 1 & 15 & 90 & 350 & 1050 \\ 1 & 31 & 301 & 1701 & 6951 \end{pmatrix}.$$

For any matrix A, let $A^{\langle j_1, j_2, \ldots, j_t\rangle}_{\langle i_1, i_2, \ldots, i_r\rangle}$ denote the submatrix of A obtained from A by deleting rows $i_1 < i_2 < \cdots < i_r$ and columns $j_1 < j_2 < \cdots < j_t$. Let t be a positive integer. It is of interest to study matrices obtained from M by deleting its first t rows or its first t columns, that is, $M_{\langle 1,2,\ldots,t\rangle}$ and $M^{\langle 1,2,\ldots,t\rangle}$. The former was done in [4] and the latter is the purpose of this article. It is easy to observe that the (i,j)-entry of these matrices is given by

$$\left(M_{\langle 1,2,\ldots,t\rangle}\right)_{i,j} = S(i+j+t-1, j),$$

and

$$\left(M^{\langle 1,2,\dots,t\rangle}\right)_{i,j} = S(i+j+t-1, j+t).$$

For example, we have

$$M_{\langle 1\rangle}(4) = \begin{pmatrix} 1 & 3 & 6 & 10 \\ 1 & 7 & 25 & 65 \\ 1 & 15 & 90 & 350 \\ 1 & 31 & 301 & 1701 \end{pmatrix} \quad \text{and} \quad M^{\langle 1\rangle}(4) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 6 & 10 & 15 \\ 7 & 25 & 65 & 140 \\ 15 & 90 & 350 & 1050 \end{pmatrix}.$$

In [4], we proved several decomposition results for the matrices $M_{\langle 1,2,\dots,t\rangle}$ and we presented the following determinant formula:

$$\det M_{\langle 1,2,\dots,t\rangle}(n) = \det\big[S(i+j+t-1, j)\big]_{1\leq i,j\leq n} = \prod_{i=1}^{n} i^{i+t-1}. \tag{1.3}$$

The first aim of this article is to present some matrix decompositions for the matrices

$$M^{\langle 1,2,\dots,t\rangle}(n) = [S(i+j+t-1, j+t)]_{1\leq i,j\leq n},$$

where $t \geq 1$ is a fixed integer (Theorem 3.1). In particular, we establish the following determinant formula (Theorem 3.2):

$$\det M^{\langle 1,2,\dots,t\rangle}(n) = \prod_{1\leq i\leq n} (i+t)^{i-1}.$$

The second aim of this paper is to give some combinatorial identities for Stirling numbers (Theorems 4.1 and 4.2).

The organization of this article is as follows. In Section 2, we introduce additional definitions and notation, and then derive preparatory results. In Section 3, we present LU-decompositions for the matrices $M^{\langle 1,2,\dots,t\rangle}(n)$. The proofs of Theorems 3.1 and 3.2, which constitute the main results of this article, are presented in this section. Finally, in Section 4, we use the results obtained in the previous sections to derive some identities for Stirling numbers.

The notation used in this article is fairly standard; the reader is referred to [3, 8]. In particular, for any matrix A, we use $A_{i,j}$ to denote its (i,j)th entry.

2. Auxiliary results

In this section, we give several auxiliary results to be used later. Before proceeding, we provide some additional definitions and introduce relevant notation.

Let $r, t \geq 1$ be two integers. Denote by $A^{[t]}(n) = [A^{[t]}_{i,j}]_{1\leq i,j\leq n}$ the $n \times n$ lower triangular matrix whose entries are given by

$$A^{[t]}_{i,j} = \begin{cases} 1, & \text{if } i = j; \\ j+t, & \text{if } i = j+1; \\ 0, & \text{otherwise.} \end{cases}$$

Also, denote by $B^{[r,t]}(n) = [B^{[r,t]}_{i,j}]_{1\le i,j\le n}$ the $n \times n$ upper triangular matrix whose entries are defined by

$$B^{[r,t]}_{i,j} = \begin{cases} 1, & \text{if } i=j,\ j\le r; \\ j+t, & \text{if } i=j,\ j>r; \\ 1, & \text{if } i=j-1,\ j>r; \\ 0, & \text{otherwise.} \end{cases}$$

For instance, we have

$$A^{[2]}(5) = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ 3 & 1 & \cdot & \cdot & \cdot \\ \cdot & 4 & 1 & \cdot & \cdot \\ \cdot & \cdot & 5 & 1 & \cdot \\ \cdot & \cdot & \cdot & 6 & 1 \end{pmatrix} \quad \text{and} \quad B^{[3,2]}(5) = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & 6 & 1 \\ \cdot & \cdot & \cdot & \cdot & 7 \end{pmatrix}.$$

We begin with the following lemma.

Lemma 2.1. *Let $t \ge 1$ be an integer. Then, for any positive integer n, we have*

$$S^{\langle 1,2,\ldots,t\rangle}_{\langle 1,2,\ldots,t\rangle}(n) = S(n) \cdot \prod_{k=1}^{t} A^{[k]}(n). \tag{2.1}$$

Proof. We proceed by induction on t. If $t = 1$, then we have to show that

$$S^{\langle 1\rangle}_{\langle 1\rangle}(n) = S(n) \cdot A^{[1]}(n). \tag{2.2}$$

For the proof of the above equation we compute the (i, j)-entry of $S(n) \cdot A^{[1]}(n)$, which is by definition $\sum_{l=1}^{i} S_{i,l}A^{[1]}_{l,j}$. Since the product of two lower triangular matrices is lower triangular, we may assume $i \ge j \ge 1$. If $i = j$, then we have

$$\big(S(n)\cdot A^{[1]}(n)\big)_{i,i} = \sum_{l=1}^{i} S_{i,l}A^{[1]}_{l,i} = S_{i,i}A^{[1]}_{i,i} = 1\cdot 1 = 1 = \big(S^{\langle 1\rangle}_{\langle 1\rangle}(n)\big)_{i,i},$$

and (2.2) holds in this case. We may now assume that $i > j$. We have

$$\begin{aligned}
\big(S(n)\cdot A^{[1]}(n)\big)_{i,j} &= \sum_{l=1}^{i} S_{i,l}A^{[1]}_{l,j} \\
&= S_{i,j}A^{[1]}_{j,j} + S_{i,j+1}A^{[1]}_{j+1,j} \quad \text{(by the structure of } A^{[1]}(n)) \\
&= S_{i,j}\cdot 1 + S_{i,j+1}\cdot (j+1) \\
&= S(i,j) + (j+1)S(i,j+1) \quad \text{(by the definition of } S(n)) \\
&= S(i+1,j+1) \quad \text{(by (1.1))} \\
&= S_{i+1,j+1} \quad \text{(by the definition of } S(n)) \\
&= \big(S^{\langle 1\rangle}_{\langle 1\rangle}(n)\big)_{i,j},
\end{aligned}$$

and the proof of (2.2) is complete.

Assume that $t > 1$. In the sequel, we claim that

$$S^{\langle 1,2,\ldots,t\rangle}_{\langle 1,2,\ldots,t\rangle}(n) = S^{\langle 1,2,\ldots,t-1\rangle}_{\langle 1,2,\ldots,t-1\rangle}(n) \cdot A^{[t]}(n). \tag{2.3}$$

By an argument similar to the preceding, we again compute the (i,j)-entry on the right-hand side of (2.3). For convenience of notation, we define $\widehat{S}(n) = [\widehat{S}_{i,j}]_{1\le i,j\le n}$ to denote the matrix $S^{\langle 1,2,\ldots,t-1\rangle}_{\langle 1,2,\ldots,t-1\rangle}(n) = [S_{i+t-1,j+t-1}]_{1\le i,j\le n}$. As before, it suffices to consider the case where $i \ge j \ge 1$. If $i = j$, then we have

$$\big(\widehat{S}(n)\cdot A^{[t]}\big)_{i,i} = \sum_{l=1}^{i} \widehat{S}(n)_{i,l} A^{[t]}_{l,i} = \widehat{S}_{i,i} A^{[t]}_{i,i} = 1\cdot 1 = 1 = \big(S^{\langle 1,2,\ldots,t\rangle}_{\langle 1,2,\ldots,t\rangle}(n)\big)_{i,i}.$$

Moreover, in the case that $i > j$, we see that

$$\begin{aligned}
\big(\widehat{S}(n)\cdot A^{[t]}(n)\big)_{i,j} &= \sum_{l=1}^{i} \widehat{S}_{i,l} A^{[t]}_{l,j} \\
&= \widehat{S}_{i,j} A^{[t]}_{j,j} + \widehat{S}_{i,j+1} A^{[t]}_{j+1,j} \quad \text{(by the structure of } A^{[t]}(n)) \\
&= \widehat{S}_{i,j}\cdot 1 + \widehat{S}_{i,j+1}\cdot (j+t) \\
&= S_{i+t-1,j+t-1} + S_{i+t-1,j+t}\cdot (j+t) \\
&= S_{i+t,j+t} \quad \text{(by (1.1) and the definition of } S(n)) \\
&= \big(S^{\langle 1,2,\ldots,t\rangle}_{\langle 1,2,\ldots,t\rangle}(n)\big)_{i,j},
\end{aligned}$$

and the claim follows. We now turn to the induction step. We have

$$\begin{aligned}
S^{\langle 1,2,\ldots,t\rangle}_{\langle 1,2,\ldots,t\rangle}(n) &= S^{\langle 1,2,\ldots,t-1\rangle}_{\langle 1,2,\ldots,t-1\rangle}(n)\cdot A^{[t]}(n) \quad \text{(by (2.3))} \\
&= \Big(S(n)\cdot \prod_{k=1}^{t-1} A^{[k]}(n)\Big)\cdot A^{[t]}(n) \quad \text{(by the inductive hypothesis)} \\
&= S(n)\cdot \prod_{k=1}^{t} A^{[k]}(n).
\end{aligned}$$

The proof is complete. □

We will also need the following identity involving Stirling numbers of the second kind which is verified through induction on n.

Lemma 2.2. *Let $n \ge m \ge 0$ be integers. Then, we have*

$$\sum_{k=m}^{n} \binom{n}{k} S(k,m) = S(n+1,m+1).$$

Proof. See (6.15) in [6, Table 251]. □

Lemma 2.3. *Let i, j and l be integers satisfying $1 \le l \le i < j$ and put $\omega = j - i$. Then, we have*

$$\sum_{k=l}^{i} \binom{j}{k}\Big((\omega+1)\binom{k}{l}+\binom{k}{l-1}\Big)S(j-k,\omega) = (\omega+1)\binom{j+1}{l}S(j+1-l,\omega+1).$$

Proof. We observe first that by the trinomial revision identity

$$\binom{j}{k}\binom{k}{l} = \binom{j}{l}\binom{j-l}{k-l}, \tag{2.4}$$

and

$$\binom{j}{k}\binom{k}{l-1} = \binom{j}{l-1}\binom{j-l+1}{k-l+1}. \tag{2.5}$$

Now, from Equation (2.4) and Lemma 2.2, we deduce that

$$\begin{aligned} \sum_{k=l}^{i}\binom{j}{k}\binom{k}{l}S(j-k,\omega) &= \binom{j}{l}\sum_{k=l}^{i}\binom{j-l}{k-l}S(j-k,\omega) \\ &= \binom{j}{l}\sum_{k=l}^{i}\binom{j-l}{j-k}S(j-k,\omega) \\ &= \binom{j}{l}S(j-l+1,\omega+1). \end{aligned} \tag{2.6}$$

Similarly, from Equation (2.5) and Lemma 2.2, we conclude that

$$\begin{aligned} &\sum_{k=l}^{i}\binom{j}{k}\binom{k}{l-1}S(j-k,\omega) \\ &= \binom{j}{l-1}\sum_{k=l}^{i}\binom{j-l+1}{k-l+1}S(j-k,\omega) \\ &= \binom{j}{l-1}\sum_{k=l}^{i}\binom{j-l+1}{j-k}S(j-k,\omega) \\ &= \binom{j}{l-1}\sum_{k=l-1}^{i}\binom{j-l+1}{j-k}S(j-k,\omega) - \binom{j}{l-1}S(j-l+1,\omega) \\ &= \binom{j}{l-1}S(j-l+2,\omega+1) - \binom{j}{l-1}S(j-l+1,\omega). \end{aligned} \tag{2.7}$$

Also, it follows from Equation (1.1) that

$$S(j-l+2,\omega+1) = S(j-l+1,\omega) + (\omega+1)S(j-l+1,\omega+1). \tag{2.8}$$

We substitute Equation (2.8) into Equation (2.7) and simplify to obtain

$$\sum_{k=l}^{i}\binom{j}{k}\binom{k}{l-1}S(j-k,\omega) = \binom{j}{l-1}(\omega+1)S(j-l+1,\omega+1). \tag{2.9}$$

Finally, using Equations (2.6), (2.9) and Pascal's identity, we observe that

$$\begin{aligned}
&\sum_{k=l}^{i}\binom{j}{k}\Big((\omega+1)\binom{k}{l}+\binom{k}{l-1}\Big)S(j-k,\omega)\\
&=(\omega+1)\sum_{k=l}^{i}\binom{j}{k}\binom{k}{l}S(j-k,\omega)+\sum_{k=l}^{i}\binom{j}{k}\binom{k}{l-1}S(j-k,\omega)\\
&=(\omega+1)\Big(\binom{j}{l}S(j-l+1,\omega+1)+\binom{j}{l-1}S(j-l+1,\omega+1)\Big)\\
&=(\omega+1)\Big(\binom{j}{l}+\binom{j}{l-1}\Big)S(j-l+1,\omega+1)\\
&=(\omega+1)\binom{j+1}{l}S(j-l+1,\omega+1),
\end{aligned}$$

and the proof is complete. $\square$

Next, we introduce the matrix $U^{[t]}(n)=[U^{[t]}_{i,j}]_{1\le i,j\le n}$ whose entries satisfy the recurrence relation:

$$U^{[t]}_{i,j}=U^{[t]}_{i,j-1}+(i-1+t)U^{[t]}_{i-1,j-1}+(j-i+1)U^{[t]}_{i-1,j}\quad(2\le i,j\le n),\tag{2.10}$$

and the initial conditions

$$U^{[t]}_{1,1}=1,\ U^{[t]}_{i,1}=0,\ U^{[t]}_{1,j}=1\quad(2\le i,j\le n).\tag{2.11}$$

The matrix $U^{[t]}(n)=[U^{[t]}_{i,j}]_{1\le i,j\le n}$ is an upper triangular matrix which plays a crucial role in the sequel. For instance, when $n=5$ and $t=3$, we have

$$U^{[3]}(5)=\begin{pmatrix}1&1&1&1&1\\ \cdot&5&11&18&26\\ \cdot&\cdot&36&127&295\\ \cdot&\cdot&\cdot&343&1695\\ \cdot&\cdot&\cdot&\cdot&4096\end{pmatrix}.$$

For the next purpose, we need the following lemma, which describes an explicit formula for $U^{[t]}_{i,j}$.

Lemma 2.4. *The matrix $U^{[t]}(n)$ is an upper triangular matrix whose entries are given by*

$$U^{[t]}_{i,j}=\sum_{k=0}^{i-1}\binom{j-1}{k}S(j-1-k,j-i)(i+t)^k,\quad \textit{for } j\ge i\ge 1.$$

In particular, the diagonal entries of $U^{[t]}(n)$ are given by

$$U^{[t]}_{i,i}=(i+t)^{i-1},\quad \textit{for}\ \ i=1,2,\dots,n.$$

Proof. Observe, first of all, that $U^{[t]}_{i,1}=0$ and $U^{[t]}_{i+1,i}=U^{[t]}_{i+1,i-1}+(i+t)U^{[t]}_{i,i-1}$ for $i\ge 2$, which implies that $U^{[t]}(n)$ is an upper triangular matrix.

Now let $Q^{[t]}(n) = [Q^{[t]}_{i,j}]_{1\le i,j\le n}$ be an upper triangular matrix given by

$$Q^{[t]}_{i,j} = \sum_{k=0}^{i-1} \binom{j-1}{k} S(j-1-k, j-i)(i+t)^k \quad \text{for } j \ge i \ge 1. \tag{2.12}$$

To prove the lemma we need only show that $U^{[t]}(n) = Q^{[t]}(n)$. In fact, we will show that the first row of $Q^{[t]}(n)$ is equal to the first row of $U^{[t]}(n)$, and for $1 < i \le j$ the entries $Q^{[t]}_{i,j}$ of the matrix $Q^{[t]}(n)$ satisfy the recursion relation

$$Q^{[t]}_{i,j} = Q^{[t]}_{i,j-1} + (i-1+t)Q^{[t]}_{i-1,j-1} + (j-i+1)Q^{[t]}_{i-1,j}. \tag{2.13}$$

Let us do the required calculations. If $i = 1$, then a direct computation gives

$$\begin{aligned} Q^{[t]}_{1,j} &= \sum_{k=0}^{0} \binom{j-1}{k} S(j-1-k, j-1)(1+t)^k \\ &= \binom{j-1}{0} S(j-1-0, j-1)(1+t)^0 = 1, \end{aligned}$$

and hence $Q^{[t]}_{1,j} = U^{[t]}_{1,j}$ for any $j \ge 1$.

If $1 < i \le j$, then we treat the different cases $i = j$ and $i < j$ separately. First, assuming that $i = j$, we see that (2.13) reduces to

$$Q^{[t]}_{i,i} = (i-1+t)Q^{[t]}_{i-1,i-1} + Q^{[t]}_{i-1,i}.$$

In order to prove this, it is routine to compute that

$$Q^{[t]}_{i,i} = \sum_{k=0}^{i-1} \binom{i-1}{k} S(i-1-k, 0)(i+t)^k = \binom{i-1}{i-1} S(0,0)(i+t)^{i-1} = (i+t)^{i-1},$$

and similarly, $Q^{[t]}_{i-1,i-1} = (i-1+t)^{i-2}$. On the other hand, we obtain

$$Q^{[t]}_{i-1,i} = \sum_{k=0}^{i-2} \binom{i-1}{k} S(i-1-k, 1)(i-1+t)^k = \sum_{k=0}^{i-2} \binom{i-1}{k} (i-1+t)^k.$$

Hence, we observe that

$$(i-1+t)Q^{[t]}_{i-1,i-1} + Q^{[t]}_{i-1,i} = \sum_{k=0}^{i-1} \binom{i-1}{k} (i-1+t)^k = (i+t)^{i-1} = Q^{[t]}_{i,i},$$

as required.

We now assume that $1 < i < j$. Using the Pascal's identity and (1.1), we first observe that

$$Q^{[t]}_{i,j} = \sum_{k=0}^{i-1} \binom{j-1}{k} S(j-1-k, j-i)(i+t)^k$$

$$
\begin{aligned}
&= S(j-1,j-i) + \sum_{k=1}^{i-1}\left(\binom{j-2}{k-1} + \binom{j-2}{k}\right) S(j-1-k,j-i)(i+t)^k \\
&= \sum_{k=0}^{i-1}\binom{j-2}{k} S(j-1-k,j-i)(i+t)^k \\
&\quad + \sum_{k=1}^{i-1}\binom{j-2}{k-1} S(j-1-k,j-i)(i+t)^k \\
&= \sum_{k=0}^{i-1}\binom{j-2}{k}\big(S(j-2-k,j-i-1) + (j-i)S(j-2-k,j-i)\big)(i+t)^k \\
&+ \sum_{k=0}^{i-2}\binom{j-2}{k} S(j-2-k,j-i)(i+t)^{k+1}.
\end{aligned}
$$

Next, using the binomial theorem and Equation (2.12), the right-hand side can be rewritten as

$$
\begin{aligned}
Q^{[t]}_{i,j} &= Q^{[t]}_{i,j-1} + (j-i)\sum_{k=0}^{i-2}\binom{j-2}{k} S(j-2-k,j-i)\Big(\sum_{l=0}^{k}\binom{k}{l}(i-1+t)^l\Big) \\
&\quad + \sum_{k=0}^{i-2}\binom{j-2}{k} S(j-2-k,j-i)\Big(\sum_{l=0}^{k+1}\binom{k+1}{l}(i-1+t)^l\Big) \\
&= Q^{[t]}_{i,j-1} + (j-i)\sum_{k=0}^{i-2}\sum_{l=0}^{k}\binom{j-2}{k}\binom{k}{l} S(j-2-k,j-i)(i-1+t)^l \\
&\quad + (i-1+t)Q^{[t]}_{i-1,j-1} \\
&\quad + \sum_{k=0}^{i-2}\sum_{l=0}^{k}\binom{j-2}{k}\binom{k+1}{l} S(j-2-k,j-i)(i-1+t)^l \\
&= Q^{[t]}_{i,j-1} + (i-1+t)Q^{[t]}_{i-1,j-1} \\
&\quad + (j-i+1)\sum_{k=0}^{i-2}\binom{j-2}{k} S(j-2-k,j-i) \\
&\quad + \sum_{k=1}^{i-2}\sum_{l=1}^{k}\binom{j-2}{k}\Big((j-i+1)\binom{k}{l} \\
&\quad + \binom{k}{l-1}\Big) S(j-2-k,j-i)(i-1+t)^l.
\end{aligned}
$$

Finally, by Lemmas 2.2 and 2.3, and the standard interchange formula (see [7, 3.16]), we get

$$
\begin{aligned}
Q_{i,j}^{[t]} &= Q_{i,j-1}^{[t]} + (i-1+t)Q_{i-1,j-1}^{[t]} + (j-i+1)S(j-1,j-i+1) \\
&\quad + \sum_{l=1}^{i-2}\sum_{k=l}^{i-2}\binom{j-2}{k}\Big((j-i+1)\binom{k}{l} \\
&\quad + \binom{k}{l-1}\Big)S(j-2-k,j-i)(i-1+t)^l \\
&= Q_{i,j-1}^{[t]} + (i-1+t)Q_{i-1,j-1}^{[t]} + (j-i+1)S(j-1,j-i+1) \\
&\quad +(j-i+1)\sum_{l=1}^{i-2}\binom{j-1}{l}S(j-1-l,j-i+1)(i-1+t)^l \\
&= Q_{i,j-1}^{[t]} + (i-1+t)Q_{i-1,j-1}^{[t]} + (j-i+1)Q_{i-1,j}^{[t]},
\end{aligned}
$$

which completes the proof of (2.13). The first statement now follows.

The second statement follows from the first. The proof is complete. □

In the following, we will discuss a factorization of $U^{[t]}(n)$. Before continuing, we present an additional notation. The *direct sum* of two matrices A and B is defined as

$$
A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.
$$

For instance, it is an easy consequence of the definition that

$$
B^{[r,t]}(n) = I_{r-1} \oplus B^{[1,t+r-1]}(n-r+1). \tag{2.14}
$$

Lemma 2.5. *Let $t \geq 1$ be an integer. Then, for any positive integer n, we have*

$$
U^{[t]}(n) = B^{[1,t]}(n) \cdot [I_1 \oplus U^{[t+1]}(n-1)]. \tag{2.15}
$$

Proof. For the sake of brevity, throughout the proof we will set $U = U^{[t]}(n) = [U_{i,j}]_{1\leq i,j\leq n}$, $B = B^{[1,t]}(n) = [B_{i,j}]_{1\leq i,j\leq n}$ and $C = I_1 \oplus U^{[t+1]}(n-1) = [C_{i,j}]_{1\leq i,j\leq n}$. Note that the matrices U, B and C are upper triangular matrices, and the following relation holds:

$$
C_{i,j} = C_{i,j-1} + (i+t-1)C_{i-1,j-1} + (j-i+1)C_{i-1,j}, \quad (2 \leq i,j \leq n). \tag{2.16}
$$

In order to prove $U = B \cdot C$, therefore, it suffices to show that

$$
(B \cdot C)_{1,j} = U_{1,j},
$$

for every $1 \leq j \leq n$, and

$$
(B\cdot C)_{i,j} = (B\cdot C)_{i,j-1} + (i+t-1)(B\cdot C)_{i-1,j-1} + (j-i+1)(B\cdot C)_{i-1,j}, \tag{2.17}
$$

for every $2 \leq i,j \leq n$.

First, using elementary calculations, we see that

$$
(B \cdot C)_{1,j} = \sum_{k=1}^{n} B_{1,k}C_{k,j} = B_{1,1}C_{1,j} + B_{1,2}C_{2,j} = C_{1,j} + C_{2,j} = 1 = U_{1,j},
$$

for every $1 \leq j \leq n$.

We assume, therefore, that $1 < i \leq j \leq n$. We consider the two cases $i = j$ and $i < j$, separately.

Case 1: $i = j$. In this case, with (2.17) in mind, we must show that

$$(B \cdot C)_{i,i} = (i + t - 1)(B \cdot C)_{i-1,i-1} + (B \cdot C)_{i-1,j}.$$

On the one hand, we have

$$(B \cdot C)_{i,i} = \sum_{k=1}^{n} B_{i,k} C_{k,i} = B_{i,i} C_{i,i} = (i + t) C_{i,i},$$

and on the other hand, we have

$$(B \cdot C)_{i-1,i-1} = (i + t - 1) C_{i-1,i-1} \text{ and } (B \cdot C)_{i-1,i} = (i + t - 1) C_{i-1,i} + C_{i,i},$$

from which it follows that

$$\begin{aligned}
&(i + t - 1)(B \cdot C)_{i-1,i-1} + (B \cdot C)_{i-1,j} \\
&= (i + t - 1)[(i + t - 1) C_{i-1,i-1} + C_{i-1,i}] + C_{i,i} \\
&= (i + t - 1) C_{i,i} + C_{i,i} \quad \text{(by (2.16))} \\
&= (i + t) C_{i,i}.
\end{aligned}$$

Comparing the above equalities yields the result in this case.

Case 2: $i < j$. To verify (2.17), we need some calculations again. The (i, j)-entry of $B \cdot C$ is

$$(B \cdot C)_{i,j} = \sum_{k=1}^{n} B_{i,k} C_{k,j} = B_{i,i} C_{i,j} + B_{i,i+1} C_{i+1,j} = (i + t) C_{i,j} + C_{i+1,j}.$$

Similarly, we get

$$\begin{aligned}
(B \cdot C)_{i,j-1} &= B_{i,i} C_{i,j-1} + B_{i,i+1} C_{i+1,j-1} = (i + t) C_{i,j-1} + C_{i+1,j-1}, \\
(B \cdot C)_{i-1,j-1} &= B_{i-1,i-1} C_{i-1,j-1} + B_{i-1,i} C_{i,j-1} \\
&= (i + t - 1) C_{i-1,j-1} + C_{i,j-1}, \\
(B \cdot C)_{i-1,j} &= B_{i-1,i-1} C_{i-1,j} + B_{i-1,i} C_{i,j} = (i + t - 1) C_{i-1,j} + C_{i,j}.
\end{aligned}$$

Substituting these expressions for $(B \cdot C)_{i,j-1}$, $(B \cdot C)_{i-1,j-1}$ and $(B \cdot C)_{i-1,j}$ into (2.17) yields

$$\begin{aligned}
&(B \cdot C)_{i,j-1} + (i + t - 1)(B \cdot C)_{i-1,j-1} + (j - i + 1)(B \cdot C)_{i-1,j} \\
&= (i + t) C_{i,j-1} + C_{i+1,j-1} + (i + t - 1)(i + t - 1) C_{i-1,j-1} \\
&\quad + (i + t - 1) C_{i,j-1} + (j - i + 1)(i + t - 1) C_{i-1,j} + (j - i + 1) C_{i,j} \\
&= (i + t - 1) \left[C_{i,j-1} + (i + t - 1) C_{i-1,j-1} + (j - i + 1) C_{i-1,j} \right] + C_{i,j-1} \\
&\quad + C_{i+1,j-1} + (i + t - 1) C_{i,j-1} + (j - i + 1) C_{i,j} \\
&= (i + t - 1) C_{i,j} + C_{i,j-1} + C_{i+1,j-1} + (i + t - 1) C_{i,j-1} + (j - i + 1) C_{i,j}
\end{aligned}$$

$$\begin{aligned}
&\quad\text{(by (2.16))}\\
&= (j+t)C_{i,j} + (i+t)C_{i,j-1} + C_{i+1,j-1}\\
&= (i+t)C_{i,j} + \big[(j-i)C_{i,j} + (i+t)C_{i,j-1} + C_{i+1,j-1}\big]\\
&= (i+t)C_{i,j} + C_{i+1,j} \quad \text{(by (2.16))}\\
&= (B \cdot C)_{i,j}.
\end{aligned}$$

This establishes (2.17) in this case, and the proof is complete. □

For example, we have

$$U^{[2]}(5) = B^{[1,2]}(5) \cdot [I_1 \oplus U^{[3]}(4)],$$

or equivalently

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ \cdot & 4 & 9 & 15 & 22 \\ \cdot & \cdot & 25 & 91 & 217 \\ \cdot & \cdot & \cdot & 216 & 1105 \\ \cdot & \cdot & \cdot & \cdot & 2401 \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & 4 & 1 & \cdot & \cdot \\ \cdot & \cdot & 5 & 1 & \cdot \\ \cdot & \cdot & \cdot & 6 & 1 \\ \cdot & \cdot & \cdot & \cdot & 7 \end{pmatrix} \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & 1 & 1 & 1 \\ \cdot & \cdot & 5 & 11 & 18 \\ \cdot & \cdot & \cdot & 36 & 127 \\ \cdot & \cdot & \cdot & \cdot & 343 \end{pmatrix}.$$

The following corollary is an immediate consequence of Lemma 2.5.

Corollary 2.6. *Let $t \geq 1$ be an integer. Then, for any positive integer n, we have*

$$U^{[t]}(n) = \prod_{r=1}^{n} B^{[r,t]}(n). \tag{2.18}$$

Proof. By repeated application of Lemma 2.5 and (2.14), it follows that

$$\begin{aligned}
U^{[t]}(n) &= B^{[1,t]}(n) \cdot [I_1 \oplus U^{[t+1]}(n-1)]\\
&= B^{[1,t]}(n) \cdot \Big(I_1 \oplus \Big(B^{[1,t+1]}(n-1) \cdot [I_1 \oplus U^{[t+2]}(n-2)]\Big)\Big)\\
&= B^{[1,t]}(n) \cdot \big(I_1 \oplus B^{[1,t+1]}(n-1)\big) \cdot \big(I_2 \oplus U^{[t+2]}(n-2)\big)\\
&= B^{[1,t]}(n) \cdot B^{[2,t]}(n) \cdot \big(I_2 \oplus U^{[t+2]}(n-2)\big)\\
&\quad\vdots\\
&= B^{[1,t]}(n) \cdot B^{[2,t]}(n) \cdot \ldots \cdot B^{[n,t]}(n),
\end{aligned}$$

as desired. □

For example, for $t = 2$ and $n = 5$, we have

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ \cdot & 4 & 9 & 15 & 22 \\ \cdot & \cdot & 25 & 91 & 217 \\ \cdot & \cdot & \cdot & 216 & 1105 \\ \cdot & \cdot & \cdot & \cdot & 2401 \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdot & \cdot & \cdot \\ \cdot & 4 & 1 & \cdot & \cdot \\ \cdot & \cdot & 5 & 1 & \cdot \\ \cdot & \cdot & \cdot & 6 & 1 \\ \cdot & \cdot & \cdot & \cdot & 7 \end{pmatrix} \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & 5 & 1 & \cdot \\ \cdot & \cdot & \cdot & 6 & 1 \\ \cdot & \cdot & \cdot & \cdot & 7 \end{pmatrix}$$
$$\times \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & 6 & 1 \\ \cdot & \cdot & \cdot & \cdot & 7 \end{pmatrix} \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & 7 \end{pmatrix}$$
$$\times \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}.$$

3. The truncated matrix $M^{\langle 1,2,\dots,t\rangle}(n)$

We are now ready to state and prove the main results of this article.

Theorem 3.1. *Let $t \geq 1$ be an integer. Then, for any positive integer n, we have*

$$M^{\langle 1,2,\dots,t\rangle}(n) = S^{\langle 1,2,\dots,t\rangle}_{\langle 1,2,\dots,t\rangle}(n) \cdot U^{[t]}(n). \tag{3.1}$$

Furthermore, we have

$$M^{\langle 1,2,\dots,t\rangle}(n) = S(n) \cdot \prod_{k=1}^{t} A^{[k]}(n) \cdot \prod_{r=1}^{n} B^{[r,t]}(n). \tag{3.2}$$

Proof. Write $\widehat{M} = \widehat{M}(n)$, $\widehat{S} = \widehat{S}(n)$, and $\widehat{U} = \widehat{U}(n)$ for $M^{\langle 1,2,\dots,t\rangle}(n)$, $S^{\langle 1,2,\dots,t\rangle}_{\langle 1,2,\dots,t\rangle}(n)$, and $U^{[t]}(n)$, respectively. It is easy to see that

$$\widehat{M}_{i,j} = S(i+j+t-1, j+t),$$

and the entries of $\widehat{M}_{i,j}$ also satisfy the recurrence relation

$$\widehat{M}_{i,j} = \widehat{M}_{i,j-1} + (j+t)\widehat{M}_{i-1,j},$$

when $i, j \geq 2$, and the initial conditions

$$\widehat{M}_{i,1} = S(i+t, 1+t) \quad \text{and} \quad \widehat{M}_{1,j} = 1,$$

when $1 \leq i, j \leq n$. Similarly, one can easily check that $\widehat{S}_{i,j} = S(i+t, j+t)$ and they also satisfy the recurrence relation

$$\widehat{S}_{i,j} = \widehat{S}_{i-1,j-1} + (j+t)\widehat{S}_{i-1,j}, \tag{3.3}$$

when $i, j \geq 2$, and the initial conditions

$$\widehat{S}_{1,1} = 1, \quad \widehat{S}_{i,1} = S(i+t, 1+t) \quad \text{and} \quad \widehat{S}_{1,j} = 0,$$

when $2 \leq i, j \leq n$. For instance, we have

$$\widehat{M}(4) = M^{\langle 1,2 \rangle}(4) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 6 & 10 & 15 & 21 \\ 25 & 65 & 140 & 266 \\ 90 & 350 & 1050 & 2646 \end{pmatrix},$$

and,

$$\widehat{S}(4) = S^{\langle 1,2 \rangle}_{\langle 1,2 \rangle}(4) = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ 6 & 1 & \cdot & \cdot \\ 25 & 10 & 1 & \cdot \\ 90 & 65 & 15 & 1 \end{pmatrix}.$$

In order to prove (3.1), namely $\widehat{M} = \widehat{S} \cdot \widehat{U}$, it suffices to show that their entries correspond to the same recurrence relations satisfying the same initial conditions, that is

$$(\widehat{S} \cdot \widehat{U})_{1,j} = \widehat{M}_{1,j} \quad \text{and} \quad (\widehat{S} \cdot \widehat{U})_{i,1} = \widehat{M}_{i,1},$$

for every $1 \leq i, j \leq n$, and

$$(\widehat{S} \cdot \widehat{U})_{i,j} = (\widehat{S} \cdot \widehat{U})_{i,j-1} + (j+t)(\widehat{S} \cdot \widehat{U})_{i-1,j}, \tag{3.4}$$

for every $2 \leq i, j \leq n$.

First, using simple calculations, we have

$$(\widehat{S} \cdot \widehat{U})_{1,j} = \sum_{k=1}^{n} \widehat{S}_{1,k} \widehat{U}_{k,j} = \widehat{S}_{1,1} \widehat{U}_{1,j} = 1 \cdot 1 = 1 = \widehat{M}_{1,j},$$

and

$$(\widehat{S} \cdot \widehat{U})_{i,1} = \sum_{k=1}^{n} \widehat{S}_{i,k} \widehat{U}_{k,1} = \widehat{S}_{i,1} \widehat{U}_{1,1} = S(i+t, 1+t) \cdot 1 = S(i+t, 1+t) = \widehat{M}_{i,1},$$

as required.

Next, to verify (3.4) formally, we argue as follows. Let us assume for the moment that $2 \leq i, j \leq n$. Since

$$\begin{aligned}
(\widehat{S} \cdot \widehat{U})_{i,j} &= \sum_{k=1}^{n} \widehat{S}_{i,k} \widehat{U}_{k,j} = \widehat{S}_{i,1} \widehat{U}_{1,j} + \sum_{k=2}^{n} \widehat{S}_{i,k} \widehat{U}_{k,j} \\
&= \widehat{S}_{i,1} \widehat{U}_{1,j-1} + \sum_{k=2}^{n} \widehat{S}_{i,k} \big[\widehat{U}_{k,j-1} + (k+t-1) \widehat{U}_{k-1,j-1} \\
&\qquad\qquad + (j-k+1) \widehat{U}_{k-1,j} \big] \\
&= (\widehat{S} \cdot \widehat{U})_{i,j-1} + \sum_{k=2}^{n} \widehat{S}_{i,k} \big[(k+t-1) \widehat{U}_{k-1,j-1} + (j-k+1) \widehat{U}_{k-1,j} \big],
\end{aligned}$$

it suffices to show that

$$\sum_{k=2}^{n} \widehat{S}_{i,k}\big[(k+t-1)\widehat{U}_{k-1,j-1}+(j-k+1)\widehat{U}_{k-1,j}\big]=(j+t)(\widehat{S}\cdot\widehat{U})_{i-1,j} \tag{3.5}$$

in order to prove (3.4). To do this, we see that using (3.3), the sum on the left-hand side of (3.5) is equal to

$$\begin{aligned}
&\sum_{k=2}^{n}\big[\widehat{S}_{i-1,k-1}+(k+t)\widehat{S}_{i-1,k}\big]\big[(k+t-1)\widehat{U}_{k-1,j-1}+(j-k+1)\widehat{U}_{k-1,j}\big]\\
&=(t+1)\widehat{S}_{i-1,1}\widehat{U}_{1,j-1}+\sum_{k=3}^{n}(k+t-1)\widehat{S}_{i-1,k-1}\widehat{U}_{k-1,j-1}\\
&\quad+(j-1)\widehat{S}_{i-1,1}\widehat{U}_{1,j}+\sum_{k=3}^{n}(j-k+1)\widehat{S}_{i-1,k-1}\widehat{U}_{k-1,j}\\
&\quad+\sum_{k=2}^{n-1}(k+t)(k+t-1)\widehat{S}_{i-1,k}\widehat{U}_{k-1,j-1}\\
&\quad+\sum_{k=2}^{n-1}(k+t)(j-k+1)\widehat{S}_{i-1,k}\widehat{U}_{k-1,j}\\
&\quad(\text{note that } \widehat{S}_{i-1,n}=0),\\
&=(j+t)\widehat{S}_{i-1,1}\widehat{U}_{1,j}+\sum_{k=2}^{n-1}(k+t)\widehat{S}_{i-1,k}\widehat{U}_{k,j-1}+\sum_{k=2}^{n-1}(j-k)\widehat{S}_{i-1,k}\widehat{U}_{k,j}\\
&\quad+\sum_{k=2}^{n-1}(k+t)(k+t-1)\widehat{S}_{i-1,k}\widehat{U}_{k-1,j-1}\\
&\quad+\sum_{k=2}^{n-1}(k+t)(j-k+1)\widehat{S}_{i-1,k}\widehat{U}_{k-1,j}\\
&\quad(\text{note that } \widehat{U}_{1,j-1}=\widehat{U}_{1,j}),\\
&=(j+t)\widehat{S}_{i-1,1}\widehat{U}_{1,j}+\sum_{k=2}^{n-1}\widehat{S}_{i-1,k}\Big[(k+t)\big[\widehat{U}_{k,j-1}+(k+t-1)\widehat{U}_{k-1,j-1}\\
&\quad+(j-k+1)\widehat{U}_{k-1,j}\big]+(j-k)\widehat{U}_{k,j}\Big]\\
&=(j+t)\widehat{S}_{i-1,1}\widehat{U}_{1,j}+\sum_{k=2}^{n-1}\widehat{S}_{i-1,k}\big[(k+t)\widehat{U}_{k,j}+(j-k)\widehat{U}_{k,j}\big]\quad(\text{by } (2.10))\\
&=(j+t)\sum_{k=1}^{n}\widehat{S}_{i-1,k}\widehat{U}_{k,j}\quad(\text{note that again } \widehat{S}_{i-1,n}=0),
\end{aligned}$$

as required. This completes the proof of (3.1). Now, the decomposition (3.2) follows immediately from Lemma 2.1 and Corollary 2.6. $\square$

Theorem 3.2. *Let t and n be positive integers. Then, we have*

$$\det M^{\langle 1,2,\ldots,t\rangle}(n) = \det\bigl[S(i+j+t-1, j+t)\bigr]_{1\le i,j\le n} = \prod_{i=1}^{n}(i+t)^{i-1}.$$

Proof. We first note that $\bigl(M^{\langle 1,2,\ldots,t\rangle}(n)\bigr)_{i,j} = S(i+j+t-1, j+t)$, for any $1 \le i, j \le n$. Now, using the decomposition (3.2) in Theorem 3.1, we obtain

$$\begin{aligned}
\det M^{\langle 1,2,\ldots,t\rangle}(n) &= \det S(n)\Bigl(\prod_{k=1}^{t}\det A^{[k]}(n)\Bigr)\Bigl(\prod_{r=1}^{n-1}\det B^{[r,t]}(n)\Bigr)\\
&= 1\Bigl(\prod_{k=1}^{t}1\Bigr)\Bigl(\prod_{r=1}^{n-1}\det B^{[r,t]}(n)\Bigr)\\
&= \prod_{r=1}^{n-1}\det B^{[r,t]}(n).
\end{aligned}$$

Observe that the matrix $B^{[r,t]}(n)$ is an upper triangular matrix with diagonal entries

$$1, 1, \ldots, 1, r+1+t, r+2+t, \ldots, n+t,$$

and so

$$\det B^{[r,t]}(n) = \prod_{l=r+1}^{n}(l+t).$$

Thus, we have

$$\det M^{\langle 1,2,\ldots,t\rangle}(n) = \prod_{r=1}^{n-1}\det B^{[r,t]}(n) = \prod_{r=1}^{n-1}\prod_{l=r+1}^{n}(l+t) = \prod_{i=1}^{n}(i+t)^{i-1},$$

as desired. $\square$

4. Some identities for Stirling numbers

In our previous work [1], we derived several combinatorial identities. In this section, we present some identities for Stirling numbers related to the results obtained in the preceding sections. We start with the following theorem.

Theorem 4.1. *Let $i, j, t \ge 1$ be integers. Then, we have*

$$\sum_{l=1}^{\min\{i,j\}}\sum_{k=0}^{l-1}\binom{j-1}{k}S(i+t, l+t)S(j-1-k, j-l)(l+t)^{k} = S(i+j+t-1, j+t).$$

Proof. Given the integers i, j, we choose an integer n large enough such that $i, j \le n$. In view of decomposition (3.1) in Theorem 3.1, we obtain

$$\left(M^{\langle 1,2,\ldots,t\rangle}(n)\right)_{i,j} = \left(S^{\langle 1,2,\ldots,t\rangle}_{\langle 1,2,\ldots,t\rangle}(n) \cdot U^{[t]}(n)\right)_{i,j}. \tag{4.1}$$

The left-hand side of (4.1) is equal to $S(i+j+t-1, j+t)$. We now compute the right-hand side of (4.1). For simplicity, write $\widehat{S} = S^{\langle 1,2,\ldots,t\rangle}_{\langle 1,2,\ldots,t\rangle}(n)$ and $\widehat{U} = U^{[t]}(n)$. Now, we obtain

$$\begin{aligned}
\left(\widehat{S} \cdot \widehat{U}\right)_{i,j} &= \sum_{l=1}^{n} \widehat{S}_{i,l} \cdot \widehat{U}_{l,j} \\
&= \sum_{l=1}^{\min\{i,j\}} \widehat{S}_{i,l}\widehat{U}_{l,j} \quad \text{(note that } \widehat{S} \text{ and } \widehat{U} \text{ are triangular matrices)} \\
&= \sum_{l=1}^{\min\{i,j\}} \sum_{k=0}^{l-1} \binom{j-1}{k} S(i+t, l+t) S(j-1-k, j-l)(l+t)^k,
\end{aligned}$$

where the last equality is obtained by Lemma 2.4 and noting that $\widehat{S}_{i,l} = S(i+t, l+t)$. The proof is now complete. $\square$

The following identity involving Stirling numbers is also satisfied.

Theorem 4.2. *Let $i, j, t \ge 1$ be integers. Then, we have*

$$\sum_{k=1}^{i} s(i+t, k+t) S(k+j+t-1, j+t) = 0,$$

if $j < i$, and

$$\sum_{k=1}^{i} s(i+t, k+t) S(k+j+t-1, j+t) = \sum_{k=0}^{i-1} \binom{j-1}{k} S(j-1-k, j-i)(i+t)^k,$$

if $j \ge i$. In particular,

$$\sum_{k=1}^{i} s(i+t, k+t) S(k+i+t-1, i+t) = (i+t)^{i-1}.$$

Proof. Fix i, j and choose the integer n such that $i, j \le n$. Since $S^{-1}(n+t) = s(n+t)$, it is easy to see that

$$S^{\langle 1,2,\ldots,t\rangle}_{\langle 1,2,\ldots,t\rangle}(n)^{-1} = s^{\langle 1,2,\ldots,t\rangle}_{\langle 1,2,\ldots,t\rangle}(n).$$

Therefore, multiplying both sides of (3.1) by $s^{\langle 1,2,\ldots,t\rangle}_{\langle 1,2,\ldots,t\rangle}(n)$ from the left, we obtain

$$s^{\langle 1,2,\ldots,t\rangle}_{\langle 1,2,\ldots,t\rangle}(n) \cdot M^{\langle 1,2,\ldots,t\rangle}(n) = U^{[t]}(n),$$

from which it follows that

$$\left(s^{\langle 1,2,\ldots,t\rangle}_{\langle 1,2,\ldots,t\rangle}(n) \cdot M^{\langle 1,2,\ldots,t\rangle}(n)\right)_{i,j} = \left(U^{[t]}(n)\right)_{i,j}. \tag{4.2}$$

For simplicity, write $\widehat{s} = s^{\langle 1,2,\ldots,t\rangle}_{\langle 1,2,\ldots,t\rangle}(n)$ and $\widehat{M} = M^{\langle 1,2,\ldots,t\rangle}(n)$. It is clear that

$$\widehat{s}_{i,j} = \begin{cases} s(i+t, j+t) & \text{if } i \geq j, \\ 0 & \text{otherwise,} \end{cases}$$

and $\widehat{M}_{i,j} = S(i+j+t-1, j+t)$. We argue, on the one hand, that the left-hand side of (4.2) reduces to

$$\begin{aligned} (4.3)\quad \left(\widehat{s}\cdot\widehat{M}\right)_{i,j} &= \sum_{k=1}^{n} \widehat{s}_{i,k}\widehat{M}_{k,j} \\ &= \sum_{k=1}^{i} \widehat{s}_{i,k}\widehat{M}_{k,j} \quad \text{(note that } \widehat{s} \text{ is a lower triangular matrix)} \\ &= \sum_{k=1}^{i} s(i+t, k+t)S(k+j+t-1, j+t). \end{aligned}$$

On the other hand, by Lemma 2.4, the right-hand side of (4.2) is equal to

$$(4.4)\qquad \left(U^{[t]}(n)\right)_{i,j} = \begin{cases} \sum\limits_{k=0}^{i-1} \binom{j-1}{k} S(j-1-k, j-i)(i+t)^k & \text{if } j > i, \\ (i+t)^{i-1} & \text{if } j = i, \\ 0 & \text{if } j < i. \end{cases}$$

The conclusion of the theorem then follows by (4.2), (4.3), and (4.4). □

In [4], we found an explicit formula for $\det M_{\langle 1,2,\ldots,t\rangle}(n)$ (see (1.3)), and in the current article we proved a determinant result for the matrix $M^{\langle 1,2,\ldots,t\rangle}(n)$ (see Theorem 3.2). One may ask if it is possible to find the determinant of the matrices $M^{\langle 1,2,\ldots,t\rangle}_{\langle 1,2,\ldots,t\rangle}(n)$. It seems that it is difficult to present a general determinant formula for these matrices. However, here, using (1.3) and Theorem 3.2, we can show that the determinant of the matrices $M^{\langle 1\rangle}_{\langle 1\rangle}(n)$ satisfy the following recurrence relation with *nonconstant coefficients* of degree 1.

Theorem 4.3. *Let n be a positive integer and let $D(n)$ denote the determinant of the truncated matrix $M^{\langle 1\rangle}_{\langle 1\rangle}(n)$. Then, we have the following recurrence relation of degree 1 for $(D(n))_{n\geq 1}$:*

$$D(n) = (n+1)^n D(n-1) + n! \prod_{i=1}^{n} (i+1)^{i-1}, \quad (n \geq 2),$$

with the initial condition $D(1) = 3$. *Also, we obtain the closed form of* $(D(n))_{n\geq 1}$ *as*

$$D(n) = \prod_{i=1}^{n} (i+1)^i \big(\frac{3}{2} + \sum_{j=3}^{n+1} \frac{1}{j}\big) \quad \textit{for all } n \geq 1.$$

Proof. It is clear that $D(1) = \det M_{\langle 1\rangle}^{\langle 1\rangle}(1) = 3$. Assume then that $n \geq 2$. Applying the Desnanot–Jacobi identity (see, for instance, [2, Theorem 3.12]) to the matrix $M(n+1)$, we obtain

$$\begin{aligned} &\det M(n+1) \det M_{\langle 1\rangle}^{\langle 1\rangle}(n-1) \\ &\qquad\qquad = \det M(n) \det M_{\langle 1\rangle}^{\langle 1\rangle}(n) - \det M^{\langle 1\rangle}(n) \det M_{\langle 1\rangle}(n). \end{aligned}$$

Now, it follows from (1.3) and Theorem 3.2 that

$$\big(\prod_{i=1}^{n+1} i^{i-1}\big) D(n-1) = \big(\prod_{i=1}^{n} i^{i-1}\big) D(n) - \big(\prod_{i=1}^{n} (i+1)^{i-1}\big)\big(\prod_{i=1}^{n} i^i\big).$$

If we divide both sides of the above equation by $\prod_{i=1}^{n} i^{i-1}$, we obtain

$$(n+1)^n D(n-1) = D(n) - n! \prod_{i=1}^{n} (i+1)^{i-1},$$

and the first statement follows. The second statement is immediate from [5, Sec. 3.1]. □

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