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COVERING THE CROSSPOLYTOPE WITH CROSSPOLYTOPES

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ABSTRACT. Let $\gamma_m^d(K)$ be the smallest positive number λ such that the convex body K can be covered by m translates of λK. Let K^d be the d-dimensional crosspolytope. It will be proved that $\gamma_m^d(K^d) = 1$ for $1 \leq m < 2d$, $d \geq 4$; $\gamma_m^d(K^d) = \frac{d-1}{d}$ for $m = 2d, 2d+1, 2d+2$, $d \geq 4$; $\gamma_m^d(K^d) = \frac{d-1}{d}$ for $m = 2d+3$, $d = 4, 5$; $\gamma_m^d(K^d) = \frac{2d-3}{2d-1}$ for $m = 2d+4$, $d = 4$ and $\gamma_m^d(K^d) \leq \frac{2d-3}{2d-1}$ for $m = 2d+4$, $d \geq 5$. Moreover the Hadwiger's covering conjecture is verified for the d-dimensional crosspolytope.

1. INTRODUCTION

Let $\mathbb{R}^d$ be the d-dimensional Euclidean space and let $\mathcal{K}^d$ be the set of all convex bodies in $\mathbb{R}^d$ with nonempty interior. Let p and q be points in $\mathbb{R}^d$. Let $[p, q]$, $|pq|$ and $\overrightarrow{pq}$ denote, respectively, the line segment, the distance and the vector with initial point p and terminal point q. If $K \in \mathcal{K}^d$, then let $c^d(K)$ denote the *covering number* of K, i.e., the smallest number of translates of the interior of K such that their union can cover K. Levi [10] proved in 1955 that

$$c^2(K) = \begin{cases} 4 & \text{if } K \text{ is a parallelogram,} \\ 3 & \text{otherwise.} \end{cases}$$

In 1957 Hadwiger [6] considered this question in any dimensions and posed

Conjecture 1 (Hadwiger's covering problem)**.** *For every $K \in \mathcal{K}^d$ we have $c^d(K) \leq 2^d$, where the equality holds if and only if K is a parallelepiped.*

In the literature, the Boltyanski's illumination problem is a similar problem (see. e.g. [4], [15], [3]). Lassak [8] proved in 1984 the Hadwiger's covering conjecture for the three-dimensional centrally symmetric bodies. Rogers and Zong [12] presented the upper bound $c^d(K) \leq \binom{2d}{d}(d\log d + d\log\log d + 5d)$ for general d-dimensional convex bodies and $c^d(K) \leq 2^d(d\log d$

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$+ d \log \log d + 5d$) for centrally symmetric d-dimensional convex bodies. Using the idea of [1] Huang et. al. [7] presented the upper bound $c^d(K) \leq c_1 4^d e^{-c_2\sqrt{d}}$ for some universal constants $c_1, c_2 > 0$. K. Bezdek [2] proved the Conjecture 1 for convex polyhedron in $\mathbb{R}^3$ having an anaffine symmetry and Dekster [5] verified it for convex bodies in $\mathbb{R}^3$ symmetric about a plane.

For $K \in \mathcal{K}^d$ and any positive integer m, let $\gamma_m^d(K)$ be the smallest positive number λ such that K can be covered by m translations of λK, i.e.,

$$\gamma_m^d(K) = \min\left\{ \lambda \in \mathbb{R}^+ : \exists (u_1, \ldots, u_m) \in \left(\mathbb{R}^d\right)^m, \text{ s.t. } K \subseteq \bigcup_{i=1}^{m} (\lambda K + u_i) \right\}.$$

This is called the *covering functional* of K with respect to m [14] or the *m-covering number* of K [9] [17]. A translate of λK^d is called a *homothetic copy* of K^d. In the following we use the short notation λK^d for a homothetic copy of K^d instead of $\lambda K^d + \mathbf{v}$ where $\mathbf{v}$ is a vector. Observe, $\gamma_m^d(K) = 1$, for all $m \leq d$, and $\gamma_m^d(K)$ is a nonincreasing step sequence for all positive integers m and all convex bodies K. Now $c^d(K) \leq m$ for some $m \in Z^+$ if and only if $\gamma_m^d(K) < 1$ [4]. Estimating covering functionals of convex bodies plays a crucial role in Chuanming Zong's quantitative program for attacking Conjecture 1 (see [17] for more details). Lassak [9] showed that for every two-dimensional convex domain K, $\gamma_4^2(K) \leq \frac{\sqrt{2}}{2}$. Zong [16] proved $\gamma_8^3(C) \leq \frac{2}{3}$ for a bounded three-dimensional convex cone C, and $\gamma_8^3(B_p) \leq \sqrt{\frac{2}{3}}$ for all the unit ball B_p of the three-dimensional l_p spaces. Wu and He [13] estimated the value of $\gamma_m^3(P)$ where P is a convex polytope. Wu et al. [14] determined the value of $\gamma_m^3(K)$ where K is the union of two compact convex sets having no interior points. (See [4] for more information.)

Let K^d be the crosspolytope in the d-dimensional Euclidean space with diameter 2, that is, $K^d = \{(x_1, \ldots, x_d) : |x_1| + \ldots + |x_d| \leq 1\}$. In 2021 Lian and Zhang [11] proved

$$\gamma_m^3(K^3) = \begin{cases} 1 & \text{if} \quad m = 1, \ldots, 5, \\ 2/3 & \text{if} \quad m = 6, \ldots, 9, \\ 3/5 & \text{if} \quad m = 10, \ldots, 13, \\ 4/7 & \text{if} \quad m = 14, \ldots, 17. \end{cases}$$

Let $K \subset \mathbb{R}^d$ be a convex body, and denote by r and s points in K such that $\overrightarrow{rs} \parallel \overrightarrow{pq}$ and $|rs| \geq |r's'|$ where $\{r', s'\} \subset K$ and $r's' \parallel pq$. The K-length of $[p, q]$, or equivalently, the K-distance of p and q is $2|pq|/|rs|$, and it is denoted by $d_K(p, q)$. If K is the Euclidean d-ball, then $d_K(p, q)$ is the Euclidean distance. Let $||p||_s$ be the s-norm of p, i.e. if $p = (p_1, \ldots, p_d)$, then $||p||_s = \sqrt[s]{\sum_{i=1}^{d} |p_i|^s}$ for $s \geq 1$. The 2-norm of $p - q$ is the Euclidean distance of the points p and q. Observe, the $d_{K^d}(p, q)$ is the distance of $p - q$ in 1-norm, i.e. $d_{K^d}(p, q) = ||p - q||_1$.

Remark 1. *Let p and q be different points in $\mathbb{R}^d$. If $[p, q]$ lies in a homothetic copy λK^d, then $\lambda \geq \frac{1}{2} d_{K^d}(p, q) = \frac{1}{2}||p - q||_1$.*

Remark 2. *Let p be a point in $\mathbb{R}^d$. If p lies in a homothetic copy λK^d and c is the centre of this homothetic copy λK^d, then $d_{K^d}(c,p) = ||c-p||_1 \le \lambda$.*

Theorem 1. *If $1 \le m < 2d$, then $\gamma_m^d(K^d) = 1$.*

Proof. Of course, if $\lambda = 1$, then there is a homothetic copy λK^d such that K^d is covered by λK^d. Thus $\gamma_m^d(K^d) \le 1$. Assume $\gamma_m^d(K^d) = \mu < 1$. Since the number of the vertices of K^d is $2d$, then there is a homothetic copy μK^d which contains two vertices of K^d. Let v_1 and v_2 be these two vertices. Observe, the opposite facets of K^d are parallel and any edge of K^d connects two vertices of opposite facets. By Remark 1, $\mu \ge \frac{1}{2}||v_1 - v_2||_1 = \frac{1}{2}d_{K^d}(v_1, v_2) = 1$, a contradiction. □

Lemma 1. *Let F be a facet of K^d. If a vertex v of F is covered by K_1^d a homothetic copy of K^d with ratio λ $(0 < \lambda < 1)$, then $F \cap K_1^d$ is contained in the homothetic image of F with ratio λ and centre v.*

Proof. Let α be the hyperplane $x_1 + \ldots + x_d = 1$. Without loss of generality it may be assumed that $v = (0, \ldots, 0, 1)$ and F is the facet on the hyperplane α. Let $\mathcal{A}$ be the affine transformation such that $K_1^d = \mathcal{A}(K^d)$. It may be assumed that $\mathcal{A}(o)$ lies in K^d. Let $w = (w_1, \ldots, w_d) = \mathcal{A}(v)$. Let us assume that $o \notin K_1^d$ (the opposite case is similar). Let F_1 be the facet of K^d on the hyperplane $x_1 + \ldots + x_{d-1} - x_d = 1$. Let $p = (p_1, \ldots, p_d)$ be a point of $F \cap \mathcal{A}(F_1)$. First we will see that $p_d \ge 1 - \lambda$. Observe, the equation of the hyperplane containing $\mathcal{A}(F_1)$ is

$$x_1 - w_1 + x_2 - w_2 + \ldots + x_{d-1} - w_{d-1} - x_d + w_d - \lambda = \lambda.$$

Since p lies on F and $\mathcal{A}(F_1)$, we have

$$p_1 + \ldots + p_d = 1 \tag{1.1}$$

and

$$p_1 + p_2 + \ldots + p_{d-1} - p_d = w_1 + \ldots + w_{d-1} - w_d + 2\lambda. \tag{1.2}$$

From (1.1)–(1.2) we have

$$p_d = \frac{1}{2} + \frac{1}{2}(-w_1 - \ldots - w_{d-1} + w_d) - \lambda \tag{1.3}$$

Observe, the point w lies in the translated image of K^d by the vector $[0, \ldots, 0, 2]^T$. Since v is covered by K_1^d, the point w lies in the halfspace bounded by the hyperplane $-x_1 - \ldots - x_{d-1} + x_d = 1$ and does not contain the origin. Thus

$$-w_1 - \ldots - w_{d-1} + w_d \ge 1.$$

Substituting this into (1.3), we have

$$p_d \ge \frac{1}{2} + \frac{1}{2} - \lambda = 1 - \lambda$$

Observe K_1^d lies in the halfspace bounded by the hyperplane $\mathcal{A}(F_1)$ and containing the vertex v. Moreover p is an arbitrary point of the intersection of this halfspace and F. This means any point from the intersection has the property that the last coordinate of the point is not less than $1-\lambda$, which completes the proof of the lemma. □

Theorem 2. *If $m = 2d$, then $\gamma_m^d(K^d) = \frac{d-1}{d}$.*

Proof. Let $\Gamma = \{\frac{d-1}{d}K^d + (\pm\frac{1}{d}, 0\ldots, 0), \ldots, \frac{d-1}{d}K^d + (0\ldots, 0, \pm\frac{1}{d})\}$. Since the homothetic copies in Γ cover the surface of K^d and the origin lies in each element in Γ, K^d is covered by Γ. Thus $\gamma_m^d(K^d) \le \frac{d-1}{d}$.
Assume $\gamma_m^d(K^d) = \mu < \frac{d-1}{d}$. Since $\mu < \frac{d-1}{d} < 1$, there is no homothetic copy containing two vertices of K^d. Let F be a facet of K^d lying on the hyperplane $x_1 + \ldots + x_d = 1$ and let c be the centre of the $(d-1)$-simplex F. Observe $c = (1/d, \ldots, 1/d)$. By Lemma 1, a homothetic copy μK^d does not contain both c and a vertex of F. Since $\frac{d-1}{d} < 1$, a homothetic copy μK^d does not contain both c and a vertex of K^d on the plane $x_1 + \ldots + x_d = -1$, a contradiction □

Observe, the proof of Theorem 2 comes from Remark 1. Indeed, the 1-norm of $||c - v||_1 \ge 2\frac{d-1}{d}$ where v is a vertex of K_1^d.

Corollary 1. *We have $c^d(K^d) \le 2d$ for $d \ge 4$. The Hadwiger's covering problem is solved for the d-dimensional crosspolytope for $d \ge 4$.*

Theorem 3. *If $m = 2d+1, 2d+2$, then $\gamma_m^d(K^d) = \frac{d-1}{d}$ for $d \ge 4$.*

Proof. Since

$$\gamma_{2d+2}^d(K^d) \le \gamma_{2d+1}^d(K^d) \le \gamma_{2d}^d(K^d) = \frac{d-1}{d},$$

from $\gamma_{2d+2}^d(K^d) = \frac{d-1}{d}$ it comes the statement of the theorem. Let it be assumed that $\gamma_{2d+2}^d(K^d) = \mu < \frac{d-1}{d}$. Let $c_1, \ldots, c_{2^d}$ be the centres of the facets of K^d and let C be the d-cube with vertices $c_1, \ldots, c_{2^d}$. By Lemma 1, $2d$ homothetic copies of K^d cover the $2d$ vertices of K^d and the points $c_1, \ldots, c_{2^d}$ are uncovered by these homothetic copies. Let S be the set containing the vertices of K^d and the centres $(1/d, \ldots, 1/d)$ and $(-1/d, \ldots, -1/d)$. Let x_1 and x_2 be two different points in S. By Remark 1, a homothetic copy of K^d with ratio μ does not contain both x_1 and x_2, a contradiction. □

Lemma 2. *Let $c_1, \ldots, c_{2^d}$ be the centres of the facets of K^d. If $\lambda < 1$ and the facets containing c_i and c_j are parallel facets, then a homothetic copy λK^d does not contain both c_i and c_j.*

Proof. Without loss of generality it may be assumed that the two hyperplanes are $\alpha_1 : x_1 + \ldots + x_d = 1$ and $\alpha_2 : x_1 + \ldots + x_d = -1$. In this case the centres are $c_i = (-1/d, \ldots, -1/d)$ and $c_j = (1/d, \ldots, 1/d)$. Now $||c_i - c_j||_1 = d_{K^d}(c_i, c_j) = 2$. By Remark 1, a homothetic copy λK^d does not contain both c_i and c_i. □

Lemma 3. *Let $c_1, \dots, c_{2^d}$ be the centres of the facets of K^d. If $\lambda < \frac{d-1}{d}$ and the facets containing c_i and c_j have only a vertex in common, then a homothetic copy λK^d does not contain both c_i and c_j.*

Proof. Without loss of generality it may be assumed that the two centres are $c_i = (1/d, -1/d, \dots, -1/d)$ and $c_j = (1/d, \dots, 1/d)$. Now the common vertex of the facets of c_i and c_j is $(1, 0, \dots, 0)$. Thus $||c_i - c_j||_1 = d_{K^d}(c_i, c_j) = 2\frac{d-1}{d}$ and by Remark 1, a homothetic copy λK^d does not contain both c_i and c_i. □

Theorem 4. *If $m = 2d + 3$, then $\gamma^d_m(K^d) = \frac{d-1}{d}$ for $d = 4, 5$.*

Proof. By Theorem 3, $\gamma^d_{2d+3}(K^d) \le \frac{d-1}{d}$. Assume that $\gamma^d_{2d+3}(K^d) = \mu < \frac{d-1}{d}$. Let $c_1, \dots, c_{2^d}$ be the centres of the facets of K^d and let C be the d-cube with vertices $c_1, \dots, c_{2^d}$. By Lemma 1, the $2d$ homothetic copies of K^d cover the $2d$ vertices of K^d and the points $c_1, \dots, c_{2^d}$ are uncovered by these homothetic copies. It will be proved that the vertices of the d-cube C are not covered by 3 homothetic copies of K^d with ratio μ. Let us assume, that the vertices of the d-cube C are covered by 3 homothetic copies of K^d with ratio μ. Observe, c_i and c_j lie on two parallel facets of K^d if and only it c_i and c_j are the endpoints of a diagonal of C. Thus if c_i and c_j are the endpoints of a diagonal of the d-cube C, then by Lemma 2, a homothetic copy μK^d does not contain both c_i and c_j. Observe, c_i and c_j lie on two facets and the two facets have only one vertex in common if and only if c_i and c_j are the endpoints of a diagonal of a facet of C. Thus if c_i and c_j are the endpoints of a diagonal of a facet of C, then by Lemma 3, a homothetic copy μK^d does not contain both c_i and c_j. We distinguish 2 cases.

Case 1: $d = 4$.
By the Pigeonhole principle there is a homothetic copy - say H_1 - that H_1 contains 6 vertices of the 4-cube C. Without loss of generality it may be assumed that $c_1 = (1/4, 1/4, 1/4, 1/4)$ lies in H_1. By Lemma 2, the vertex $(-1/4, -1/4, -1/4, -1/4)$ does not lie in H_1. By Lemma 3, the vertices $(1/4, -1/4, -1/4, -1/4)$, $(-1/4, 1/4, -1/4, -1/4)$, $(-1/4, -1/4, 1/4, -1/4)$ and $(-1/4, -1/4, -1/4, 1/4)$ do not lie in H_1. Let $c_2 = (-1/4, 1/4, 1/4, 1/4)$, $c_3 = (1/4, -1/4, 1/4, 1/4)$, $c_4 = (1/4, 1/4, -1/4, 1/4)$, $c_5 = (1/4, 1/4, 1/4, -1/4)$, $c_6 = (-1/4, -1/4, /4, 1/4)$, $c_7 = (-1/4, 1/4, -1/4, 1/4)$, $c_8 = (-1/4, 1/4, 1/4, -1/4)$, $c_9 = (1/4, -1/4, -1/4, /4)$, $c_{10} = (1/4, -1/4, 1/4, -1/4)$, $c_{11} = (1/4, 1/4, -1/4, -1/4)$ and $S^4 = \{c_2, c_3, c_4, c_5\}$. We distinguish 5 subcases.

Subcase 1.1: The homothetic copy H_1 contains the vertices c_2, c_3, c_4 and c_5.

By Lemma 3, H_1 does not contain the vertices $c_6, \dots, c_{11}$. Thus H_1 does not contain 6 vertices of C, a contradiction.

Subcase 1.2: The homothetic copy H_1 contains exactly three vertices from S^4.

Without loss of generality it may be assumed that H_1 contains c_2, c_3 and c_4. By Lemma 3, H_1 does not contain the vertices $c_6, \ldots, c_{11}$. Thus H_1 does not contain 6 vertices of C, a contradiction.

SUBCASE 1.3: The homothetic copy H_1 contains exactly two vertices from S^4.

Without loss of generality it may be assumed that H_1 contains c_2 and c_3. By Lemma 3, H_1 does not contain the vertices $c_7, \ldots, c_{11}$. Thus H_1 does not contain 6 vertices of C, a contradiction.

SUBCASE 1.4: The homothetic copy H_1 contains exactly one vertex from S^4.

Without loss of generality it may be assumed that H_1 contains c_2. By Lemma 3, H_1 does not contain the vertices $c_9, \ldots, c_{11}$. Thus H_1 does not contain 6 vertices of C, a contradiction.

SUBCASE 1.5: The homothetic copy H_1 does not contain any vertex from S^4.

By Lemma 2, H_1 does not contain both c_6 and c_{11}. By Lemma 2, H_1 does not contain both c_7 and c_{10}. Thus H_1 does not contain 6 vertices of C, a contradiction.

CASE 2: $d = 5$.

By the Pigeonhole principle there is a homothetic copy - say H_1 - that H_1 contains 11 vertices of the 5-cube C. Without loss of generality it may be assumed that $c_1 = (1/5, 1/5, 1/5, 1/5, 1/5)$ lies in H_1. By Lemma 2, the vertex $(-1/5, -1/5, -1/5, -1/5, -1/5)$ does not lie in H_1. By Lemma 3, the vertices $(1/5, -1/5, -1/5, -1/5, -1/5)$, $(-1/5, 1/5, -1/5, -1/5, -1/5)$, $(-1/5, -1/5, 1/5, -1/5, -1/5)$, $(-1/5, -1/5, -1/5, 1/5, -1/5)$ and $(-1/5, -1/5, -1/5, -1/5, 1/5)$ do not lie in H_1. Let $c_2 = (-1/5, 1/5, 1/5, 1/5, 1/5)$, $c_3 = (1/5, -1/5, 1/5, 1/5, 1/5)$, $c_4 = (1/5, 1/5, -1/5, 1/5, 1/5)$, $c_5 = (1/5, 1/5, 1/5, -1/5, 1/5)$, $c_6 = (1/5, 1/5, 1/5, 1/5, -1/5)$, $c_7 = (-1/5, -1/5, 1/5, 1/5, 1/5)$, $c_8 = (-1/5, 1/5, -1/5, 1/5, 1/5)$, $c_9 = (-1/5, 1/5, 1/5, -1/5, 1/5)$, $c_{10} = (-1/5, 1/5, 1/5, 1/5, -1/5)$, $c_{11} = (1/5, -1/5, -1/5, 1/5, 1/5)$, $c_{12} = (1/5, -1/5, 1/5, -1/5, 1/5)$, $c_{13} = (1/5, -1/5, 1/5, 1/5, -1/5)$, $c_{14} = (1/5, 1/5, -1/5, -1/5, 1/5)$, $c_{15} = (1/5, 1/5, -1/5, 1/5, -1/5)$, $c_{16} = (1/5, 1/5, 1/5, -1/5, -1/5)$, $c_{17} = (-1/5, -1/5, -1/5, 1/5, 1/5)$, $c_{18} = (-1/5, -1/5, 1/5, -1/5, 1/5)$, $c_{19} = (-1/5, -1/5, 1/5, 1/5, -1/5)$, $c_{20} = (-1/5, 1/5, -1/5, -1/5, 1/5)$, $c_{21} = (-1/5, 1/5, -1/5, 1/5, -1/5)$, $c_{22} = (-1/5, 1/5, 1/5, -1/5, -1/5)$, $c_{23} = (1/5, -1/5, -1/5, -1/5, 1/5)$, $c_{24} = (1/5, -1/5, -1/5, 1/5, -1/5)$, $c_{25} = (1/5, -1/5, 1/5, -1/5, -1/5)$, $c_{26} = (1/5, 1/5, -1/5, -1/5, -1/5)$ and $S^5 = \{c_2, c_3, c_4, c_5, c_6\}$. We distinguish 6 subcases.

SUBCASE 2.1: The homothetic copy H_1 contains the vertices c_2, c_3, c_4, c_5 and c_6.

By Lemma 3, H_1 does not contain $c_{17}, \ldots, c_{26}$. Let it be assumed that H_1 contains c_7. (The cases H_1 contains $c_8, \ldots, c_{15}$, or c_{16} are similar.) By Lemma 3, H_1 does not contain c_{14}, c_{15} or c_{16}. By Lemma 3, H_1 does not contain both c_8 and c_{12}. By Lemma 3, H_1 does not contain both c_9 and

c_{13}. By Lemma 3, H_1 does not contain both c_{10} and c_{11}. Thus H_1 does not contain 11 vertices of C, a contradiction.

Subcase 2.2: The homothetic copy H_1 contains exactly four vertices from S^5.

Without loss of generality it may be assumed that H_1 contains c_2, c_3, c_4 and c_5. By Lemma 3, H_1 does not contain the vertices $c_{17}, \dots, c_{26}$.
Let us assume that H_1 contains c_7. (The cases that H_1 contains c_8, c_9, c_{11}, c_{12} or c_{14} are similar.) By Lemma 3, H_1 does not contain c_{14}, c_{15} or c_{16}. By Lemma 3, H_1 does not contain both c_8 and c_{12}. By Lemma 3, H_1 does not contain both c_9 and c_{11}. Thus H_1 does not contain 11 vertices of C, a contradiction.
Let us assume that H_1 contains c_{10} (and does not contain the vertices $c_7, c_8, c_9, c_{11}, c_{12}$ or c_{14}). (The cases that H_1 contains c_{13}, c_{15} or c_{16} are similar.) Now, H_1 contains at most the vertices $c_1, \dots, c_5$, c_{10}, c_{13}, c_{15}, c_{16}, thus H_1 does not contain 11 vertices of C, a contradiction.

Subcase 2.3: The homothetic copy H_1 contains exactly three vertices from S^5.

Without loss of generality it may be assumed that H_1 contains c_2, c_3 and c_4. By Lemma 3, H_1 does not contain the vertices $c_{18}, \dots, c_{26}$.
Let us assume that H_1 contains c_7. (The cases H_1 contains c_8 or c_{11} are similar.) By Lemma 3, H_1 does not contain c_{14}, c_{15} or c_{16}. By Lemma 3, H_1 does not contain both c_8 and c_{12}. By Lemma 3, H_1 does not contain both c_9 and c_{13}. By Lemma 3, H_1 does not contain both c_{10} and c_{11}. Thus H_1 does not contain 11 vertices of C, a contradiction.
Let us assume that H_1 contains c_9 (and does not contain the vertices c_7, c_8 or c_{11}). (The cases H_1 contains c_{10}, c_{12}, c_{13}, c_{14} or c_{15} are similar.) By Lemma 3, H_1 does not contain c_{11}, c_{13} or c_{15}. Since H_1 contains at most the vertices $c_1, \dots, c_4$, c_9, c_{10}, c_{12}, c_{14}, c_{16}, c_{17}, H_1 does not contain 11 vertices of C, a contradiction.
Let us assume that H_1 contains c_{16} (and does not contain the vertices $c_7, \dots, c_{14}$ or c_{15}). Since H_1 contains at most the vertices $c_1, \dots, c_4$, c_{16}, c_{17}, H_1 does not contain 11 vertices of C, a contradiction.
Let us assume that H_1 contains c_{17} (and does not contain the vertices $c_7, \dots, c_{15}$ or c_{16}). Since H_1 contains at most the vertices $c_1, \dots, c_4$, c_{17}, H_1 does not contain 11 vertices of C, a contradiction.

Subcase 2.4: The homothetic copy H_1 contains exactly two vertices from S^5.

Without loss of generality it may be assumed that H_1 contains c_2 and c_3. By Lemma 3, H_1 does not contain the vertices $c_{20}, \dots, c_{26}$.
Let us assume that H_1 contains c_7. By Lemma 3, H_1 does not contain c_{14}, c_{15} or c_{16}. By Lemma 3, H_1 does not contain both c_8 and c_{12}. By Lemma 3, H_1 does not contain both c_9 and c_{13}. By Lemma 3, H_1 does not contain both c_{10} and c_{11}. Thus H_1 does not contain 11 vertices of C, a contradiction.

Let us assume that H_1 contains c_8 (and does not contain the vertex c_7). (The cases H_1 contains c_9, c_{10}, c_{11}, c_{12} or c_{13} are similar.) By Lemma 3, H_1 does not contain c_{12}, c_{13} or c_{16}. By Lemma 3, H_1 does not contain both c_9 and c_{11}. By Lemma 3, H_1 does not contain both c_{10} and c_{14}. Thus H_1 does not contain 11 vertices of C, a contradiction.
Let us assume that H_1 contains c_{14} (and does not contain the vertices $c_7, \ldots, c_{12}$ or c_{13}). (The cases H_1 contains c_{15} or c_{16} are similar.) Since H_1 contains at most the vertices c_1, c_2, c_3, $c_{14}, \ldots, c_{19}$, H_1 does not contain 11 vertices of C, a contradiction.
Let us assume that H_1 contains c_{17} (and does not contain the vertices $c_7, \ldots, c_{15}$ or c_{16}). (The cases H_1 contains c_{18} or c_{19} are similar.) Since H_1 contains at most the vertices c_1, c_2, c_3, c_{17}, c_{18}, c_{19}, H_1 does not contain 11 vertices of C, a contradiction.

Subcase 2.5: The homothetic copy H_1 contains exactly one vertex from S^5.

Without loss of generality it may be assumed that H_1 contains c_2. By Lemma 3, H_1 does not contain the vertices $c_{23}, \ldots, c_{26}$.
Let us assume that H_1 contains c_7. (The cases H_1 contains c_8, c_9 or c_{10} are similar.) By Lemma 3, H_1 does not contain c_{14}, c_{15} or c_{16}. By Lemma 3, H_1 does not contain both c_8 and c_{12}. By Lemma 3, H_1 does not contain both c_9 and c_{13}. By Lemma 3, H_1 does not contain both c_{10} and c_{11}. By Lemma 3, H_1 does not contain both c_{17} and c_{22}. By Lemma 3, H_1 does not contain both c_{18} and c_{21}. Thus H_1 does not contain 11 vertices of C, a contradiction.
Let us assume that H_1 contains c_{11} (and does not contain the vertices $c_7, \ldots, c_9$ or c_{10}). (The cases H_1 contains c_{12}, c_{13}, c_{14}, c_{15}, c_{16} are similar.) By Lemma 3, H_1 does not contain c_{16}. By Lemma 2, H_1 does not contain c_{22}. By Lemma 2, H_1 does not contain both c_{12} and c_{21}. By Lemma 2, H_1 does not contain c_{13} and c_{20}. Thus H_1 does not contain 11 vertices of C, a contradiction.
Let us assume that H_1 contains c_{17} (and does not contain the vertices $c_7, \ldots, c_{15}$ or c_{16}). (The cases H_1 contains $c_{18}, \ldots, c_{22}$ are similar.) Since H_1 contains at most the vertices c_1, c_2, $c_{17}, \ldots, c_{22}$, H_1 does not contain 11 vertices of C, a contradiction.

Subcase 2.6: The homothetic copy H_1 does not contain any vertex from S^5.

Let us assume that H_1 contains c_7. (The cases H_1 contains $c_8, \ldots, c_{16}$ are similar.) By Lemma 3, H_1 does not contain c_{14}, c_{15} or c_{16}. By Lemma 2, H_1 does not contain c_{26}. By Lemma 3, H_1 does not contain both c_8 and c_{12}. By Lemma 3, H_1 does not contain both c_9 and c_{13}. By Lemma 3, H_1 does not contain both c_{10} and c_{11}. By Lemma 3, H_1 does not contain both c_{17} and c_{25}. By Lemma 3, H_1 does not contain both c_{18} and c_{24}. By Lemma 3, H_1 does not contain both c_{19} and c_{20}. By Lemma 3, H_1 does not contain both c_{21} and c_{23}. Thus H_1 does not contain 11 vertices of C, a contradiction.

Let us assume that H_1 contains c_{17} (and does not contain the vertices $c_7, \ldots, c_{15}$ or c_{16}). (The cases H_1 contains $c_{18}, \ldots, c_{26}$ are similar.) By Lemma 3, H_1 does not contain c_{22}, c_{25} or c_{26}. Since H_1 contain at most the vertices c_1, c_2, c_{17}, c_{18}, c_{19}, c_{20}, c_{21}, c_{23}, c_{24}, H_1 does not contain 11 vertices of C, a contradiction. $\square$

Since $\gamma^d_{2d+3}(K^d) \leq \gamma^d_{2d+2}(K^d)$ and $\gamma^d_{2d+2}(K^d) = \frac{d-1}{d}$ for $d \geq 6$, $\gamma^d_{2d+2}(K^d) \leq \frac{d-1}{d}$ for $d \geq 6$.

Conjecture 2. *If $m = 2d+3$, then $\gamma^d_m(K^d) = \frac{d-1}{d}$ for $d \geq 6$.*

Theorem 5. *If $m = 2d+4$, then $\gamma^d_m(K^d) = \frac{2d-3}{2d-1}$ for $d = 4$ and $\gamma^d_m(K^d) \leq \frac{2d-3}{2d-1}$ for $d \geq 5$.*

Proof. First it will be proved that $\gamma^d_m(K^d) \leq \frac{2d-3}{2d-1}$ for $d \geq 4$. Let $\lambda = \frac{2d-3}{2d-1}$ and consider the following homothetic copies of K^d. $K^d_1 = \lambda K^d + \left[\frac{2}{2d-1}, 0, 0, \ldots, 0\right]^T$, $K^d_2 = \lambda K^d + \left[0, \frac{2}{2d-1}, 0, \ldots, 0\right]^T$, …, $K^d_d = \lambda K^d + \left[0, \ldots, 0, \frac{2}{2d-1}\right]^T$, $K^d_{d+1} = \lambda K^d + \left[-\frac{2}{2d-1}, 0, 0, \ldots, 0\right]^T$, …, $K^d_{2d} = \lambda K^d + \left[0, \ldots, 0, -\frac{2}{2d-1}\right]^T$, $K^d_{2d+1} = \lambda K^d + \left[\frac{1.5}{2d-1}, \frac{1.5}{2d-1}, 0, 0 \ldots, 0\right]^T$, $K^d_{2d+2} = \lambda K^d + \left[-\frac{1.5}{2d-1}, \frac{1.5}{2d-1}, 0, 0 \ldots, 0\right]^T$, $K^d_{2d+3} = \lambda K^d + \left[-\frac{1.5}{2d-1}, -\frac{1.5}{2d-1}, 0, 0 \ldots, 0\right]^T$, $K^d_{2d+4} = \lambda K^d + \left[\frac{1.5}{2d-1}, -\frac{1.5}{2d-1}, 0, 0 \ldots, 0\right]^T$. Now it will be proved that the surface of K^d is covered by the homothetic copies $K^d_1, \ldots, K^d_{2d+4}$. Consider a facet F of K^d. Without loss of generality it can be assumed that F lies on the hyperplane $x_1 + \ldots + x_d = 1$. The homothetic image of F with ratio λ and centre $(1, 0, 0 \ldots, 0)$ $((0, 1, 0 \ldots, 0), \ldots, (0, \ldots, 0, 1)$, resp.) is covered by K^d_1 ($K^d_2, \ldots, K^d_d$, resp.). Let $n_1 = \left(\frac{1}{2d-1}, \frac{2}{2d-1}, \frac{2}{2d-1}, \ldots, \frac{2}{2d-1}\right)$, $n_2 = \left(\frac{2}{2d-1}, \frac{1}{2d-1}, \frac{2}{2d-1}, \ldots, \frac{2}{2d-1}\right), \ldots,$ $n_d = \left(\frac{2}{2d-1}, \ldots, \frac{2}{2d-1}, \frac{1}{2d-1}\right)$. The convex hull of the points $n_1, \ldots, n_d$ is uncovered by $K^d_1, \ldots, K^d_d$. Since

$$\left\|\left(\frac{1.5}{2d-1}, \frac{1.5}{2d-1}, 0, 0 \ldots, 0\right) - n_i\right\|_1$$

$$\leq \max\left(2\frac{0.5}{2d-1} + (d-2)\frac{2}{2d-1}, 2\frac{0.5}{2d-1} + (d-3)\frac{2}{2d-1} + \frac{1}{2d-1}\right)$$

$$= \frac{2d-3}{2d-1} = \lambda$$

for $i = 1, \ldots, d$, the points $n_1, \ldots, n_d$ are covered by K^d_{2d+1}. Since K^d_{2d+1} and the convex hull of the points $n_1, \ldots, n_d$ are convex bodies, K^d_{2d+1} covers the convex hull of the points $n_1, \ldots, n_d$. Thus the facet F is covered by $K^d_1, \ldots, K^d_d$, K^d_{2d+1}. Similarly any other facet of K^d is covered by

$K_1^d, \ldots, K_{2d+4}^d$. Since the origin lies in each homothetic copy $K_1^d, \ldots, K_{2d+4}^d$, K^d is covered by $K_1^d, \ldots, K_{2d+4}^d$ and $\gamma_{2d+4}^d(K^d) \leq \frac{2d-3}{2d-1}$ for $d \geq 4$.

Now consider the case $d = 4$. Let it be assumed that $\gamma_{12}^4(K^d) = \mu < \frac{5}{7}$. Let $n_{1,1} = (1/7, 2/7, 2/7, 2/7)$, $n_{1,2} = (2/7, 1/7, 2/7, 2/7)$, $n_{1,3} = (2/7, 2/7, 1/7, 2/7)$, $n_{1,4} = (2/7, 2/7, 2/7, 1/7)$, $n_{2,1} = (-1/7, 2/7, 2/7, 2/7)$, ..., $n_{10,3} = (2/7, -2/7, 1/7, -2/7)$, ..., $n_{12,3} = (-2/7, -2/7, -1/7, 2/7)$, ..., $n_{14,1} = (-1/7, 2/7, -2/7, -2/7)$ ..., $n_{16,4} = (-2/7, -2/7, -2/7, -1/7)$. By Lemma 1, the 8 homothetic copies of K^4 with ratio μ cover the 8 vertices of K^4 and the points $n_{1,1}, \ldots, n_{16,4}$ are uncovered by these homothetic copies. It will be proved that four homothetic copies of K^4 with ratio μ does not cover the points $n_{1,1}, \ldots, n_{16,4}$. Since $||n_{1,1} - n_{10,3}||_1 = ||n_{1,1} - n_{12,3}||_1 = ||n_{1,1} - n_{14,1}||_1 = ||n_{10,3} - n_{12,3}||_1 = ||n_{10,3} - n_{14,1}||_1 = ||n_{12,3} - n_{14,1}||_1 = \frac{10}{7} = 2\lambda$, the points $n_{1,1}$, $n_{10,3}$, $n_{12,3}$ and $n_{14,1}$ are not covered by four homothetic copies of K^4 with ratio μ, a contradiction. $\square$

Conjecture 3. *If $m = 2d + 4$, then $\gamma_m^d(K^d) = \frac{2d-3}{2d-1}$ for $d \geq 5$.*

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