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COMBINATORICS OF CERTAIN CLASSES OF PLANE PARTITIONS

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Abstract. In this paper, we study new restricted plane partitions and connect them with associated lattice paths by using n-color partitions. We also obtain generating functions and recurrence relations for certain classes of plane partitions.

1. Introduction

The theory of partitions can be seen as one of the most beautiful branches of combinatorics. Euler was the first person to make a real development in the area of partitions by giving many important properties of the partition function in his book "Introduction in Analysin Infinitorium". Since partitions are sequences of positive integers, one can see them as "one dimensional" objects. Plane partitions are defined by MacMahon [17] as a natural generalization of partitions to two dimensions. Presently, plane partitions are studied in connection with many diverse areas of mathematics. Interested readers are referred to [15, 16] for an extensive and detailed study of this field. First, we recall the following definitions:

Definition 1.1 (Euler [14]). *A partition of a positive integer n is a finite nonincreasing sequence of positive integers whose sum is n.*

Example 1.2. *The partitions of* 3 *are* $3,\ 2+1,\ 1+1+1$.

Definition 1.3 (MacMahon [17]). *A plane partition of a positive integer ν is an array of nonnegative integers*

$$\begin{matrix} m_{11} & m_{12} & m_{13} & \dots \\ m_{21} & m_{22} & m_{23} & \dots \\ \vdots & \vdots & \vdots & \end{matrix}$$

for which $\sum_{i,j} m_{ij} = \nu$ and rows and columns are in nonincreasing order.

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Example 1.4. *There are six plane partitions of* 3*, viz.,*

$$3,\ 2\quad 1,\ \begin{matrix}2\\1\end{matrix},\ 1\quad 1\quad 1,\ \begin{matrix}1&1\\1&\end{matrix},\ \begin{matrix}1\\1.\\1\end{matrix}$$

Remark: The entries m_{ij} are called the parts of a plane partition. A plane partition is called symmetric if $m_{ij} = m_{ji}$ for all i and j.

MacMahon in his book [18] showed that if $PP(\nu)$ denotes the number of plane partitions of a positive integer ν, then

$$1 + \sum_{\nu=1}^{\infty} PP(\nu)q^{\nu} = \prod_{n=1}^{\infty}(1-q^n)^{-n}. \tag{1.1}$$

Partitions with "n copies of n" referred to as n-color partitions were initially defined by Agarwal and Andrews [9] for the purpose of interpreting several q-series combinatorially. However, these partitions had been used indirectly in many studies of plane partitions before Andrews and Agarwal began studying them (see, for instance, Chaundy [19], Cheema and Gordon [20], and Sagan [21]). Now a full-fledged theory (almost parallel to the theory of the classical partitions) is being developed for them. For example, analogous to MacMahon's combinatorial interpretations of the Rogers–Ramanujan identities given in [18], several q-series identities have been interpreted combinatorially using n-color partitions in [1, 2, 3, 6, 7, 10, 12]. Conjugate and self-conjugate n-color partitions have been studied in [11], n-color perfect partitions were introduced in [13].

Definition 1.6 (Agarwal and Andrews [9])**.** *An n-color partition (or, a partition with "n copies of n") is a partition in which a part of size n, $n \geq 1$, can come in n different colors denoted by subscripts $n_1, n_2, \ldots, n_n$ and the parts satisfy the order*

$$1_1 < 2_1 < 2_2 < 3_1 < 3_2 < 3_3 < 4_1 < 4_2 < 4_3 < 4_4 < 5_1 < 5_2 < \ldots.$$

Example 1.7. *There are* 13 *n-color partitions of* 4*, viz.,*

$$\begin{aligned}&4_1, 4_2, 4_3, 4_4,\\&3_1+1_1, 3_2+1_1, 3_3+1_1,\\&2_1+2_1, 2_1+2_2, 2_2+2_2,\\&2_1+1_1+1_1, 2_2+1_1+1_1, 1_1+1_1+1_1+1_1.\end{aligned}$$

Definition 1.8 (Agarwal and Balasubramanian [11])**.** *Let $\pi = (a_1)_{b_1} + (a_2)_{b_2} + \cdots + (a_r)_{b_r}$ be an n-color partition. The n-color partition obtained by replacing each part $(a_i)_{b_i}$ of π by its conjugate $(a_i)_{a_i-b_i+1}$, is known as the conjugate of π and is denoted by π^c. An n-color partition is said to be self-conjugate if it is identical with its conjugate.*

Example 1.9. $2_1 + 2_2 + 1_1$ *is a self-conjugate n-color partition of* 5*.*

Agarwal and Bressoud have introduced lattice paths in [10] to give a combinatorial interpretation of multiple basic hypergeometric series. In [22], Anand and Agarwal defined a new class of lattice paths which they called associated lattice paths. The authors in [22] gave a graphical representation of n-color partitions in terms of associated lattice paths. First we recall the description of lattice paths as follows:

All paths are of finite length and lie in the first quadrant. They will begin on the y-axis or x-axis and terminate on the x-axis. Only three moves are allowed at each step.

Northeast ($\nearrow$): from (i, j) to $(i + 1, j + 1)$,
Southeast ($\searrow$): from (i, j) to $(i + 1, j - 1)$, only allowed if $j > 0$,
Horizontal ($\rightarrow$): from $(i, 0)$ to $(i + 1, 0)$, only allowed along x-axis.

Associated lattice paths are those paths of finite length in the first quadrant which begin on the x-axis and terminate on the x-axis. The northeast and southeast steps are the same as in the case of lattice paths. However, the horizontal step is a step from (i, j) to $(i + 1, j)$, only allowed when the preceding step is a northeast step and the following step is a southeast step. Next, we will be using the following terminology to describe associated lattice paths:

(1) *Truncated Isosceles Trapezoidal Section (TITS)*: A section of path which starts on the x-axis with northeast steps followed by horizontal steps and then followed by southeast steps ending on the x-axis forms what we call a Truncated Isosceles Trapezoidal Section. Since the lower base of the trapezoid lies on the x-axis and is not a part of the path, the term truncated is used.
(2) *Weight of a TITS*: To define this, we shall represent every TITS by an ordered pair $\{a, b\}$ where a denotes its altitude and b the length of the upper base. The weight of the TITS with ordered pair $\{a, b\}$ is a units.
(3) *Weight of an associated lattice path*: It is the sum of weights of its TITSs.

Example 1.10. *The following associated lattice path has three TITS with* $\{1, 2\}, \{2, 4\}$ *and* $\{1, 1\}$ *as corresponding ordered pairs.*

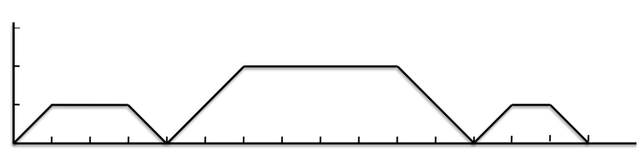


FIGURE 1. An associated lattice path having weight 4

The following Theorem from [22] gives a one-to-one correspondence between a class of associated lattice paths of weight ν and n-color partitions of a positive integer ν:

Theorem 1.11. *Let $\chi(\nu)$ denote the number of associated lattice paths of weight ν such that for any TITS with ordered pair $\{a, b\}$, b does not exceed a and TITSs are arranged in order of nondecreasing altitudes and the TITSs with the same altitude are ordered by the length of their upper base. Let $P(\nu)$ denote the number of n-color partitions of ν. Then*

$$\chi(\nu) = P(\nu), \quad \textit{for all } \nu \geq 0.$$

Using generating functions, Agarwal and Andrews [9] proved that the number of n-color partitions of ν equals the number of plane partitions of ν. Agarwal [4] established a bijection between the set of plane partitions of ν and the set of n-color partitions of ν via infinite matrices defined by Bender and Knuth in [23] and referred to as Bender–Knuth matrices. However, a new bijective correspondence between n-color partitions and plane partitions is constructed in [24] that exhibits very beautiful properties. First we give an outline of the procedure used in [24] to obtain a plane partition from a given n-color partition.

Consider an n-color partition $\pi = (m_1)_{n_1} + (m_2)_{n_2} + \cdots + (m_k)_{n_k}$. We associate each part $(m_i)_{n_i}$ of π to a point $(n_i, m_i - n_i + 1)$, $1 \leq i \leq k$. In this way, the partition π corresponds to a multiset $S = \{(n_1, m_1 - n_1 + 1), (n_2, m_2 - n_2 + 1), \ldots, (n_k, m_k - n_k + 1)\}$. Conversely such a multiset S corresponds to a unique n-color partition. The product order is defined on multiset S as $(a, b) \geq (c, d)$ if and only if $a \geq b$ and $c \geq d$ which is clearly a partial order on S. Recall that a chain on a multiset is defined to be a sub-multiset such that every distinct pair of elements is comparable. Let c_t denote the cardinality of the largest sub-multiset, say $\mathcal{C}_t$, of S obtained by taking the union of t chains. By convention, we define $c_0 = 0$. The sequence $(\lambda_t)_{t\geq 1} = (c_t - c_{t-1})_{t\geq 1}$ is nonincreasing and thus a partition of k. We denote $\lambda(\pi) = (\lambda_1, \lambda_2, \ldots)$ and call it as the shape of partition π.

We now obtain a multiset $S^{(i,j)}$ from the multiset S by deleting all those points whose first coordinate is less than i or the second coordinate is less than j. Let $\pi^{(i,j)}$ denote the corresponding n-color partition. Further, $c_t^{(i,j)}$ will denote the cardinality of the corresponding sub-multiset $\mathcal{C}_t^{(i,j)}$. Let $\lambda^{(i,j)} = (\lambda_1^{(i,j)}, \lambda_2^{(i,j)}, \ldots)$ be the shape of the n-color partition $\pi^{(i,j)}$, where $\lambda_t^{(i,j)} = c_t^{(i,j)} - c_{t-1}^{(i,j)}, t \geq 1$. Next we construct a plane partition with (i, j)th part as n_{ij} by writing these shapes diagonally as follows:

$$n_{ij} = \begin{cases} \lambda_j^{(i-j+1,1)} & \text{if } i \geq j \\ \lambda_i^{(1,j-i+1)} & \text{if } i \leq j. \end{cases} \tag{1.2}$$

Example 1.12. *Let $\pi = 5_3 + 4_3 + 4_3 + 4_3 + 3_3 + 3_3 + 4_2 + 3_2 + 2_2 + 3_1 + 3_1 + 2_1 + 2_1 + 1_1 + 1_1 + 1_1$ be an n-color partition of 45. Then the multiset S is given by*

$$\begin{aligned} S =& \{(3,3), (3,2), (3,2), (3,2), (3,1), (3,1), (2,3), (2,2), (2,1), (1,3), (1,3), \\ & (1,2), (1,2), (1,1), (1,1), (1,1)\}. \end{aligned}$$

Now we obtain the shape $\lambda(\pi)$ *as follows:*

We obtain $\mathcal{C}_1$ *by only* 1 *chain*

$$\{(3,3),(3,2),(3,2),(3,2),(3,1),(3,1),(2,1),(1,1),(1,1),(1,1)\}.$$

Hence,

$$\mathcal{C}_1 = \{(3,3),(3,2),(3,2),(3,2),(3,1),(3,1),(2,1),(1,1),(1,1),(1,1)\}.$$

In this way, $c_1 = |\mathcal{C}_1| = 10$*. Now,* $\mathcal{C}_2$ *is obtained by taking a union of* 2 *chains*

$$\begin{aligned}&\{(3,3),(3,2),(3,2),(3,2),(3,1),(3,1),(2,1),(1,1),(1,1),(1,1)\}\\ &\cup\{(2,3),(1,3),(1,3),(1,2),(1,2)\}.\end{aligned}$$

Thus,

$$\begin{aligned}\mathcal{C}_2 =&\{3,3),(3,2),(3,2),(3,2),(3,1),(3,1),(2,3),(2,1),(1,3),(1,3),(1,2),\\ &(1,2),(1,1),(1,1),(1,1)\}.\end{aligned}$$

This gives $c_2 = |\mathcal{C}_2| = 15$*. Next,* $\mathcal{C}_3$ *is obtained by taking a union of* 3 *chains*

$$\begin{aligned}&\{(3,3),(3,2),(3,2),(3,2),(3,1),(3,1),(2,1),(1,1),(1,1),(1,1)\}\\ &\cup\{(2,3),(1,3),(1,3),(1,2),(1,2)\}\cup\{(2,2)\}.\end{aligned}$$

Hence,

$$\begin{aligned}\mathcal{C}_3 =&\{(3,3),(3,2),(3,2),(3,2),(3,1),(3,1),(2,3),(2,2),(2,1),(1,3),(1,3),\\ &(1,2),(1,2),(1,1),(1,1),(1,1)\}.\end{aligned}$$

This shows $c_3 = |\mathcal{C}_3| = 16$*.*

Hence $\lambda(\pi) = (\lambda_1, \lambda_2, \lambda_3) = (10, 5, 1)$*.*

Also note that $\lambda^{(1,1)} = \lambda(\pi) = (10, 5, 1)$*.*

Next, we obtain $\lambda^{(1,2)}$ *by using* $S^{(1,2)} = \{(3,3),(3,2),(3,2),(3,2),(2,3),(2,2),(1,3),(1,3),(1,2),(1,2)\}$*. In this case,* $\mathcal{C}_i^{(1,2)}$*s are given as follows:*

We have only 1 *chain*

$$\{(3,3),(3,2),(3,2),(3,2),(2,2),(1,2),(1,2)\}$$

to get

$$\mathcal{C}_1^{(1,2)} = \{(3,3),(3,2),(3,2),(3,2),(2,2),(1,2),(1,2)\}.$$

Hence $c_1^{(1,2)} = |\mathcal{C}_1^{(1,2)}| = 7$*. Next, the union of* 2 *chains*

$$\{(3,3),(3,2),(3,2),(3,2),(2,2),(1,2),(1,2)\}\cup\{(2,3),(1,3),(1,3)\}$$

gives us

$$\mathcal{C}_2^{(1,2)} = \{(3,3),(3,2),(3,2),(3,2),(2,3),(2,2),(1,3),(1,3),(1,2),(1,2)\}.$$

This shows $c_2^{(1,2)} = |\mathcal{C}_2^{(1,2)}| = 10$*. Thus*

$$\lambda^{(1,2)} = (\lambda_1^{(1,2)}, \lambda_2^{(1,2)}) = (7,3).$$

Similarly

$$\lambda^{(2,1)} = (7,2);\ \ \lambda^{(1,3)} = 4;\ \ \lambda^{(3,1)} = 6.$$

Now, using equation (1.2)*, we get the following plane partition:*

$$\begin{array}{ccc} 10 & 7 & 4 \\ 7 & 5 & 3 \\ 6 & 2 & 1. \end{array}$$

Let $\mathcal{P}(U, V; \nu)$ denote the set of all n-color partitions of ν of the form $\sum_i (m_i)_{n_i}$ such that $m_i \in U,\ n_i \in V$ and $P(U, V; \nu) = |\mathcal{P}(U, V; \nu)|$. We shall denote the set of all even positive integers and the set of all odd positive integers by E and O, respectively. Let $R(\nu)$ denote the number of n-color partitions of ν such that even parts appear with even subscripts and odd parts appear with odd subscripts. Further, we denote the number of self-conjugate n-color partitions of a positive integer ν by $SC(\nu)$. Next, if $\mathcal{A}(\nu)$ denotes the set of a particular type of partitions of ν, then $A(\nu)$ will denote the number of partitions in $\mathcal{A}(\nu)$ and $GA(q)$ will denote the generating function for $A(\nu)$. Using standard techniques of partition theory, the following generating functions are proved in [8]:

$$1 + \sum_{\nu=1}^{\infty} P(O, O; \nu) q^{\nu} = \prod_{n=1}^{\infty} \frac{1}{(1 - q^{2n-1})^n} \tag{1.3}$$

$$1 + \sum_{\nu=1}^{\infty} P(E, E; \nu) q^{\nu} = \prod_{n=1}^{\infty} \frac{1}{(1 - q^{2n})^n} \tag{1.4}$$

$$1 + \sum_{\nu=1}^{\infty} P(E, O; \nu) q^{\nu} = \prod_{n=1}^{\infty} \frac{1}{(1 - q^{2n})^n} \tag{1.5}$$

$$1 + \sum_{\nu=1}^{\infty} P(O, E; \nu) q^{\nu} = \prod_{n=1}^{\infty} \frac{1}{(1 - q^{2n-1})^{n-1}} \tag{1.6}$$

$$1 + \sum_{\nu=1}^{\infty} R(\nu) q^{\nu} = \prod_{n=1}^{\infty} \frac{1}{(1 - q^{n})^{[\frac{n+1}{2}]}} \tag{1.7}$$

where [ ] denotes the greatest integer function.

It was proved in [5] that the number of self-conjugate n-color partitions of a positive integer ν has the following generating function:

$$1 + \sum_{\nu=1}^{\infty} SC(\nu) q^{\nu} = \prod_{n=1}^{\infty} \frac{1}{(1 - q^{2n-1})(1 - q^{2n})^{[\frac{n}{2}]}}. \tag{1.8}$$

The main results of this paper are contained in Section 2 and 3. Section 2 consists of results related to new restricted plane partitions which include generating functions and recurrence relations. In Section 3, a connection between certain classes of plane partitions and restricted associated lattice paths is established. In Section 4, we provide the first few values of all new restricted plane partition functions. These values are obtained on a computer using the results of Section 2.

2. Restricted plane partitions

In this section, we give generating functions and recurrence relations for new restricted plane partitions.

Throughout the section, we denote

$$\pi = \begin{matrix} l_{11} & l_{12} & \cdots \\ l_{21} & l_{22} & \cdots \\ \vdots & \vdots & \end{matrix}$$

to be a plane partition of ν. Recall that $\sigma_k(n)$ denotes the sum of kth powers of divisors of n.

Theorem 2.1. *Let $A_1(\nu)$ denote the number of plane partitions π of ν such that*

(1) the number of columns as well as the number of rows of π is odd,
(2) if i and j are of opposite parity then

$$l_{ij} = \begin{cases} l_{(i+1)j} & \text{if } i > j \\ l_{i(j+1)} & \text{if } i < j \end{cases}.$$

Then

$$GA_1(q) = 1 + \sum_{\nu=1}^{\infty} A_1(\nu) q^{\nu} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{2n-1})^n}, \tag{2.1}$$

$$A_1(0) = 1, A_1(1) = 1,$$

and for $\nu \geq 2$, we have

$$A_1(\nu) = \frac{1}{\nu} \left\{ \sum_{m=1}^{\nu} \frac{1}{2}(\sigma_2(m) + \sigma_1(m)) A_1(\nu - m) - \sum_{k=1}^{[\nu/2]} (2\sigma_2(k) + \sigma_1(k)) A_1(\nu - 2k) \right\}. \tag{2.2}$$

Theorem 2.2. *Let $A_2(\nu)$ denote the number of plane partitions π of ν such that*

(1) the number of columns as well as the number of rows of π is even,
(2) if i and j are of same parity then

$$l_{ij} = l_{(i+1)j} \text{ if } i \geq j,$$

(3) if i and j are of opposite parity then

$$l_{ij} = l_{i(j+1)} \text{ if } i < j.$$

Then

$$GA_2(q) = 1 + \sum_{\nu=1}^{\infty} A_2(\nu) q^{\nu} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{2n})^n}, \tag{2.3}$$

$$A_2(0) = 1, A_2(1) = 0,$$

$$A_2(\nu) = \frac{1}{\nu}\left\{\sum_{k=1}^{[\nu/2]}(\sigma_2(k))A_2(\nu-2k)\right\} \text{ for } \nu \geq 2. \tag{2.4}$$

Theorem 2.3. *Let $A_3(\nu)$ denote the number of plane partitions π of ν such that*

(1) the number of columns of π is even and the number of rows of π is odd,

(2) if i and j are of opposite parity then

$$l_{ij} = l_{(i+1)j} \text{ if } i > j,$$

(3) if i and j are of same parity then

$$l_{ij} = l_{i(j+1)} \text{ if } i \leq j.$$

Then

$$GA_3(q) = 1 + \sum_{\nu=1}^{\infty} A_3(\nu)q^{\nu} = \prod_{n=1}^{\infty}\frac{1}{(1-q^{2n})^{n}}, \tag{2.5}$$

$$A_3(0) = 1, A_3(1) = 0,$$

$$A_3(\nu) = \frac{1}{\nu}\left\{\sum_{k=1}^{[\nu/2]}(\sigma_2(k))A_3(\nu-2k)\right\} \text{ for } \nu \geq 2. \tag{2.6}$$

Theorem 2.4. *Let $A_4(\nu)$ denote the number of plane partitions π of ν such that*

(1) the number of columns of π is even and the number of rows of π is odd,

(2) if i and j are of opposite parity then

$$l_{ij} = \begin{cases} l_{(i+1)j} & \text{if } i > j \\ l_{i(j+1)} & \text{if } i < j \end{cases}.$$

Then

$$GA_4(q) = 1 + \sum_{\nu=1}^{\infty} A_4(\nu)q^{\nu} = \prod_{n=1}^{\infty}\frac{1}{(1-q^{2n})^{n-1}}, \tag{2.7}$$

$$A_4(0) = 1, A_4(1) = 0,$$

and for $\nu \geq 2$, we have

$$A_4(\nu) = \frac{1}{\nu}\left\{\sum_{m=1}^{\nu}\frac{1}{2}(\sigma_2(m)+\sigma_1(m))A_4(\nu-m) - \sum_{k=1}^{[\nu/2]}(2\sigma_2(k)-\sigma_1(k))A_4(\nu-2k)\right\}. \tag{2.8}$$

Theorem 2.5. *Let $A_5(\nu)$ denote the number of plane partitions π of ν such that*

(1) the number of columns of π is odd,
(2) if i and j are of opposite parity then $l_{ij} = l_{i(j+1)}$ for $i < j$.

Then

$$GA_5(q) = 1 + \sum_{\nu=1}^{\infty} A_5(\nu) q^{\nu} = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{[\frac{n+1}{2}]}}, \tag{2.9}$$

$$A_5(0) = 1, A_5(1) = 0,$$

and for $\nu \geq 2$, we have

(2.10)

$$A_5(\nu) = \frac{1}{\nu}\left\{ \sum_{m=1}^{\nu} \frac{1}{2}(\sigma_2(m) + \sigma_1(m))A_5(\nu - m) - \sum_{k=1}^{[\nu/2]} (\sigma_1(k))A_5(\nu - 2k) \right\}.$$

Theorem 2.6. *Let $A_6(\nu)$ denote the number of symmetric plane partitions of ν, then*

$$GA_6(q) = 1 + \sum_{\nu=1}^{\infty} A_6(\nu) q^{\nu} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{2n-1})(1-q^{2n})^{[\frac{n}{2}]}}, \tag{2.11}$$

$$A_6(0) = 1, A_6(1) = 1, A_6(2) = 1, A_6(3) = 2,$$

and for $\nu \geq 4$, we have

(2.12)

$$\begin{aligned} A_6(\nu) =& \frac{1}{\nu}\Bigg\{ \sum_{m=1}^{\nu} (\sigma_1(m))A_6(\nu - m) - \sum_{k=1}^{[\nu/2]} (\sigma_2(k) - 3\sigma_1(k))A_6(\nu - 2k) \\ &+ \sum_{\ell=1}^{[\nu/4]} (2\sigma_1(\ell))A_6(\nu - 4\ell)\Bigg\}. \end{aligned}$$

Since the proofs of Theorems 2.1–2.6 are similar, we give the details of the proof of Theorem 2.1 and omit the proofs of the remaining theorems.

Bijective proof of Theorem 2.1. First we show that $A_1(\nu) = P(O, O; \nu)$. Let $\omega = (a_1)_{b_1} + (a_2)_{b_2} + \ldots (a_r)_{b_r}$ be an n-color partition in $\mathcal{P}(O, O; \nu)$ or we can say that a_i and b_i are odd $\forall\, 1 \leq i \leq r$. Now we obtain a plane partition, say π corresponding to the n-color partition ω using the procedure given in Section 1. In the multiset $S = \{(b_1, a_1 - b_1 + 1), (b_2, a_2 - b_2 + 1), \ldots, (b_r, a_r - b_r + 1)\}$ corresponding to the partition ω, all points have odd coordinates. Note that the number of columns and the number of rows in the corresponding plane partition π are the largest $a_i - b_i + 1$ and b_i, respectively. This proves part 1.

Further, since there is no even ordinate in the points of S, the corresponding multiset $S^{(i,j)}$ will be the same as $S^{(i,j+1)}$, for j even. Hence we obtain the relation

$$\lambda^{(1,j)} = \lambda^{(1,j+1)},\ j \text{ even}.$$

Now using equation (1.2), we get that for $i < j$, $l_{ij} = l_{i(j+1)}$ where i and j are of opposite parity. Similarly, using the fact that there is no even abscissa in the points of S, we obtain

$$\lambda^{(i,1)} = \lambda^{(i+1,1)},\ i \text{ even}.$$

Hence for $i > j$, we get $l_{ij} = l_{(i+1)j}$ where i and j are of opposite parity. In this way, we get that the plane partition π is in $\mathcal{A}_1(\nu)$. Now using equation (1.3), we get (2.1). To prove (2.2), we differentiate (2.1) logarithmically both sides with respect to q.

$$\begin{aligned}\frac{qGA_1'(q)}{GA_1(q)} &= \sum_{n=1}^{\infty} \frac{n(2n-1)q^{2n-1}}{1-q^{2n-1}} \\ &= \sum_{n=1}^{\infty} \left(\frac{n(n+1)}{2} \frac{q^n}{1-q^n} - \frac{2n(2n+1)}{2} \frac{q^{2n}}{1-q^{2n}} \right).\end{aligned}$$

Using $\sum_{n=1}^{\infty} \sigma_k(n) q^n = \sum_{n=1}^{\infty} \frac{n^k q^n}{1-q^n}$, we get

$$\frac{qGA_1'(q)}{GA_1(q)} = \frac{1}{2}\sum_{n=1}^{\infty}(\sigma_2(n)+\sigma_1(n))q^n - \sum_{n=1}^{\infty}(2\sigma_2(n)+\sigma_1(n))q^{2n},$$

and hence

$$\begin{aligned}\sum_{\nu=0}^{\infty} \nu A_1(\nu) q^{\nu} = \left(\sum_{\nu=0}^{\infty} A_1(\nu) q^{\nu}\right) \Bigg(\frac{1}{2}\sum_{n=1}^{\infty}(\sigma_2(n)+\sigma_1(n))q^n - \\ \sum_{n=1}^{\infty}(2\sigma_2(n)+\sigma_1(n))q^{2n} \Bigg).\end{aligned}$$

Equating coefficients of q^{ν} on both sides, we get the result. □

3. Graphical Representation

In view of Theorem 1.11 and the fact that the number of plane partitions of a positive integer ν is the same as the number of n-color partitions of ν, we conclude that

$$PP(\nu) = P(\nu) = \chi(\nu).$$

In this Section, we connect the restricted plane partitions introduced in Section 2 with certain classes of associated lattice paths.

Theorem 3.1. *Let $B_1(\nu)$ denote the number of associated lattice paths of weight ν such that for any TITS with ordered pair $\{a, b\}$, both a and b are odd, b does not exceed a, where the TITSs are arranged as in Theorem* 1.11. *Then*

$$B_1(\nu) = A_1(\nu).$$

Theorem 3.2. *Let $B_2(\nu)$ denote the number of associated lattice paths of weight ν such that for any TITS with ordered pair $\{a, b\}$, both a and b are even, b does not exceed a, where the TITSs are arranged as in Theorem* 1.11. *Then*

$$B_2(\nu) = A_2(\nu).$$

Theorem 3.3. *Let $B_3(\nu)$ denote the number of associated lattice paths of weight ν such that for any TITS with ordered pair $\{a, b\}$, a is even and b is odd, b does not exceed a, where the TITSs are arranged as in Theorem* 1.11. *Then*

$$B_3(\nu) = A_3(\nu).$$

Theorem 3.4. *Let $B_4(\nu)$ denote the number of associated lattice paths of weight ν such that for any TITS with ordered pair $\{a, b\}$, a is odd and b is even, b does not exceed a, where the TITSs are arranged as in Theorem* 1.11. *Then*

$$B_4(\nu) = A_4(\nu).$$

Theorem 3.5. *Let $B_5(\nu)$ denote the number of associated lattice paths of weight ν such that for any TITS with ordered pair $\{a, b\}$, a and b have same parity, b does not exceed a, where the TITSs are arranged as in Theorem* 1.11. *Then*

$$B_5(\nu) = A_5(\nu).$$

Theorem 3.6. *Let $B_6(\nu)$ denote the number of associated lattice paths of weight ν such that either pairs of TITS occur with $\{a, b\}, \{a, a-b+1\}$ as the corresponding ordered pairs, b does not exceed a or any TITS has ordered pair $\{2b-1, b\}$, where the TITSs are arranged as in Theorem* 1.11. *Then*

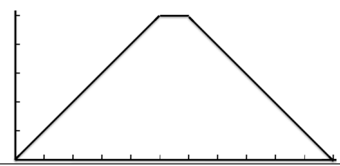
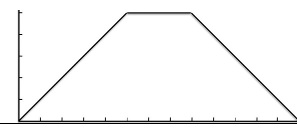
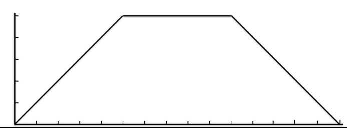
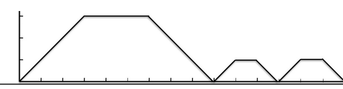
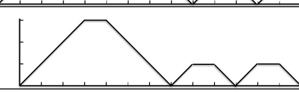
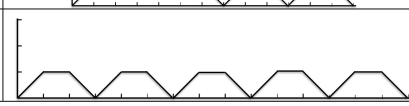
$$B_6(\nu) = A_6(\nu).$$

Here we give the details of the proof of Theorem 3.1. The proofs of the Theorems 3.2-3.6 can be obtained on similar lines.

Proof of Theorem 3.1. Let $\mathcal{B}_1(\nu)$ denote the set of associated lattice paths enumerated by $B_1(\nu)$. Let α be an associated lattice path in $\mathcal{B}_1(\nu)$. Each TITS with ordered pair $\{a, b\}$ in α is mapped to an n-color part a_b. Since b does not exceed a and both a and b are odd, the corresponding n-color partition will belong to $\mathcal{P}(O, O; \nu)$. Reversing the steps, we get that each n-color partition in $\mathcal{P}(O, O; \nu)$ corresponds to an associated lattice path in $\mathcal{B}_1(\nu)$. This correspondence along with Theorem 2.1 shows that there is a one-to-one correspondence between $\mathcal{A}_1(\nu)$ and $\mathcal{B}_1(\nu)$. $\square$

We illustrate the equality of $A_1(\nu)$, $B_1(\nu)$ and $P(O, O; \nu)$ for $\nu = 5$ with the help of Table 1.

Table 1. Illustration in the proof of Theorem 2.1 and 3.1

n-color partitions in $\mathcal{P}(O, O; 5)$	Plane partitions in $\mathcal{A}_1(5)$	Associated lattice paths in $\mathcal{B}_1(5)$
5_1	1 1 1 1 1	
5_3	1 1 1 1 1	
5_5	1 1 1 1 1	
$3_3 1_1 1_1$	3 1 1	
$3_1 1_1 1_1$	3 1 1	
$1_1 1_1 1_1 1_1 1_1$	5	

4. Integer sequences

In this section, we give values of restricted plane partition functions $A_i(\nu)$ for $1 \leq i \leq 6$ and $1 \leq \nu \leq 10$ obtained on a computer using the results of Section 2.

$$A_1(\nu) : 1,\ 1,\ 3,\ 3,\ 6,\ 9,\ 13,\ 19,\ 28,\ 42, \ldots$$
$$A_2(\nu) : 0,\ 1,\ 0,\ 3,\ 0,\ 6,\ 0,\ 13,\ 0,\ 24, \ldots$$
$$A_3(\nu) : 0,\ 1,\ 0,\ 3,\ 0,\ 6,\ 0,\ 13,\ 0,\ 24, \ldots$$
$$A_4(\nu) : 0,\ 0,\ 1,\ 0,\ 2,\ 1,\ 3,\ 2,\ 5,\ 6, \ldots$$
$$A_5(\nu) : 1,\ 2,\ 4,\ 7,\ 12,\ 21,\ 34,\ 56,\ 90,\ 143, \ldots$$
$$A_6(\nu) : 1,\ 1,\ 2,\ 3,\ 4,\ 6,\ 8,\ 12,\ 16,\ 22, \ldots$$

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