

APPLICATIONS VIA SERIES ACCELERATIONS OF NEW
IDENTITIES INVOLVING CATALAN-TYPE NUMBERS

JOHN M. CAMPBELL AND EMRAH KILIÇ

ABSTRACT. We introduce infinite families of terminating hypergeometric identities involving generalizations of Catalan numbers, generalizing results introduced by Chu and Kılıç, and we apply Wilf–Zeilberger (WZ) pairs associated with our new identities via a series acceleration method. We apply a WZ pair introduced in our article to prove an identity for accelerating the convergence for a family of ${}_3F_2(1)$ -series with three real parameters from 1 to $\frac{1}{4}$, and we apply this identity to generalize Ramanujan-like series for $\frac{1}{\pi}$, $\frac{\sqrt{2}}{\pi}$, $\frac{\sqrt{3}}{\pi}$, and $\frac{\sqrt{2\pm\sqrt{2}}}{\pi}$ that are due to Chu et al. A fast-converging series for π^2 due to Guillera is also a special case of our acceleration identity. We also apply another WZ pair introduced in this article to prove an identity for accelerating the convergence of a ${}_3F_2(1)$ -family with three real parameters from 1 to $\frac{1}{16}$, and we apply this result, via a series bisection, to formulate a new WZ proof of Ramanujan’s series for $\frac{1}{\pi}$ of convergence rate $\frac{1}{4}$. A number of our finite sums involving Catalan-type numbers are such that up-to-date versions of the Maple Computer Algebra System cannot compute WZ pairs for such sums, which is representative of the computationally challenging nature of our results.

1. INTRODUCTION

The k th Catalan number may be defined so that $C_k = \binom{2k}{k} \frac{1}{k+1}$. The Catalan numbers are famously ubiquitous in combinatorics, and a popular way of illustrating this is with the many different combinatorial interpretations of C_k recorded in Stanley’s classic *Enumerative combinatorics* text [24]. The purpose of this article is to build on the results from the recent article [11] by introducing infinite families of summation identities involving generalizations of Catalan numbers and by applying such identities via an acceleration method that had been formulated by Guillera in [13] (cf. [15]) and by Campbell in [2, 3, 4] and that is based on the Wilf–Zeilberger (WZ) method [19]. In addition to [11], our work is inspired by a number of past

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articles concerning finite sums involving Catalan or Catalan-type numbers, including [6, 7, 10, 16, 17, 18, 23, 25, 26].

An advantage of proving our identities involving generalized Catalan numbers using the WZ method, compared to bijective or classically oriented approaches, is given by the series accelerations that we apply in Sections 2 and 3 and that are based on WZ pairs involved in our proofs. We highlight the following new formulas that are introduced in this article via our acceleration method and that are closely related to and heavily inspired by the work of Ramanujan [21], Chu and Zhang [12], and Chu [9], referring to Section 1.1 for preliminaries on our notation concerning hypergeometric series:

$$(1.1) \quad \frac{15}{\pi} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \left[-\frac{5}{6}, -\frac{5}{6}, \frac{5}{6}, \frac{5}{6} \right]_{\frac{1}{2}, 1, 1, 1} (25 - 108n^2),$$

$$(1.2) \quad \frac{2\sqrt{2}}{\pi} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \left[-\frac{3}{4}, -\frac{3}{4}, \frac{3}{4}, \frac{3}{4} \right]_{\frac{1}{2}, 1, 1, 1} (3 - 16n^2),$$

$$(1.3) \quad \frac{3\sqrt{3}}{\pi} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \left[-\frac{2}{3}, -\frac{2}{3}, \frac{2}{3}, \frac{2}{3} \right]_{\frac{1}{2}, 1, 1, 1} (4 - 27n^2),$$

$$(1.4) \quad \frac{3\sqrt{3}}{2\pi} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \left[-\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right]_{\frac{1}{2}, 1, 1, 1} (1 - 27n^2),$$

$$(1.5) \quad \frac{28\sqrt{2-\sqrt{2}}}{\pi} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \left[-\frac{7}{8}, -\frac{7}{8}, \frac{7}{8}, \frac{7}{8} \right]_{\frac{1}{2}, 1, 1, 1} (49 - 192n^2),$$

$$(1.6) \quad \frac{20\sqrt{2+\sqrt{2}}}{\pi} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \left[-\frac{5}{8}, -\frac{5}{8}, \frac{5}{8}, \frac{5}{8} \right]_{\frac{1}{2}, 1, 1, 1} (25 - 192n^2).$$

In Section 2, we introduce an infinite family of generalizations of the Catalan sum identity

$$(1.7) \quad \sum_{k=0}^n \binom{2n-2k}{n-k} \binom{2n}{2k} C_k = (2n+1) C_n^2$$

from [11], and our WZ pair associated with our proof of (1.7) is used to obtain our identity for accelerating the convergence of a family of ${}_3F_2(1)$ -series with three real parameters from 1 to $\frac{1}{4}$ (cf. [2, 4]). We apply this series acceleration to prove closed forms as in the motivating results shown in (1.1)–(1.6). We also apply this series acceleration to obtain a new proof of Guillerá's formula

$$(1.8) \quad \frac{\pi^2}{4} = \sum_{n=0}^{\infty} \frac{2^{4n}(3n+2)}{(2n+1)^3 \binom{2n}{n}^3}$$

introduced in [15]. Furthermore, a WZ pair that we have computed to generalize the Catalan sum identity

$$(1.9) \quad \sum_{k=0}^n \binom{2n}{2k} C_k C_{n-k} = C_n C_{n+1}$$

given by Chu and Kiliç in [11] may be applied, via our acceleration method, to formulate a new WZ proof of the famous formula

$$(1.10) \quad \frac{4}{\pi} = \sum_{n=0}^{\infty} \left(\frac{1}{4} \right)^n \left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1 \end{matrix} \right]_n (6n+1)$$

due to Ramanujan [21]. Our WZ proof of Ramanujan's formula in (1.10) is inequivalent to and dramatically different from previously known WZ proofs of (1.10) [5, 14].

1.1. Preliminaries. We recall that the Γ -function [20, §2] may be defined so that $\Gamma(x) = \int_0^{\infty} u^{x-1} e^{-u} du$ for $\Re(x) > 0$, and that the Pochhammer symbol is such that $(x)_n = \frac{\Gamma(n+x)}{\Gamma(x)}$, with $(x)_n = x(x+1) \cdots (x+n-1)$ for $n \in \mathbb{N}$. Classical properties concerning the Γ -function such as the Legendre duplication formula

$$(1.11) \quad \Gamma(2z) = (2\pi)^{-1/2} 2^{2z-1/2} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right)$$

are to be later involved in our work. We also adopt the notational conventions such that

$$(1.12) \quad \left[\begin{matrix} \alpha, \beta, \dots, \gamma \\ A, B, \dots, C \end{matrix} \right]_n = \frac{(\alpha)_n (\beta)_n \cdots (\gamma)_n}{(A)_n (B)_n \cdots (C)_n}$$

and such that

$$\Gamma \left[\begin{matrix} \alpha, \beta, \dots, \gamma \\ A, B, \dots, C \end{matrix} \right] = \frac{\Gamma(\alpha) \Gamma(\beta) \cdots \Gamma(\gamma)}{\Gamma(A) \Gamma(B) \cdots \Gamma(C)}.$$

In regard to the notation in (1.12), we let generalized hypergeometric series be denoted in a manner that agrees with classic texts such as [1, 22], writing

$${}_pF_q \left[\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{matrix} \middle| x \right] = \sum_{k=0}^{\infty} \frac{x^k}{k!} \left[\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{matrix} \right]_k.$$

Given an identity of the form

$$(1.13) \quad \sum_k \text{summand}(n, k) = \text{closed}(n),$$

where the summand in (1.13) is hypergeometric and vanishes for all integers n outside of a bounded interval, and where the closed-form evaluation on the right-hand side of (1.13) is hypergeometric and nonzero, by writing

$$(1.14) \quad F(n, k) := \frac{\text{summand}(n, k)}{\text{closed}(n)},$$

and by employing the WZ method [19], we may determine a hypergeometric function $G(n, k)$ so that the discrete difference equation

$$(1.15) \quad F(n+1, k) - F(n, k) = G(n, k+1) - G(n, k)$$

is satisfied, with $G(n, 0) = 0$ and $\lim_{k \rightarrow \infty} G(n, k) = 0$. So, by summing both sides of (1.15) with respect to k , a telescoping phenomenon allows us to prove an equivalent form of (1.13). The companion function G may be written as $G(n, k) = F(n, k)R(n, k)$ for a rational function $R(n, k)$ referred to as a *WZ proof certificate*, so it remains to evaluate $R(n, k)$ to prove (1.14).

2. SERIES OF CONVERGENCE RATE $\frac{1}{4}$

Part (d) of Theorem 1 in [11] is as in (1.7). In the below Theorem, we provide an infinite family of generalizations of (1.7), with the $c = 1$ case of the below Theorem being equivalent to (1.7). We are to later apply a WZ pair associated with the binomial identity in (2.1), to obtain the series acceleration identity highlighted in Theorem 2.2.

Theorem 2.1. *The identity*

$$(2.1) \quad \sum_{k=0}^n \binom{2n}{2k} \binom{2n-2k}{n-k} \binom{2k}{k} \frac{1}{(k+1)_c} = \frac{\binom{2n}{n}(c+2n)!}{((c+n)!)^2}$$

holds for all $n \in \mathbb{N}_0$ and for a parameter c that is not a negative integer.

Proof. Setting

$$(2.2) \quad F(n, k) := \frac{\binom{2k}{k}((c+n)!)^2 \binom{2n}{2k} \binom{2n-2k}{n-k}}{\binom{2n}{n}(c+2n)!(k+1)_c},$$

it remains to prove that $\sum_k F(n, k)$ constantly equals 1, for c as specified. The Maple Computer Algebra System is able to compute the desired WZ proof certificate, yielding

$$(2.3) \quad R(n, k) = \frac{k(k+c)(kc+2kn+2k-2cn-2c-3n^2-6n-3)}{(k-n-1)^2(c+2n+1)(c+2n+2)},$$

so as to form a WZ pair (F, G) , with $G(n, k) = F(n, k)R(n, k)$. $\square$

Observe that the WZ pair (F, G) specified in the proof of Theorem 2.1 is such that, according to the WZ method, the arguments n and k of F and G are nonnegative integers. However, by rewriting F and G in terms of the Γ -function, with

$$(2.4) \quad F(n, k) = \Gamma \left[\begin{matrix} n+1, n+1, c+n+1, c+n+1 \\ k+1, c+k+1, n-k+1, n-k+1, c+2n+1 \end{matrix} \right],$$

we find that the same difference equation in (1.15) holds for real n, k , and c such that $n+1$ and $c+n+1$ are not in $\mathbb{Z}_{\leq 0}$ and such that the lower Γ -arguments in (2.4) are not in $\mathbb{Z}_{\leq 0}$. We may thus define $F^{(p)}(n, k) = F(n+p, k)$ for a real parameter p such that the Γ -arguments of $F(n+p, k)$,

according to (2.4), are not in $\mathbb{Z}_{\leq 0}$. This leads us to the difference equation in (2.5), following a similar approach as in [2].

As in [2], we may rewrite the difference equation in (1.15) as

$$(2.5) \quad F(n+p+1, k) - F(n+p, k) = G(n+p, k+1) - G(n+p, k)$$

for a real parameter p . This leads us toward the following acceleration formula based on the WZ pair given in the proof of Theorem 2.1. The recent paper [2] also concerned accelerations derived from WZ pairs associated with generalizations of the Catalan sums from [11], but the series accelerations from [2] cannot be used to obtain series for constants involving $\frac{1}{\pi}$, which are a main object of study in our work.

Theorem 2.2. *Let k , p , and c be parameters such that c is a real number distinct from negative integers and $p+1$ and $p+c+1$ and $2p+c+1$ are distinct from zero and negative integers, and we again set $G(n, k)$ as the product of (2.2) and (2.3). Then*

$$(2.6) \quad \Gamma \left[\begin{matrix} p+1, p+1, p+c+1, p+c+1 \\ k+1, k+c+1, p-k+1, p-k+1, 2p+c+1 \end{matrix} \right] \times {}_3F_2 \left[\begin{matrix} 1, k-p, k-p \\ k+1, k+c+1 \end{matrix} \middle| 1 \right]$$

equals

$$\sum_{n=0}^{\infty} \Gamma \left[\begin{matrix} n+p+1, n+p+1, c+n+p+1, c+n+p+1 \\ k+1, c+k+1, n-k+p+2, n-k+p+2, c+2n+2p+3 \end{matrix} \right] \times \\ k(c+k)(ck-2cn-2cp-2c+2kn+2kp+2k-3n^2-6np- \\ 6n-3p^2-6p-3)+1,$$

if the above series are convergent.

Proof. Summing both sides of the difference equation in (2.5) with respect to n , a telescoping phenomenon then gives us that

$$F(m+p+1, k) - F(p, k) = \sum_{n=0}^m (G(n+p, k+1) - G(n+p, k))$$

for $m \in \mathbb{N}_0$. Setting $m \rightarrow \infty$, the vanishing of $F(m+p+1, k)$ is immediate from the Γ -evaluation in (2.4). So, we obtain that

$$-F(p, k) = \sum_{n=0}^{\infty} (G(n+p, k+1) - G(n+p, k)).$$

A telescoping argument then gives us that

$$(2.7) \quad \sum_{n=0}^{\ell-1} -F(p, k+n) = \sum_{n=0}^{\infty} (G(n+p, k+\ell) - G(n+p, k))$$

for $\ell \in \mathbb{N}_0$. The given constraints on k , p , and c allow us to set $\ell \rightarrow \infty$, and, by writing

$$\sum_{n=0}^{\infty} -F(p, k+n) = \lim_{\ell \rightarrow \infty} \sum_{n=0}^{\infty} G(n+p, k+\ell) - \sum_{n=0}^{\infty} G(n+p, k),$$

we find that the above limit involving ℓ reduces to -1 by following an identical argument from [2, 4]. Evaluating

$$(2.8) \quad \sum_{n=0}^{\infty} -F(p, k+n)$$

in terms of the ${}_3F_2(1)$ -series in (2.6) then gives us the desired result. $\square$

Adopting notation from [2], by setting the tuple (k, p, c) in Theorem 2.2 with specific rational values, this often gives us fast-converging series that are closely related to the work of Chu and Zhang in [12]. We refer the interested reader to [2] for details as to how we may obtain closed forms from the WZ identity in (2.7); as below, we specify values for the entries of the tuple (k, p, c) for a given closed form.

Following a similar approach as in [2], to generate and prove evaluations for series of convergence rate $\frac{1}{4}$, we apply the following steps:

- (1) Systematically search for tuples (k, p, c) of rational numbers such that the $F(p, k+n)$ -series in (2.8) evaluates in closed form;
- (2) Evaluate $\sum_{n=0}^{\infty} G(n+p, k)$ as a scalar multiple of a single ${}_pF_q$ -series; and
- (3) If possible, use contiguity relations to reduce the ${}_pF_q$ -series to an affine combination involving a ${}_{p'}F_{q'}$ -series derived from the original ${}_pF_q$ -series, for $p' < p$ and $q' < q$.

Built-in Maple algorithms are able to complete the second step in an automatic way, and built-in Mathematica algorithms are able to complete the final step in an automatic way. With regard to our below applications of the new result in Theorem 2.2, we refer to [2] for background on the term *Chu-style series*.

2.1. Chu-style series for $\frac{1}{\pi}$. In addition to the famous Ramanujan formula in (1.10), the following closely related formulas introduced by Chu and Zhang in [12] and reproved by Chu in [9] using a different approach relative to [12]

are main sources of motivation in regard to our research:

$$(2.9) \quad \frac{18}{\pi} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \left[\begin{matrix} \frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6} \\ 1, 1, 1, \frac{3}{2} \end{matrix} \right]_n (108n^2 + 72n + 5),$$

$$(2.10) \quad \frac{8\sqrt{2}}{\pi} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \left[\begin{matrix} \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4} \\ 1, 1, 1, \frac{3}{2} \end{matrix} \right]_n (48n^2 + 32n + 3),$$

$$(2.11) \quad \frac{9\sqrt{3}}{2\pi} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \left[\begin{matrix} \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \\ 1, 1, 1, \frac{3}{2} \end{matrix} \right]_n (27n^2 + 18n + 2).$$

Setting

$$(2.12) \quad (k, p, c) = \left(\frac{1}{6}, \frac{1}{6}, \frac{2}{3}\right)$$

in Theorem 2.2, this gives us a new and WZ-based proof of the result included as Example 83 in [12] and reproved in [9] and reproduced in (2.9). Explicitly, setting k and p and c to be as in (2.12), Theorem 2.2 gives us that

$$72\pi - 20\pi^2 = \sum_{n=0}^{\infty} \frac{(108n^2 + 288n + 185) \left((n + \frac{1}{6})!\right)^2 \left((n + \frac{5}{6})!\right)^2}{((n+1)!)^2 (2n+3)!},$$

and we proceed to rewrite the right-hand side as

$$\frac{185}{6} \left(\left(\frac{1}{6}\right)!\right)^2 \left(\left(\frac{5}{6}\right)!\right)^2 {}_7F_6 \left[\begin{matrix} 1, \frac{7}{6}, \frac{7}{6}, \frac{11}{6}, \frac{11}{6}, \frac{7}{3} - \frac{\sqrt{21}}{18}, \frac{7}{3} + \frac{\sqrt{21}}{18} \\ 2, 2, 2, \frac{5}{2}, \frac{4}{3} + \frac{\sqrt{21}}{18}, \frac{4}{3} - \frac{\sqrt{21}}{18} \end{matrix} \middle| \frac{1}{4} \right],$$

and contiguity relations (see [2] for details) give us that the above ${}_7F_6$ -series may be rewritten so as to obtain

$$\frac{47952 (\sqrt{21} - 6) (6 + \sqrt{21}) \left(\left(\frac{1}{6}\right)!\right)^2 \left(\left(\frac{5}{6}\right)!\right)^2}{5 (\sqrt{21} - 24) (24 + \sqrt{21})} \times$$

$$\left({}_6F_5 \left[\begin{matrix} \frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6}, \frac{4}{3} - \frac{\sqrt{7}}{6}, \frac{4}{3} + \frac{\sqrt{7}}{6} \\ 1, 1, \frac{3}{2}, \frac{1}{3} - \frac{\sqrt{7}}{6}, \frac{1}{3} + \frac{\sqrt{7}}{6} \end{matrix} \middle| \frac{1}{4} \right] - 1 \right),$$

which gives us that

$$\frac{18}{5\pi} = {}_6F_5 \left[\begin{matrix} \frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6}, \frac{4}{3} - \frac{\sqrt{7}}{6}, \frac{4}{3} + \frac{\sqrt{7}}{6} \\ 1, 1, \frac{3}{2}, \frac{1}{3} - \frac{\sqrt{7}}{6}, \frac{1}{3} + \frac{\sqrt{7}}{6} \end{matrix} \middle| \frac{1}{4} \right],$$

which is equivalent to (2.9). Our WZ proof of the series for $\frac{1}{\pi}$ in (2.9) related to the work of Ramanujan, together with the large amount of past literature related to Ramanujan's seminal article [21] on fast-converging series for $\frac{1}{\pi}$, inspires us to further apply Theorem 2.2 in the determination of new, Ramanujan-inspired series for $\frac{1}{\pi}$.

Example 2.3. Setting $(k, p, c) = \left(-\frac{5}{6}, -\frac{5}{6}, \frac{5}{3}\right)$ in Theorem 2.2, we obtain the motivating example highlighted in (1.1).

Example 2.4. Setting $(k, p, c) = \left(-\frac{5}{6}, \frac{1}{6}, -\frac{1}{3}\right)$ in Theorem 2.2, we may obtain that:

$$\frac{108}{35\pi} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \left[-\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{6} \right]_n (1 - 72n - 108n^2).$$

Example 2.5. Setting $(k, p, c) = \left(-\frac{5}{6}, \frac{1}{6}, \frac{2}{3}\right)$ in Theorem 2.2, we may obtain that:

$$\frac{648}{5\pi} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \left[\frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6} \right]_n (108n^2 + 144n + 41).$$

Example 2.6. Setting $(k, p, c) = \left(-\frac{1}{6}, \frac{11}{6}, \frac{4}{3}\right)$ in Theorem 2.2, we may obtain that:

$$\frac{18}{\pi} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \left[-\frac{1}{6}, -\frac{1}{6}, \frac{7}{6}, \frac{7}{6} \right]_n (7 - 72n - 108n^2).$$

2.2. Chu-style series for $\frac{\sqrt{2}}{\pi}$. Setting $(k, p, c) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$ in Theorem 2.2, we obtain that

$$128\sqrt{2} - 48\pi = \sum_{n=0}^{\infty} \frac{2^{-4n} (48n^2 + 128n + 83) ((2n + \frac{3}{2})!)^2}{((n+1)!)^2 (2n+3)!},$$

and we may rewrite the right-hand side as

$$\frac{83}{6} \left(\left(\frac{3}{2} \right)! \right)^2 {}_7F_6 \left[\begin{matrix} 1, \frac{5}{4}, \frac{5}{4}, \frac{7}{4}, \frac{7}{4}, \frac{7}{3} - \frac{\sqrt{7}}{12}, \frac{7}{3} + \frac{\sqrt{7}}{12} \\ 2, 2, 2, \frac{5}{2}, \frac{4}{3} + \frac{\sqrt{7}}{12}, \frac{4}{3} - \frac{\sqrt{7}}{12} \end{matrix} \middle| \frac{1}{4} \right],$$

and we may apply contiguity relations to obtain

$$\frac{1328(-4 + \sqrt{7})(4 + \sqrt{7})\pi}{(-16 + \sqrt{7})(16 + \sqrt{7})} \left(-1 + {}_6F_5 \left[\begin{matrix} \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{4}{3} - \frac{\sqrt{7}}{12}, \frac{4}{3} + \frac{\sqrt{7}}{12} \\ 1, 1, \frac{3}{2}, \frac{1}{3} - \frac{\sqrt{7}}{12}, \frac{1}{3} + \frac{\sqrt{7}}{12} \end{matrix} \middle| \frac{1}{4} \right] \right),$$

which gives us the equality

$$\frac{8\sqrt{2}}{3\pi} = {}_6F_5 \left[\begin{matrix} \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{4}{3} - \frac{\sqrt{7}}{12}, \frac{4}{3} + \frac{\sqrt{7}}{12} \\ 1, 1, \frac{3}{2}, \frac{1}{3} - \frac{\sqrt{7}}{12}, \frac{1}{3} + \frac{\sqrt{7}}{12} \end{matrix} \middle| \frac{1}{4} \right],$$

and this is equivalent to the formula given as Example 82 in [12] and reproved in [9] and reproduced in (2.10). Our WZ proof for this formula motivates our further applying Theorem 2.2 in the determination of new series for $\frac{\sqrt{2}}{\pi}$

of convergence rate $\frac{1}{4}$. This is also inspired by the remarkable result

$$\frac{2\sqrt{2}}{\pi} = \sum_{n=0}^{\infty} \left(-\frac{1}{8}\right)^n \left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1 \end{matrix}\right]_n (6n+1)$$

proved by Guillera in [14] and later proved by Chu in [8].

Example 2.7. Setting $(k, p, c) = \left(-\frac{3}{4}, -\frac{3}{4}, \frac{3}{2}\right)$ in Theorem 2.2, we obtain the motivating example highlighted in (1.2).

Example 2.8. Setting $(k, p, c) = \left(-\frac{3}{4}, \frac{1}{4}, -\frac{1}{2}\right)$ in Theorem 2.2, we may obtain that:

$$\frac{32\sqrt{2}}{15\pi} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \left[\begin{matrix} -\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4} \\ \frac{1}{2}, 1, 2, 2 \end{matrix}\right]_n (1 - 32n - 48n^2).$$

Example 2.9. Setting $(k, p, c) = \left(-\frac{3}{4}, \frac{1}{4}, \frac{1}{2}\right)$ in Theorem 2.2, we may obtain that:

$$\frac{128\sqrt{2}}{3\pi} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \left[\begin{matrix} \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4} \\ 1, \frac{3}{2}, 2, 2 \end{matrix}\right]_n (48n^2 + 64n + 19).$$

Example 2.10. Setting $(k, p, c) = \left(-\frac{1}{4}, \frac{7}{4}, \frac{3}{2}\right)$ in Theorem 2.2, we may obtain that:

$$\frac{8\sqrt{2}}{\pi} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \left[\begin{matrix} -\frac{1}{4}, -\frac{1}{4}, \frac{5}{4}, \frac{5}{4} \\ 1, 1, 1, \frac{3}{2} \end{matrix}\right]_n (5 - 32n - 48n^2).$$

2.3. Chu-style series for $\frac{\sqrt{3}}{\pi}$. Setting

$$(2.13) \quad (k, p, c) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

in Theorem 2.2, this gives us a new and WZ-based proof of the result included as Example 81 in [12] and reproved by Chu in [9] and reproduced in (2.11). Explicitly, setting the parameters k , p , and c in Theorem 2.2 to be as in (2.13), this gives us that

$$\frac{729\sqrt{3}}{2\pi} - 162 = \sum_{n=0}^{\infty} \frac{3^{-6n} (27n^2 + 72n + 47) ((3n+2)!)^2}{(n+1)^2 (n!)^4 (2n+3)!},$$

and we may rewrite the right-hand side as

$$(2.14) \quad \frac{94}{3} {}_7F_6 \left[\begin{matrix} 1, \frac{4}{3}, \frac{4}{3}, \frac{5}{3}, \frac{5}{3}, \frac{7}{3} + \frac{\sqrt{3}}{9}, \frac{7}{3} - \frac{\sqrt{3}}{9} \\ 2, 2, 2, \frac{5}{2}, \frac{4}{3} - \frac{\sqrt{3}}{9}, \frac{4}{3} + \frac{\sqrt{3}}{9} \end{matrix} \middle| \frac{1}{4} \right],$$

and we may apply contiguity relations to reduce the ${}_7F_6$ -expression in (2.14) in the manner indicated as follows:

$$\frac{3807(\sqrt{3}-3)(3+\sqrt{3}) \left({}_6F_5 \left[\begin{matrix} \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{4}{3} - \frac{1}{3\sqrt{3}}, \frac{4}{3} + \frac{1}{3\sqrt{3}} \\ 1, 1, \frac{3}{2}, \frac{1}{3} - \frac{1}{3\sqrt{3}}, \frac{1}{3} + \frac{1}{3\sqrt{3}} \end{matrix} \middle| \frac{1}{4} \right] - 1 \right)}{(\sqrt{3}-12)(12+\sqrt{3})}.$$

This gives us that

$$\frac{729\sqrt{3}}{2\pi} = \frac{3807(\sqrt{3}-3)(3+\sqrt{3}) {}_6F_5 \left[\begin{matrix} \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{4}{3} - \frac{1}{3\sqrt{3}}, \frac{4}{3} + \frac{1}{3\sqrt{3}} \\ 1, 1, \frac{3}{2}, \frac{1}{3} - \frac{1}{3\sqrt{3}}, \frac{1}{3} + \frac{1}{3\sqrt{3}} \end{matrix} \middle| \frac{1}{4} \right]}{(\sqrt{3}-12)(12+\sqrt{3})},$$

which is equivalent to the Chu–Zhang formula in (2.11). Our new WZ proof of this formula motivates our further applications of the WZ method in the determination of new series for $\frac{\sqrt{3}}{\pi}$. This is also inspired by series formulas for $\frac{\sqrt{3}}{\pi}$ due to Chu as in the following result from [8]:

$$\frac{15\sqrt{3}}{\pi} = \sum_{n=0}^{\infty} \left(\frac{1}{16} \right)^n \left[\begin{matrix} \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \\ 1, 1, 1, \frac{11}{12}, \frac{17}{12} \end{matrix} \right]_n (135n^2 + 75n + 8).$$

Example 2.11. Setting $(k, p, c) = \left(-\frac{2}{3}, -\frac{2}{3}, \frac{4}{3}\right)$ in Theorem 2.2, we obtain the motivating example highlighted in (1.3).

Example 2.12. Setting $(k, p, c) = \left(-\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$ in Theorem 2.2, we may obtain that:

$$\frac{27\sqrt{3}}{16\pi} = \sum_{n=0}^{\infty} \left(\frac{1}{4} \right)^n \left[\begin{matrix} -\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}, \frac{1}{3} \\ \frac{1}{2}, 1, 2, 2 \end{matrix} \right]_n (1 - 18n - 27n^2).$$

Example 2.13. Setting $(k, p, c) = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$ in Theorem 2.2, we may obtain that:

$$\frac{81\sqrt{3}}{4\pi} = \sum_{n=0}^{\infty} \left(\frac{1}{4} \right)^n \left[\begin{matrix} \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \\ 1, \frac{3}{2}, 2, 2 \end{matrix} \right]_n (27n^2 + 36n + 11).$$

Example 2.14. Setting $(k, p, c) = \left(-\frac{1}{3}, \frac{5}{3}, \frac{2}{3}\right)$ in Theorem 2.2, we obtain the motivating example highlighted in (1.4).

2.4. Chu-style series for $\frac{\sqrt{2-\sqrt{2}}}{\pi}$. Setting

$$(2.15) \quad (k, p, c) = \left(\frac{1}{8}, \frac{1}{8}, \frac{3}{4} \right)$$

in Theorem 2.2, this can be used to obtain a new WZ proof of Chu's formula in

$$(2.16) \quad \frac{32\sqrt{2-\sqrt{2}}}{\pi} = \sum_{n=0}^{\infty} \left(\frac{1}{4} \right)^n \left[\begin{matrix} \frac{1}{8}, \frac{1}{8}, \frac{7}{8}, \frac{7}{8} \\ 1, 1, 1, \frac{3}{2} \end{matrix} \right]_n (192n^2 + 128n + 7),$$

which was highlighted as a main result in [9]. Explicitly, setting (k, p, c) to be as in (2.15), Theorem 2.2 gives us that

$$-\frac{4\left(32\sqrt{2-\sqrt{2}}-7\pi\right)\pi}{\sqrt{2}-2} = \sum_{n=0}^{\infty} \frac{(192n^2 + 512n + 327) \left((n + \frac{1}{8})!\right)^2 \left((n + \frac{7}{8})!\right)^2}{((n+1)!)^2(2n+3)!},$$

and we may rewrite the right-hand side as

$$\frac{109}{2} \left(\left(\frac{1}{8}\right)!\right)^2 \left(\left(\frac{7}{8}\right)!\right)^2 {}_7F_6 \left[\begin{matrix} 1, \frac{9}{8}, \frac{9}{8}, \frac{15}{8}, \frac{15}{8}, \frac{7}{3} + \frac{\sqrt{43}}{24}, \frac{7}{3} - \frac{\sqrt{43}}{24} \\ 2, 2, 2, \frac{5}{2}, \frac{4}{3} + \frac{\sqrt{43}}{24}, \frac{4}{3} - \frac{\sqrt{43}}{24} \end{matrix} \middle| \frac{1}{4} \right],$$

and contiguity relations then give us that this is equivalent to

$$\frac{1339392(\sqrt{43}-8)(8+\sqrt{43})\left(\left(\frac{1}{8}\right)!\right)^2\left(\left(\frac{7}{8}\right)!\right)^2}{49(\sqrt{43}-32)(32+\sqrt{43})} \left({}_6F_5 \left[\begin{matrix} \frac{1}{8}, \frac{1}{8}, \frac{7}{8}, \frac{7}{8}, \frac{4}{3} - \frac{\sqrt{43}}{24}, \frac{4}{3} + \frac{\sqrt{43}}{24} \\ 1, 1, \frac{3}{2}, \frac{1}{3} - \frac{\sqrt{43}}{24}, \frac{1}{3} + \frac{\sqrt{43}}{24} \end{matrix} \middle| \frac{1}{4} \right] - 1 \right),$$

which leads us to the formula

$$\frac{32\sqrt{2-\sqrt{2}}}{7\pi} = {}_6F_5 \left[\begin{matrix} \frac{1}{8}, \frac{1}{8}, \frac{7}{8}, \frac{7}{8}, \frac{4}{3} - \frac{\sqrt{43}}{24}, \frac{4}{3} + \frac{\sqrt{43}}{24} \\ 1, 1, \frac{3}{2}, \frac{1}{3} - \frac{\sqrt{43}}{24}, \frac{1}{3} + \frac{\sqrt{43}}{24} \end{matrix} \middle| \frac{1}{4} \right],$$

which is equivalent to the formula in (2.16) introduced in [9].

Example 2.15. Setting $(k, p, c) = \left(-\frac{7}{8}, -\frac{7}{8}, \frac{7}{4}\right)$ in Theorem 2.2, we obtain the motivating example highlighted in (1.5).

Example 2.16. Setting $(k, p, c) = \left(-\frac{7}{8}, \frac{1}{8}, -\frac{1}{4}\right)$ in Theorem 2.2, we may obtain that:

$$\frac{256\sqrt{2-\sqrt{2}}}{63\pi} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \left[\begin{matrix} -\frac{1}{8}, -\frac{1}{8}, \frac{1}{8}, \frac{1}{8} \\ \frac{1}{2}, 1, 2, 2 \end{matrix} \right]_n (1 - 128n - 192n^2).$$

Example 2.17. Setting $(k, p, c) = \left(-\frac{7}{8}, \frac{1}{8}, \frac{3}{4}\right)$ in Theorem 2.2, we may obtain that:

$$\frac{2048\sqrt{2-\sqrt{2}}}{7\pi} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \left[\begin{matrix} \frac{1}{8}, \frac{1}{8}, \frac{7}{8}, \frac{7}{8} \\ 1, \frac{3}{2}, 2, 2 \end{matrix} \right]_n (192n^2 + 256n + 71).$$

Example 2.18. Setting $(k, p, c) = \left(-\frac{1}{8}, \frac{15}{8}, \frac{5}{4}\right)$ in Theorem 2.2, we may obtain that:

$$\frac{32\sqrt{2-\sqrt{2}}}{\pi} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \left[\begin{matrix} -\frac{1}{8}, -\frac{1}{8}, \frac{9}{8}, \frac{9}{8} \\ 1, 1, 1, \frac{3}{2} \end{matrix} \right]_n (9 - 128n - 192n^2).$$

2.5. Chu-style series for $\frac{\sqrt{2+\sqrt{2}}}{\pi}$. Setting $(k, p, c) = (\frac{3}{8}, \frac{3}{8}, \frac{1}{4})$ in Theorem 2.2, this gives us the equality

$$\frac{128\pi \left(1 - \frac{15\pi}{32\sqrt{2+\sqrt{2}}}\right)}{\sqrt{2+\sqrt{2}}} = \sum_{n=0}^{\infty} \frac{(335 + 512n + 192n^2) \left((n + \frac{3}{8})!\right)^2 \left((n + \frac{5}{8})!\right)^2}{((n+1)!)^2 (2n+3)!},$$

and we rewrite the right-hand side as

$$\frac{335}{6} \left(\left(\frac{3}{8}\right)!\right)^2 \left(\left(\frac{5}{8}\right)!\right)^2 {}_7F_6 \left[\begin{matrix} 1, \frac{11}{8}, \frac{11}{8}, \frac{13}{8}, \frac{13}{8}, \frac{7}{3} + \frac{\sqrt{19}}{24}, \frac{7}{3} - \frac{\sqrt{19}}{24} \\ 2, 2, 2, \frac{5}{2}, \frac{4}{3} - \frac{\sqrt{19}}{24}, \frac{4}{3} + \frac{\sqrt{19}}{24} \end{matrix} \middle| \frac{1}{4} \right],$$

which we may reduce, via contiguity relations, to

$$\frac{274432 (\sqrt{19} - 8) (\sqrt{19} + 8) \left(\left(\frac{3}{8}\right)!\right)^2 \left(\left(\frac{5}{8}\right)!\right)^2}{45 (\sqrt{19} - 32) (\sqrt{19} + 32)} \left(-1 + {}_6F_5 \left[\begin{matrix} \frac{3}{8}, \frac{3}{8}, \frac{5}{8}, \frac{5}{8}, \frac{4}{3} - \frac{\sqrt{19}}{24}, \frac{4}{3} + \frac{\sqrt{19}}{24} \\ 1, 1, \frac{3}{2}, \frac{1}{3} - \frac{\sqrt{19}}{24}, \frac{1}{3} + \frac{\sqrt{19}}{24} \end{matrix} \middle| \frac{1}{4} \right] \right),$$

which gives us the equality

$$\frac{32\sqrt{2+\sqrt{2}}}{15\pi} = {}_6F_5 \left[\begin{matrix} \frac{3}{8}, \frac{3}{8}, \frac{5}{8}, \frac{5}{8}, \frac{4}{3} - \frac{\sqrt{19}}{24}, \frac{4}{3} + \frac{\sqrt{19}}{24} \\ 1, 1, \frac{3}{2}, \frac{1}{3} - \frac{\sqrt{19}}{24}, \frac{1}{3} + \frac{\sqrt{19}}{24} \end{matrix} \middle| \frac{1}{4} \right],$$

giving us a new WZ proof of the formula due to Chu from [9] such that

$$\frac{32\sqrt{2+\sqrt{2}}}{\pi} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \left[\begin{matrix} \frac{3}{8}, \frac{3}{8}, \frac{5}{8}, \frac{5}{8} \\ 1, 1, 1, \frac{3}{2} \end{matrix} \right]_n (192n^2 + 128n + 15).$$

Example 2.19. Setting $(k, p, c) = (-\frac{5}{8}, -\frac{5}{8}, \frac{5}{4})$ in Theorem 2.2, we obtain the motivating example highlighted in (1.6).

Example 2.20. Setting $(k, p, c) = (-\frac{5}{8}, \frac{3}{8}, -\frac{3}{4})$ in Theorem 2.2, we obtain that:

$$\frac{768\sqrt{2+\sqrt{2}}}{55\pi} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \left[\begin{matrix} -\frac{3}{8}, -\frac{3}{8}, \frac{3}{8}, \frac{3}{8} \\ \frac{1}{2}, 1, 2, 2 \end{matrix} \right]_n (9 - 128n - 192n^2).$$

Example 2.21. Setting $(k, p, c) = (-\frac{5}{8}, \frac{3}{8}, \frac{1}{4})$ in Theorem 2.2, we may obtain that:

$$\frac{2048\sqrt{2+\sqrt{2}}}{15\pi} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \left[\begin{matrix} \frac{3}{8}, \frac{3}{8}, \frac{5}{8}, \frac{5}{8} \\ 1, \frac{3}{2}, 2, 2 \end{matrix} \right]_n (192n^2 + 256n + 79).$$

Example 2.22. Setting $(k, p, c) = (-\frac{3}{8}, \frac{13}{8}, \frac{7}{4})$ in Theorem 2.2, we may obtain that:

$$\frac{32\sqrt{2+\sqrt{2}}}{\pi} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \left[\begin{matrix} -\frac{3}{8}, -\frac{3}{8}, \frac{11}{8}, \frac{11}{8} \\ 1, 1, 1, \frac{3}{2} \end{matrix} \right]_n (33 - 128n - 192n^2).$$

2.6. A new proof of Guillera's formula. The formula in (1.8) due to Guillera in [15] may be written in so that

$$(2.17) \quad \frac{\pi^2}{4} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \left[\begin{matrix} 1, 1, 1 \\ \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \end{matrix} \right]_n (3n+2),$$

and this equivalent formulation was later proved in [12]. Guillera proved (2.17) by setting

$$F(n, k) = 8B(n, k)n$$

and

$$G(n, k) = B(n, k)(6n + 4k + 1),$$

where

$$B(n, k) = \frac{1}{2^{8n} 2^{4k}} \frac{((2k)!)^2 ((2n)!)^3}{((n+k)!)^2 (k!)^2 (n!)^4},$$

so as to obtain the acceleration formula

$$8a \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2}{(a+1)_n^2} = \sum_{n=0}^{\infty} \frac{1}{2^{2n}} \frac{\left(a + \frac{1}{2}\right)_n^3}{(a+1)_n^3} (6(n+a) + 1)$$

for a parameter a [15].

Theorem 2.23. *Guillera's formula for π^2 holds.*

Proof. In Theorem 2.2, we set (k, p, c) as $(\frac{1}{2}, 1, -1)$. This directly gives us the formula

$$81 - \frac{27\pi^2}{4} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \left[\begin{matrix} 1, 1, 2 \\ \frac{5}{2}, \frac{5}{2}, \frac{5}{2} \end{matrix} \right]_n (6n^2 + 18n + 13).$$

So, it remains to prove that

$$(2.18) \quad 3 = \sum_{n=0}^{\infty} \left(\frac{4^{-n} (1)_n^2 (2)_n (13 + 18n + 6n^2)}{\left(\frac{5}{2}\right)_n^3 27} + \frac{4^{-n} (1)_n^2 (1)_n (3n + 2)}{\left(\frac{3}{2}\right)_n^3} \right).$$

Taking the finite sum corresponding to the above series, and letting the upper parameter be denoted as m , this finite sum may be evaluated as in with the following Mathematica output, and this may easily be verified inductively.

```
(2^(-8 - 2*m))*(-32375*Pi^(3/2)*Gamma[2 + m]^2*Gamma[3 + m]*
Gamma[(5 + 2*m)/2]^3 - 83600*m*Pi^(3/2)*Gamma[2 + m]^2*
Gamma[3 + m]*Gamma[(5 + 2*m)/2]^3 - 89490*m^2*Pi^(3/2)*
Gamma[2 + m]^2*Gamma[3 + m]*Gamma[(5 + 2*m)/2]^3 - 50852*m^3*
Pi^(3/2)*Gamma[2 + m]^2*Gamma[3 + m]*Gamma[(5 + 2*m)/2]^3 -
16184*m^4*Pi^(3/2)*Gamma[2 + m]^2*Gamma[3 + m]*Gamma[(5 + 2*
m)/2]^3 - 2736*m^5*Pi^(3/2)*Gamma[2 + m]^2*Gamma[3 + m]*
Gamma[(5 + 2*m)/2]^3 - 192*m^6*Pi^(3/2)*Gamma[2 + m]^2*
Gamma[3 + m]*Gamma[(5 + 2*m)/2]^3 - 35000*Pi^(3/2)*Gamma[2 +
m]^3*Gamma[(7 + 2*m)/2]^3 - 83000*m*Pi^(3/2)*Gamma[2 + m]^3*
```

$$\begin{aligned}
& \Gamma[(7 + 2m)/2]^3 - 78000m^2\pi^{(3/2)}\Gamma[2 + m]^3 * \\
& \Gamma[(7 + 2m)/2]^3 - 36320m^3\pi^{(3/2)}\Gamma[2 + m]^3 * \\
& \Gamma[(7 + 2m)/2]^3 - 8384m^4\pi^{(3/2)}\Gamma[2 + m]^3 * \\
& \Gamma[(7 + 2m)/2]^3 - 768m^5\pi^{(3/2)}\Gamma[2 + m]^3 * \\
& \Gamma[(7 + 2m)/2]^3 + 2097 \cdot 2^{(8 + 2m)}\Gamma[(5 + 2m)/2]^3 * \\
& \Gamma[(7 + 2m)/2]^3 + 1833 \cdot 2^{(9 + 2m)}m\Gamma[(5 + 2m)/2]^3 * \\
& \Gamma[(7 + 2m)/2]^3 + 297 \cdot 2^{(11 + 2m)}m^2\Gamma[(5 + 2m)/2]^3 * \\
& \Gamma[(7 + 2m)/2]^3 + 339 \cdot 2^{(9 + 2m)}m^3\Gamma[(5 + 2m)/2]^3 * \\
& \Gamma[(7 + 2m)/2]^3 + 9 \cdot 2^{(11 + 2m)}m^4\Gamma[(5 + 2m)/2]^3 * \\
& \Gamma[(7 + 2m)/2]^3 * \Gamma[(7 + 2m)/2]^3) / ((699 + 1222m + 792m^2 + 226m^3 + 24m^4)\Gamma[(5 + 2m)/2]^3 * \\
& \Gamma[(7 + 2m)/2]^3)
\end{aligned}$$

Setting $m \rightarrow \infty$, we may evaluate the resultant limit as in the left-hand side of (2.18). $\square$

By mimicking the uses of contiguity relations as above, special cases of Theorem 2.2 can also be applied to obtain new WZ proofs of Ramanujan's formula in (1.10), in much the same way as in Section 3 below. For example, the $(k, p, c) = (\frac{1}{2}, \frac{1}{2}, 1)$ case of Theorem 2.2 can be used to establish that

$$\frac{16}{\pi} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{3}{2} \\ 1, 1, 2 \end{matrix} \right]_n (12n^2 + 16n + 3),$$

and, by taking a vanishing linear combination with both sides of Ramanujan's formula in (1.10), the partial sums of the vanishing series we obtain are evaluable in closed form via Gosper's algorithm. A somewhat different approach, without the use of contiguity relations, is applied in Section 3 below to obtain a new WZ proof of Ramanujan's series for $\frac{1}{\pi}$ of convergence rate $\frac{1}{4}$.

3. SERIES OF CONVERGENCE RATE $\frac{1}{16}$

We proceed to consider the new generalization given below of a Catalan sum identity introduced in [11], letting $m \in \mathbb{N}$:

$$\begin{aligned}
(3.1) \quad & \sum_{k=0}^n \binom{2n}{2k} \binom{2n-2k}{n-k} \binom{2k}{k} \prod_{t=1}^m \frac{1}{(k+t)(n-k+t)} \\
& = \binom{2n}{n}^2 \prod_{t=1}^m \frac{(2n+2t-1)}{(n/2+t)(n+t)(n+2t-1)}.
\end{aligned}$$

The special case for $m = 1$ agrees with the summation identity in (1.9) given as part (a) of Theorem 1 in [11]. Our generalization of the Chu–Kiliç formula in (1.9) is of particular interest because of how we apply our WZ proof for (3.1) to obtain a series acceleration formula that we apply to obtain a new proof of Ramanujan's formula in (1.10). By rewriting the discrete products

in (3.1) using the Γ -function, and by rewriting the parameter m as a real variable c , this leads us to the reformulation in Theorem 3.1 below.

Theorem 3.1. *The identity*

$$(3.2) \quad \sum_{k=0}^n \frac{\binom{2n}{2k} \binom{2n-2k}{n-k} \binom{2k}{k}}{(1+k)_c (1-k+n)_c} = \frac{\binom{2n}{n}^2 \left(n + \frac{1}{2}\right)_c}{\left(\frac{n}{2} + 1\right)_c \left(\frac{n+1}{2}\right)_c (n+1)_c}$$

holds for $n \in \mathbb{N}_0$ and a parameter c such that c is not a negative integer.

Proof. In order to obtain a bivariate, hypergeometric expression on the right-hand side of (3.2), we need to consider the parity of $n \in \mathbb{N}_0$. For the even case, we apply the mapping $n \mapsto 2n$, and then divide the resultant summand of (3.2) by the corresponding hypergeometric evaluations, and we denote the resultant expression as $F(n, k)$, which is given by the following Maple input.

```
binomial(2*k,k)*binomial(4*n,2*k)*binomial(-2*k+4*n,-k+2*n)*
pochhammer(1+n,c)*pochhammer(1/2+n,c)*pochhammer(1+2*
n,c)/binomial(4*n,2*n)^2/pochhammer(1+k,c)/pochhammer(1/2+2*
n,c)/pochhammer(1-k+2*n,c)
```

So, by the WZ method, it remains to compute a WZ proof certificate $R(n, k)$ corresponding to $F(n, k)$. Such a proof certificate is as given by the following Maple output.

```
1/2*(4*c^3*k-12*c^3*n-8*c^2*k^2+48*c^2*k*n-80*c^2*n^2+4*c*k^3-
44*c*k^2*n+148*c*k*n^2-172*c*n^3+8*k^3*n-56*k^2*n^2+136*k*n^3-
120*n^4-10*c^3+38*c^2*k-126*c^2*n-34*c*k^2+230*c*k*n-400*c*
n^2+6*k^3-86*k^2*n+316*k*n^2-372*n^3-49*c^2+88*c*k-305*c*n-33*
k^2+242*k*n-426*n^2-76*c+61*k-213*n-39)*k*(k+c)/(1-k+2*n+
c)/(2-k+2*n+c)/(1+4*n+2*c)/(3+4*n+2*c)/(-1+k-2*n)/(-2+k-2*n)
```

For the odd case for (3.2), we apply the mapping $n \mapsto 2n + 1$ to both sides of (3.2). In this case, we set $F(n, k)$ as in the Maple input below.

```
binomial(2*k,k)*binomial(2+4*n,2*k)*binomial(-2*k+4*n+2,1-k+2*
n)*pochhammer(1+n,c)*pochhammer(3/2+n,c)*pochhammer(2+2*
n,c)/binomial(2+4*n,1+2*n)^2/pochhammer(1+k,c)/pochhammer(3/2+
2*n,c)/pochhammer(-k+2+2*n,c)
```

In this case, the corresponding proof certificate is as below.

```
(2*c^3*k-6*c^3*n-4*c^2*k^2+24*c^2*k*n-40*c^2*n^2+2*c*k^3-22*c*
k^2*n+74*c*k*n^2-86*c*n^3+4*k^3*n-28*k^2*n^2+68*k*n^3-60*n^4-
8*c^3+31*c^2*k-103*c^2*n-28*c*k^2+189*c*k*n-329*c*n^2+5*k^3-
71*k^2*n+260*k*n^2-306*n^3-66*c^2+120*c*k-417*c*n-45*k^2+330*
k*n-582*n^2-175*c+139*k-489*n-153)*k*(k+c)/(2-k+2*n+c)/(3-k+2*
n+c)/(3+4*n+2*c)/(5+4*n+2*c)/(k-2*n-3)/(-2+k-2*n)
```

□

We apply the WZ pair associated with the even case of the above Theorem to prove the following transform, following a similar approach relative to [4]. As before, we may let $F(n, k)$ and $G(n, k)$ have non-integer arguments, writing

$$F(n, k) = 2^{-2c-4n} \sqrt{\pi} \Gamma \left[\begin{matrix} 2n+1, c+2n+1, 2c+2n+1 \\ k+1, c+k+1, c+2n+\frac{1}{2}, 2n-k+1, c+2n-k+1 \end{matrix} \right].$$

Theorem 3.2. *For real parameters k, p , and c such that $2p+1$ and $c+2p+1$ and $k+1$ and $c+k+1$ and $c+2p+\frac{1}{2}$ and $2p-k+1$ and $c-k+2p+1$ are not in $\mathbb{Z}_{\leq 0}$,*

$$\begin{aligned} & \Gamma \left[\begin{matrix} 2p+1, c+2p+1, 2c+2p+1 \\ k+1, c+k+1, c+2p+\frac{1}{2}, 2p-k+1, c-k+2p+1 \end{matrix} \right] \times \\ & {}_3F_2 \left[\begin{matrix} 1, k-2p, k-2p-c \\ k+1, c+k+1 \end{matrix} \middle| 1 \right] = \frac{2^{2c+4p}}{\sqrt{\pi}} + \frac{1}{\Gamma(k+1)\Gamma(c+k+1)} \times \\ & \sum_{n=0}^{\infty} \Gamma \left[\begin{matrix} 2n+2p+1, c+2n+2p+1, 2c+2n+2p+1 \\ c+2n+2p+\frac{1}{2}, 2n-k+2p+1, c-k+2n+2p+1 \end{matrix} \right] \times \\ & \frac{R(n+p, k)}{2^{4n}} \end{aligned}$$

if the above series converge, where R denotes the WZ proof certificate corresponding to the even case in the proof of Theorem 3.1.

Proof. We rewrite the difference equation in (1.15) so as to obtain that

$$(3.3) \quad F(n+p+1, k) - F(n+p, k) = G(n+p, k+1) - G(n+p, k)$$

for a real parameter p . Summing with respect to $n \in \mathbb{N}_0$, a telescoping sum obtained from the left-hand side of (3.3) gives us that

$$F(m+p+1, k) - F(p, k) = \sum_{n=0}^m (G(n+p, k+1) - G(n+p, k))$$

for $m \in \mathbb{N}_0$. Setting $m \rightarrow \infty$, we obtain that

$$-F(p, k) = \sum_{n=0}^{\infty} (G(n+p, k+1) - G(n+p, k)).$$

A telescoping phenomenon described in [3, 4, 13] then gives us that

$$\sum_{n=0}^{m-1} -F(p, k+n) = \sum_{n=0}^{\infty} (G(n+p, k+m) - G(n+p, k))$$

for $m \in \mathbb{N}_0$. Evaluating the infinite series $\sum_{n=0}^{\infty} -F(p, k+n)$ in terms of the above indicated ${}_3F_2(1)$ -series then gives us the desired result. We may obtain that $\lim_{m \rightarrow \infty} \sum_{n=0}^{\infty} G(n+p, k+m) = -1$ for real parameters k, p , and c , letting G be as in Theorem 3.2 as in our proof of Theorem 2.2. $\square$

3.1. A new proof of Ramanujan's formula. Our new WZ proof of Ramanujan's formula for $\frac{1}{\pi}$ is of combinatorial interest in terms of how it is derived, via Theorem 3.2, using a family of finite sums involving Catalan-type numbers and generalizing the Catalan sum given by Chu and Kılıç and reproduced in (1.9).

Theorem 3.3. *Ramanujan's formula in (1.10) holds.*

Proof. In Theorem 3.2, we set $(k, p, c) = (\frac{1}{2}, \frac{1}{4}, 1)$. From a vanishing upper parameter in the above ${}_3F_2$ -series, this series reduces to 1, so that

$$1 - \frac{15\pi}{64} = \frac{1}{2^{13}\sqrt{\pi}} \sum_{n=0}^{\infty} \left(\frac{1}{16}\right)^n \Gamma \left[\begin{matrix} 2n + \frac{3}{2}, 2n + \frac{3}{2}, 2n + \frac{3}{2} \\ 2n + 4, 2n + 4, 2n + 4 \end{matrix} \right] \times \\ (2n + 3)(4n + 3)^2(4n + 5)(3840n^4 + 19072n^3 + \\ 34880n^2 + 27784n + 8121).$$

According to the Legendre duplication formula in (1.11), the infinite series evaluation shown above is equivalent to

$$\frac{524288}{\pi} - 122880 = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{16n} \binom{4n}{2n}^3 \frac{1}{(n+1)^3(2n+1)^3(2n+3)^2} \times \\ (4n+1)^3(4n+3)^2(4n+5)(3840n^4 + 19072n^3 + \\ 34880n^2 + 27784n + 8121),$$

which, in turn, we rewrite as

$$(3.4) \quad \frac{4}{\pi} - \frac{15}{16} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{16n} \binom{4n}{2n}^3 \frac{1}{131072(n+1)^3(2n+1)^3(2n+3)^2} \times \\ (4n+1)^3(4n+3)^2(4n+5)(3840n^4 + 19072n^3 + \\ 34880n^2 + 27784n + 8121).$$

Applying a series bisection to the Ramanujan series in (1.10), we find that the series in (1.10) is equivalent to

$$(3.5) \quad \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{16n} \binom{4n}{2n}^3 \frac{3840n^4 + 5888n^3 + 3168n^2 + 672n + 39}{32(2n+1)^3},$$

so it remains to prove an evaluation for (3.5). So, since we have proved the equality in (3.4), it remains to prove that the expression

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{16n} \binom{4n}{2n}^3 \times \left(\frac{3840n^4 + 5888n^3 + 3168n^2 + 672n + 39}{32(2n+1)^3} - \frac{1}{131072(n+1)^3(2n+1)^3(2n+3)^2} \times (4n+1)^3(4n+3)^2(4n+5)(3840n^4 + 19072n^3 + 34880n^2 + 27784n + 8121) \right)$$

reduces to $\frac{15}{16}$. The corresponding finite sum

$$\sum_{n=0}^m \left(\frac{1}{2}\right)^{16n} \binom{4n}{2n}^3 \times \left(\frac{3840n^4 + 5888n^3 + 3168n^2 + 672n + 39}{32(2n+1)^3} - \frac{1}{131072(n+1)^3(2n+1)^3(2n+3)^2} \times (4n+1)^3(4n+3)^2(4n+5)(3840n^4 + 19072n^3 + 34880n^2 + 27784n + 8121) \right)$$

for $m \in \mathbb{N}_0$ admits the closed form for $m \in \mathbb{N}_0$ given by the following Mathematica output, and this may be checked using induction, noting that the above summand is univariate.

$$(2^{(-20 - 16*m)}*(135*2^{(16 + 16*m)} + 45*2^{(18 + 16*m)}*m + 15*2^{(18 + 16*m)}*m^2 - 1247*\text{Binomial}[4*(1 + m), 2*(1 + m)]^3 - 2224*m*\text{Binomial}[4*(1 + m), 2*(1 + m)]^3 - 736*m^2*\text{Binomial}[4*(1 + m), 2*(1 + m)]^3 + 512*m^3*\text{Binomial}[4*(1 + m), 2*(1 + m)]^3 + 256*m^4*\text{Binomial}[4*(1 + m), 2*(1 + m)]^3))/(3 + 2*m)^2$$

Setting $m \rightarrow \infty$, it is almost immediate that the limit of the above closed form gives the desired evaluation as $\frac{15}{16}$. $\square$

In contrast to the above WZ proof, Guillera's WZ proof in [14] of Ramanujan's formula in (1.10) was obtained via Zeilberger's EKHAD package by setting

$$G_1(n, k) = \frac{(-1)^n(-1)^k}{2^{6n}2^{2k}} \frac{\binom{2n}{n}^3 \binom{2k}{k}^2}{2^{2k} \binom{n-1/2}{k} \binom{n+k}{n}} (4n+1)$$

with $\frac{4n^2}{(4n+1)(2n-2k-1)}$ as a rational certificate, so that by setting $G_2(n, k) = F_1(n+1, n+k) + G_1(n, n+k)$, a theorem due to Zeilberger allows us to write $\sum_{n=0}^{\infty} G_2(n, k) = \sum_{n=0}^{\infty} G_1(n, k) = \text{constant}$, so that (1.10) may be obtained by setting $k = 0$.

3.2. Another proof of Guillera's formula. Theorem 3.2 may be applied to formulate another new WZ proof of Guillera's formula for π^2 , in addition to the proof introduced in Section 2.6. In particular, by setting $(k, p, c) = (\frac{1}{2}, 0, 0)$ in Theorem 3.2, we can show that π^2 equals

$$\frac{8}{27} \sum_{n=0}^{\infty} \left(\frac{1}{16} \right)^n \left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, 1, 1 \\ \frac{5}{4}, \frac{5}{4}, \frac{5}{4}, \frac{7}{4}, \frac{7}{4}, \frac{7}{4} \end{matrix} \right]_n (240n^4 + 608n^3 + 564n^2 + 225n + 32),$$

and this is easily seen to be equivalent to a series bisection applied to Guillera's formula.

3.3. Further results. By setting $p = \frac{k}{2} - \frac{m}{2}$ and $c = -1 + 2m$ and $k = 1$ in Theorem 3.2, and by invoking Gauss's ${}_2F_1(1)$ -identity, we can show that

$$\frac{1}{\binom{2m}{m}} = \sum_{n=0}^{\infty} \frac{m! (7m^2 + 52mn + 29m + 60n^2 + 62n + 12) (3m + 2n - 1)!}{(3m - 1)! (m + 2n + 2)! \binom{2m+4n+4}{m+2n+2}}.$$

This may be of interest in the context of double sum identities as in

$$\frac{17\pi^4}{4860} = \sum_{\substack{m \geq 1 \\ n \geq 0}} \frac{\binom{3m+2n-1}{2m-2}}{\binom{3m}{m} \binom{2m+4n+4}{m+2n+2}} \frac{7m^2 + 52mn + 29m + 60n^2 + 62n + 12}{m^4(2m-1)(m+2n+2)}.$$

Theorem 3.2 may also be applied to obtain series of convergence rate $\frac{1}{16}$ that are of a similar quality relative to results introduced in [12]. For example, by setting $(k, p, c) = (\frac{1}{6}, \frac{1}{6}, \frac{2}{3})$ and $(k, p, c) = (\frac{7}{6}, \frac{1}{6}, \frac{2}{3})$ and by taking an appropriate linear combination, we can show that

$$\begin{aligned} \frac{\pi}{\sqrt{3}} &= \frac{16}{1276275} \sum_{n=0}^{\infty} \left(\frac{1}{16} \right)^n \left[\begin{matrix} \frac{2}{3}, 1, \frac{7}{6}, \frac{4}{3}, \frac{3}{2}, \frac{11}{6} \\ \frac{19}{12}, \frac{7}{4}, \frac{23}{12}, \frac{25}{12}, \frac{9}{4}, \frac{29}{12} \end{matrix} \right]_n \times \\ &\quad (155520n^6 + 917568n^5 + 2220048n^4 + \\ &\quad 2821680n^3 + 1990500n^2 + 741277n + \\ &\quad 114432), \end{aligned}$$

and this recalls the formula

$$\frac{\pi}{\sqrt{3}} = \frac{3}{20} \sum_{n=0}^{\infty} \left(\frac{1}{16} \right)^n \left[\begin{matrix} 1, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4} \\ \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{7}{6}, \frac{11}{6} \end{matrix} \right]_n (5n+3)(12n^2+15n+4)$$

introduced in [12].

4. FURTHER CATALAN SUMS

In this concluding section, we introduce further summations that are terminating and involve Catalan or Catalan-type numbers, inspired by the results from [11] and our above applications of the finite summation identities generalizing (1.7) and (1.9).

4.1. Identities for which state-of-the-art implementations of the WZ method do not apply. Let us consider the following natural-looking summation identity involving even-indexed Catalan numbers:

$$(4.1) \quad \sum_{k=0}^n \binom{4n-4k}{2n-2k} C_{2k} = 4^n \binom{2n}{n}.$$

Given the simplicity of the above summand, which consists only of a factor given by a hypergeometric binomial expression and an even-indexed Catalan number, it is quite remarkable that the 2022 version of Maple is not able to apply the WZ method to prove (4.1). This is representative of the computationally challenging nature of our results. Without a WZ pair that both satisfies the defining difference equation in (1.15) and corresponds to the hypergeometric identity in (4.1), we cannot apply acceleration methods as in Sections 2 and 3. So, we instead use Zeilberger's algorithm [19, §6], as opposed to the WZ method, to prove (4.1).

Theorem 4.1. *The hypergeometric identity in (4.1) holds true.*

Proof. We set $F(n, k)$ as the summand of (4.1) divided by the right-hand side of (4.1). We set $R(n, k)$ as the rational function given by the expression

$$\frac{k(2k+1)(n+1)(4n-4k+3)(4k-4n-1)(16k^2-4k(4n+7)+2n+1)}{16(2n+1)(2n+3)(2k-2n-3)(2k-2n-1)(k-n-2)(k-n-1)},$$

and we set $G(n, k) = F(n, k)R(n, k)$. We may then verify that the discrete difference equation

$$(4n^2 + 16n + 15)F(n+2, k) + (-8n^2 - 24n - 19)F(n+1, k) + (4n^2 + 8n + 4)F(n, k) = G(n, k+1) - G(n, k)$$

holds true. So, summing both sides with respect to k , a telescoping phenomenon yields

$$\begin{aligned} & (4n^2 + 16n + 15) \sum_{k=0}^{n-1} F(n+2, k) + \\ & (-8n^2 - 24n - 19) \sum_{k=0}^{n-1} F(n+1, k) + \\ & (4n^2 + 8n + 4) \sum_{k=0}^{n-1} F(n, k) = G(n, n). \end{aligned}$$

By rewriting this difference equation to obtain a recursion for $\sum_{k=0}^n F(n, k)$, a routine induction argument then gives us the desired result. $\square$

With regard to (4.1), identities of a similar nature are included as Theorems below. Again, the Maple implementation of the WZ method cannot, remarkably, be applied to obtain any WZ proof for the finite summation identities shown below.

Theorem 4.2. *The identity*

$$(4.2) \quad \sum_{k=0}^n \frac{C_{n-k} \binom{2k+2m+1}{k+2m+1}}{k+2m+2} = \frac{(2m+3) \binom{2m+2n+2}{n}}{(2m+2)(2m+n+3)}$$

holds true for $n \in \mathbb{N}_0$.

Proof. We let $F(n, k)$ denote the hypergeometric function given by the summand of (4.2) divided by the right-hand side of (4.2). We let $R(n, k)$ denote the rational function

$$\frac{k(n+1)(k+2m+2)(2k-2n-1)(2k+2m^2+3m-3n-4)}{(k-n-2)(m+n+2)(2m+2n+3)}$$

and we write $G(n, k) = R(n, k)F(n, k)$. We may verify that the difference equation

$$\begin{aligned} & (-2m^2n - 4m^2 - 3mn + n^2 - 6m + 2n) F(n+1, k) + \\ & (2m^2n + 2m^2 + 3mn - n^2 + 3m - 2n - 1) F(n, k) \\ & G(n, k+1) - G(n, k) \end{aligned}$$

holds. Summing both sides of this discrete difference equation with respect to k , a telescoping effect yields

$$\begin{aligned} & (-2m^2n - 4m^2 - 3mn + n^2 - 6m + 2n) \sum_{k=0}^{n-1} F(n+1, k) + \\ & (2m^2n + 2m^2 + 3mn - n^2 + 3m - 2n - 1) \sum_{k=0}^{n-1} F(n, k) = \\ & G(n, n). \end{aligned}$$

Rewriting this difference equation so as to obtain a recursion for $\sum_{k=0}^n F(n, k)$, a routine induction argument then gives us the desired result. $\square$

As indicated above, the Maple implementation of the WZ method also cannot be applied to the following.

Theorem 4.3. *The identity*

$$(4.3) \quad \sum_{k=0}^n \frac{C_{n-k} \binom{2k+2m}{k+2m}}{k+2m+1} = \frac{(2m+2) \binom{2n+2m+1}{n}}{(2m+1)(n+2m+2)}$$

holds true for $n \in \mathbb{N}_0$.

Proof. We set $F(n, k)$ as being given by the summand of (4.3) divided by the right-hand side of (4.3), and we set $R(n, k)$ as the below indicated rational function:

$$\frac{k(n+1)(k+2m+1)(2k-2n-1)(2k+2m^2+m-3n-5)}{(k-n-2)(m+n+1)(2m+2n+3)}.$$

We also set $G(n, k) = F(n, k)R(n, k)$. We may verify that the following difference equation holds:

$$\begin{aligned} & (-2m^2n - 4m^2 - mn + n^2 - 2m + 3n + 2) F(n+1, k) + \\ & (2m^2n + 2m^2 + mn - n^2 + m - 3n - 2) F(n, k) = \\ & G(n, k+1) - G(n, k). \end{aligned}$$

Summing both sides with respect to k , we again find a telescoping phenomenon, and this gives us that:

$$\begin{aligned} & (-2m^2n - 4m^2 - mn + n^2 - 2m + 3n + 2) \sum_{k=0}^{n-1} F(n+1, k) + \\ & (2m^2n + 2m^2 + mn - n^2 + m - 3n - 2) \sum_{k=0}^{n-1} F(n, k) = \\ & G(n, n). \end{aligned}$$

By rewriting this difference equation so as to produce a recurrence relation for $\sum_{k=0}^n F(n, k)$, and again we may apply a routine inductive argument. $\square$

4.2. Another generalization of Catalan numbers. Our computational experiments based on further generalizations of Catalan numbers have led us to discover the following result.

Theorem 4.4. *The identity*

$$(4.4) \quad \sum_{k=0}^n \binom{2n-2k}{n-k} \binom{2k+2m-1}{k+2m-1} \frac{2m}{2m+k} = \binom{2n+2m}{n}$$

holds for $n \in \mathbb{N}_0$.

Proof. Let $F(n, k)$ denote the summand on the left of (4.4) divided by the right-hand side of (4.4). The WZ method produces the proof certificate

$$R(n, k) = \frac{k(-2c-k)(2k-2n-1)}{(c+n+1)(2c+2n+1)(k-n-1)},$$

with $G(n, k) = R(n, k)F(n, k)$. $\square$

By writing

$$F(m+p+1, k) - F(p, k) = \sum_{n=0}^m (G(n+p, k+1) - G(n+p, k))$$

for the WZ pair (F, G) as in the proof of Theorem 4.4, we find that the expression

$$\lim_{m \rightarrow \infty} F(m + p + 1, k)$$

is not, in general, vanishing, so that we cannot apply the same kinds of WZ-based acceleration techniques as above. For the same WZ pair (F, G) as in Theorem 4.4, we find that

$$(4.5) \quad \frac{2^{-1-2c-2k}(1+2c)\Gamma(1+2c+2k)}{\Gamma(1+k)\Gamma(2+2c+k)} - F(p, k) = \sum_{n=0}^{\infty} (G(n+p, k+1) - G(n+p, k)).$$

We encourage the exploration as to how (4.5) may be applied for the purposes of series accelerations.

4.3. On the convolution of even-indexed Catalan numbers.

Theorem 4.5. *The identity*

$$(4.6) \quad \sum_{k=0}^n C_{2n-2k} C_{2k} = 4^n C_n$$

holds for $n \in \mathbb{N}_0$.

Proof. Let $F(n, k)$ denote the summand on the left of (4.6) divided by the right-hand side of (4.6). The WZ method yields the proof certificate

$$R(n, k) = \frac{k(2k+1)(4k-4n-3)(4k-4n-1)(2k-3n-4)}{4(n+1)(2n+1)(2n+3)(2k-2n-3)(k-n-1)},$$

again with $F(n, k) = R(n, k)G(n, k)$. $\square$

In this case, we cannot apply the same kinds of acceleration methods from the previous sections, since

$$(4.7) \quad \frac{2^{-\frac{3}{2}-4k} \binom{4k}{2k}}{1+2k} - F(p, k) = \sum_{n=0}^{\infty} (G(n+p, k+1) - G(n+p, k)).$$

We encourage the exploration as to how (4.7) may be applied for the purposes of series accelerations.

4.4. A generalization of an identity due to Mikić. In [18], Mikić proved that

$$\sum_{k=0}^n (-1)^k \binom{n}{k} C_{n-k} C_k = \frac{1}{\frac{n}{2} + 1} \left(\frac{n}{2} \right)^2.$$

We generalize this as below for a real parameter m .

Theorem 4.6. *The identity*

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{2n-2k}{n-k} \binom{2k}{k} \binom{n}{k} \prod_{t=1}^m \frac{1}{(k+t)(n-k+t)} \\ &= \begin{cases} \binom{n}{n/2}^2 \prod_{t=1}^m \frac{(n+2t-1)}{(n/2+t)(n+t)(2t-1)} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

holds for $n \in \mathbb{N}_0$.

Proof. We begin by rewriting the finite product $\prod_{t=1}^m \frac{(n+2t-1)}{(n/2+t)(n+t)(2t-1)}$ with the Pochhammer symbol. So, we may replace the integer parameter m with a real parameter c . Now, we are to apply the mapping $n \mapsto 2n$ and then set $F(n, k)$ as the resultant summand divided by the resultant closed form. We set $R(n, k)$ as the rational function given by the following Mathematica input.

```
(k*(c + k)*(1 - 2*k + 4*n)*(-5 - 2*c + c^2 + 5*k + 2*c*k -
k^2 - 23*n - 9*c*n + 16*k*n + 4*c*k*n - 2*k^2*n - 34*n^2 - 8*
c*n^2 + 12*k*n^2 - 16*n^3))/(2*(-2 + k - 2*n)*(-1 + k - 2*n)*
(1 + 2*n)*(1 + 2*c + 2*n)*(1 + c - k + 2*n)*(2 + c - k + 2*n))
```

Setting $G(n, k) = F(n, k)R(n, k)$, this gives us a WZ pair. For the vanishing case, we may again apply Zeilberger's algorithm. Explicitly, we set $F(n, k)$ as

```
((-1)^k*Binomial[2*k, k]*Binomial[1 + 2*n, k]*Binomial[-2*k +
2*(1 + 2*n), 1 - k + 2*n])/(Pochhammer[1 + k, c]*
Pochhammer[2 - k + 2*n, c])
```

and $R(n, k)$ as

```
(8*k*(c + k)*(-3 + 2*k - 4*n)*(1 + n)*(54 + 17*c - 2*c^2 - 32*
k - 8*c*k + 4*k^2 + 138*n + 34*c*n - 56*k*n - 8*c*k*n + 4*k^2*
n + 116*n^2 + 16*c*n^2 - 24*k*n^2 + 32*n^3))/((-3 + k - 2*n)*
(-2 + k - 2*n)*(-3 - c + k - 2*n)*(-2 - c + k - 2*n))
```

and $G(n, k) = F(n, k)R(n, k)$ and we may verify that the difference equation whereby

```
(-128 - 128*c - 384*n - 256*c*n - 384*n^2 - 128*c*n^2 - 128*
n^3)*F[n, k] + (18 + 27*c + 13*c^2 + 2*c^3 + 42*n + 42*c*n +
10*c^2*n + 32*n^2 + 16*c*n^2 + 8*n^3)*F[1 + n, k]
```

equals $G(n, k+1) - G(n, k)$ holds. $\square$

For the WZ pair (F, G) involved in our proof of Theorem 4.6, the desired vanishing condition for $\lim_{n \rightarrow \infty} F(n, k)$ does not, in general, hold, so that our acceleration methods cannot be applied directly.

A result of a similar nature relative to Theorem 4.6 that we have discovered experimentally is given below:

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{2n}{n-k} \binom{n+k}{2k} \binom{2k}{k} \prod_{t=1}^m \frac{1}{(k+t)(n-k+t)} \\ &= \begin{cases} (-1)^{\binom{n}{2}} \binom{n}{n/2} \binom{2n}{n} \prod_{t=1}^m \frac{1}{(n+t)(n/2+t)} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

This may be proved in much the same way as in Theorem 4.6.

4.5. Accelerations yielding series of convergence rate $\frac{1}{27}$. For even n ,

$$\begin{aligned} & \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \binom{4n}{2k} \binom{2k}{k} \binom{4n-2k}{2n-k} \prod_{t=1}^m \frac{1}{(k+t)(2n-k+t)} \\ &= \frac{(-1)^n (3n+m)! (4n)!}{(n+m)! (2n+m)! (2n+m)! n! n!}, \end{aligned}$$

which, by taking $m = 1$, generalizes the result of Theorem 3(a) in [11]:

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2n}{2k} C_k C_{n-k} = \begin{cases} (-1)^{\binom{n}{2}} \binom{3n/2+1}{n/2} C_n C_{n/2} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

In a separate article [2], a closely related generalization of Theorem 3(a) in [11] was applied to obtain the series of convergence rate $\frac{1}{27}$. We may obtain a similar series of convergence rate $\frac{1}{27}$ using the WZ pair associated with the above displayed generalization of Theorem 3(a) in [11]. For the sake of brevity, we refer to [2] for details, and we leave it to a separate project to further explore further generalizations of Theorem 3(a) in [11].

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DEPARTMENT OF MATHEMATICS, TORONTO METROPOLITAN UNIVERSITY, 350
 VICTORIA STREET, TORONTO, ON, CANADA
E-mail address: jmaxwellcampbell@gmail.com

DEPARTMENT OF MATHEMATICS, TOBB UNIVERSITY OF ECONOMICS AND
 TECHNOLOGY, SOĞUTOZU AV. 06560, ANKARA, TURKEY

DEPARTMENT OF TECHNICAL SCIENCES, WESTERN CASPIAN UNIVERSITY, BAKU,
 AZERBAIJAN
E-mail address: ekilic@etu.edu.tr