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THE EXISTENCE OF A HYPERSOLID IN $\mathbb{E}^d$ WHOSE HEESCH NUMBER IS $d-1$

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Abstract. In a recent article, it was shown that the Heesch number in $\mathbb{E}^d$ is asymptotically unbounded for $d \to \infty$, by showing that, for each d of the form 2^k, there exists a hypersolid in $\mathbb{E}^d$ whose Heesch number equals $d-1$. We here show that the same holds not only for d of the form 2^k, but for any d, $d \geqslant 2$.

1. Introduction

Within combinatorial geometry, problems on *tessellations* (or *tilings*) occupy one of the central spots. A tessellation of the Euclidean plane $\mathbb{E}^2$ is defined as a set $\mathscr{T}$ consisting of closed topological discs, where the elements of $\mathscr{T}$ have pairwise disjoint interiors and $\bigcup \mathscr{T} = \mathbb{E}^2$. The elements of $\mathscr{T}$ are called *tiles*. The monograph [8] is taken for a very thorough compendium of various problems of tilings, as well as the theoretical background.

The *Heesch number* of a figure (introduced in [9]) represents a kind of measure that expresses, loosely speaking, how "far" we can advance toward a tiling of the whole plane using the given figure (the greater the Heesch number is, the "further" we can advance; and the Heesch number is infinite if and only if the plane can be tiled by congruent copies of the given figure). In an intuitive sense (it will be formally defined in the following section), the Heesch number counts the number of times the given figure can be completely surrounded by its congruent copies. See Figure 1 for an example of a figure whose Heesch number is 3. This figure, being the first such constructed example, is often shown in the literature, and it is attributed to Ammann, but the source of its original publication is somewhat harder to trace. As per [11], we learn that the original source is [2].

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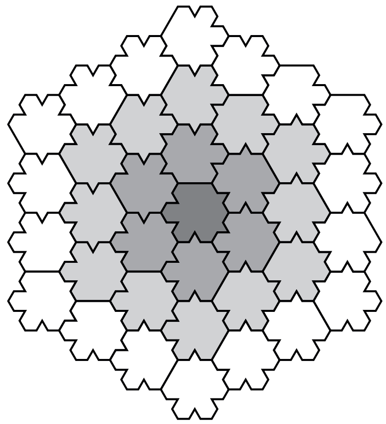

Figure 1. A figure with Heesch number 3.

The question of whether the set of all possible finite Heesch numbers is bounded from above is known as *Heesch's problem.* For almost full 20 years, the "record-holder" (in the Euclidean plane) had been a figure whose Heesch number is 5 [10], which was finally surpassed two years ago, when a figure whose Heesch number is 6 has been discovered [3]. Some different versions of the problem, such as the problem posed in the hyperbolic plane, and the problem posed for sets of more figures, have been solved ([12], respectively [4, 5]); in both these cases, it turns out that the upper bound does not exist, that is, it is possible to construct a figure (respectively a set of figures) whose Heesch number is as large as we please.

Until recently, all the research on the Heesch number has been done pretty much exclusively within the two-dimensional space, that is, the plane (however, see [1] for an attempt to extend this concept to arbitrary finitely generated groups and associated Cayley graphs). In [6], d-dimensional Heesch's problem is solved in the asymptotic sense. Namely, it is shown that, if we let $d \to \infty$, then there is no uniform upper bound on the set of all possible finite Heesch numbers in the space $\mathbb{E}^d$; in other words, given any nonnegative integer n, we can find a dimension d (depending on n) in which there exists a hypersolid whose Heesch number is finite and greater than n. In particular, it is shown that, for each d of the form 2^k, there exists a hypersolid in $\mathbb{E}^d$ whose Heesch number equals $d-1$. The existence of hypersolids with Heesch number $d-1$ (or larger) in dimensions d not of the form 2^k was left as an open question. We here complete the picture by showing that for any d, $d \geqslant 2$, there exists a hypersolid in $\mathbb{E}^d$ whose Heesch number equals $d-1$.

An extended abstract of this article has been previously published in [7].

2. Definitions and results

We first give a formal definition of the Heesch number.

Definition 2.1. We say that a hypersolid C (a topological d-ball) in $\mathbb{E}^d$ can be *surrounded* n *times* if and only if there exist finite collections $\mathscr{C}_1, \mathscr{C}_2, \dots,$ $\mathscr{C}_n$ of isometric copies of C such that:

- every two different hypersolids from $\{C\} \cup \bigcup_{i=1}^{n} \mathscr{C}_i$ have disjoint interiors;
- for each i, $1 \leqslant i \leqslant n$, each hypersolid from $\mathscr{C}_i$ has a common boundary point with some hypersolid from $\mathscr{C}_{i-1}$ (where by convention, we let $\mathscr{C}_0 = \{C\}$);
- for each i, $1 \leqslant i \leqslant n$, $\bigcup \left(\bigcup_{j=0}^{i} \mathscr{C}_j \right)$ is a closed topological d-ball such that $\bigcup \left(\bigcup_{j=0}^{i-1} \mathscr{C}_j \right)$ is completely contained in its interior.

The collection $\mathscr{C}_i$ is called the i^{th} *corona*.

Definition 2.2. The *Heesch number* of a given hypersolid C (a topological d-ball) in $\mathbb{E}^d$ is the maximal nonnegative integer n such that C can be surrounded n times. If such a maximum does not exist, then we define the Heesch number to be infinite.

We shall now define a hypersolid in $\mathbb{E}^d$ whose Heesch number is $d-1$. This will be the same hypersolid defined in [6], for which it was shown there that, if $d = 2^k$, then its Heesch number equals $d-1$. For the reader's convenience, we repeat the definition here. We start from a unit hypercube, and mark some of its facets (which are $(d-1)$-dimensional unit hypercubes) by "bumps" and "nicks" (arranged in a particular way that will be described in a moment), where each bump matches each nick. In particular, each bump or nick can be taken to be a right hypercone whose base is an $(n-1)$-dimensional (small) hyperball placed in the center of a facet of the considered hypercube, and whose axis is orthogonal to the facet; we call *bumps*, respectively *nicks*, such hypercones erected outwards, respectively inwards (with respect to the interior of the considered hypercube).

Definition 2.3. A *basic hypercube* is a hypersolid obtained in the described way that has d facets with bumps and $d-1$ facets with nicks (and 1 facet not marked by either), where, additionally, all d facets with bumps have a common vertex.

It easily follows from the definition that any two basic hypercubes are isometric. Figure 2 presents a picture of a 3-dimensional basic hypercube and a 4-dimensional basic hypercube.

The following lemma is also taken from [6] (note that its proof there does not rely on the fact that $d = 2^k$, and thus the lemma is valid in any space $\mathbb{E}^d$).

Lemma 2.4. *The Heesch number of a basic hypercube is at most $d-1$.*

Therefore, we are left to show the other inequality, that is, to show how to surround a basic hypercube $d-1$ times by its isometric copies. The constraint $d = 2^k$ has an essential role in the construction from [6] and it

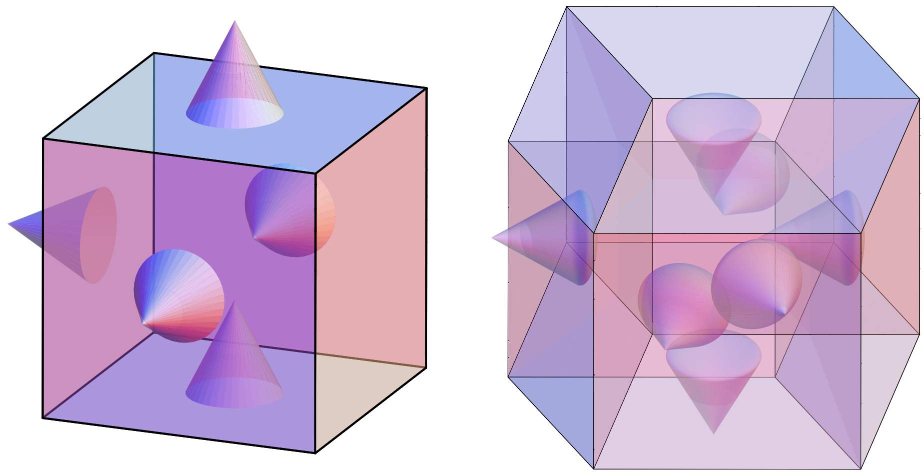

FIGURE 2. A 3-dimensional and a 4-dimensional basic hypercube.

does not seem possible to make some amendments to the idea from there that would eliminate this constraint. That is why we here devise a completely different approach, that is not dependent on the form of d.

We introduce two more types of marked hypercubes, called *neutral δ-hypercubes* and *spikey δ-hypercubes*, where $\delta \in \{1, 2, \dots, d\}$. Each of them is a δ-dimensional unit hypercube, with facets marked by bumps and nicks as follows. A *neutral δ-hypercube* has δ facets marked by bumps and the other δ facets marked by nicks, where all δ facets with bumps have a common vertex (and, clearly, the same holds for the δ facets with nicks). A *spikey δ-hypercube* can be obtained from a neutral δ-hypercube by replacing one nick by a bump (in other words, a spikey δ-hypercube has $\delta + 1$ facets marked by bumps and $\delta - 1$ facets marked by nicks, where all $\delta - 1$ facets with nicks have two common vertices).

Neutral/spikey δ-hypercubes will be represented by a $\delta \times 2$ matrix

$$\begin{bmatrix} b_{1,0} & b_{1,1} \\ b_{2,0} & b_{2,1} \\ \vdots & \vdots \\ b_{\delta,0} & b_{\delta,1} \end{bmatrix}$$

with $b_{i,j} \in \{1, -1\}$, which is interpreted as follows: the i^{th} row describes the two facets orthogonal to the i^{th} coordinate axis (it will always be the case that the edges of the considered hypercubes are parallel to the coordinate axes), in the order in which they are met when traveling the axis from $-\infty$ to ∞; $b_{i,j} = 1$, respectively -1, means that there is a bump, respectively nick, on the corresponding facet. Then neutral δ-hypercubes have exactly once 1 and once -1 in each row, while for spikey δ-hypercubes we have the same with exactly one exception, namely, there is exactly one row with two 1s.

Note: Actually, neutral 1-hypercubes and spikey 1-hypercubes do not have a geometrical interpretation, since bumps and nicks cannot be geometrically realized in one dimension. We can (intuitively) visualize them as shown in Figure 5, left-top (there are 16 of them shown there, where some are stacked together). In any case, this drawback does not affect the proof: we can simply work with them as abstract objects, using them to build some two-dimensional (and higher-dimensional) hypercubes in a way that will be seen, and this is when the geometrical interpretation will fall into place.

We shall show that a hypercube of side $2d$ in $\mathbb{E}^d$ can be arranged (up to bumps and nicks) from $(2d)^d$ basic hypercubes (this is actually more than we need; for our purpose, it would be enough to obtain a hypercube of side $2d-1$). Of course, in such an arrangement, every two basic hypercubes that have a common facet must have the corresponding facets marked in a matching way. Before we delve into all the wearisome ingredients of the proof, we present an overview of our construction.

A total of 2^d basic hypercubes can be stacked in such a way that they form twice as big hypercube, with each of its facets decorated by 2^{d-1} bumps/nicks, where each facet is marked either solely by bumps, or solely by nicks; furthermore, this structure resembles a spikey d-hypercube. And as it will turn out that d^d spikey d-hypercubes can be arranged in the form of a (hyper)cubical structure of side d, by rescaling that arrangement by the factor 2 and replacing each spikey d-hypercube by 2^d basic hypercubes (see the beginning of this paragraph), we get a hypercube of side $2d$ built from $(2d)^d$ basic hypercubes, which was needed. (Indeed, note that, if we now remove all the basic hypercubes that have a coordinate with the maximal possible value, that leaves a configuration in which the central basic hypercube is surrounded by $d-1$ coronas of basic hypercubes.) All this is what Lemma 2.8 is about. See Figure 3 for an illustration of the construction in 2D, and see Figure 4 for an illustration of the initial part of the construction in 3D (up to the point when spikey d-hypercubes are arranged to form the larger cube, that is, the phase that corresponds to the top row in Figure 3; the further steps would be rather unintelligible from a drawing, but for the reader to get some intuition, one spikey cube in the last illustration is shown replaced by 8 basic cubes).

This recap may seem quite straightforward, but there is one highly technical step contained within it: how to arrange d^d spikey d-hypercubes in a cubical form. We show this beforehand, and then in the proof of Lemma 2.8 just refer to this as a black box. And the outline of this step is as follows. Say that $d=3$. We need to build a $3\times3\times3$ cube. We shall build it by stacking three $3\times3\times1$ “blocks.” A 2-dimensional 3×3 projection of such a block is formed of some squares each of which has either 2 bumps and 2 nicks (it is *neutral*), or 3 bumps and 1 nick (it is *spikey*); then, when the thickness is added along the third coordinate (in order to make a $3\times3\times1$ block), each neutral square will have two bumps added along that coordinate, while each spikey square will have one bump and one nick added along that coordinate.

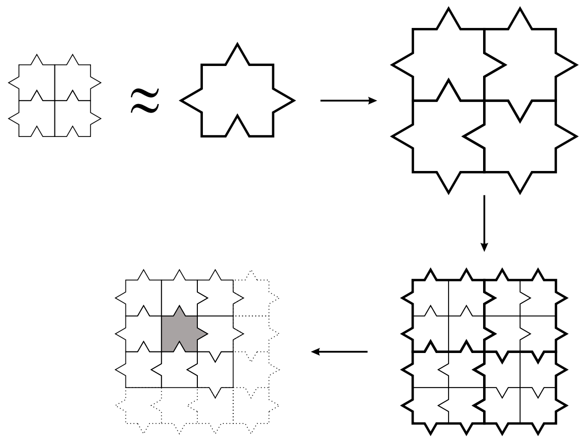

FIGURE 3. Overview of the construction for $d = 2$.

These projections are the motivation for the notion of *layers* introduced in Definition 2.5 (and, for each dimension δ, there will be considered a total of d different layers in dimension δ, called (δ, i)-layers for $i = 0, 1, \ldots, d-1$, which differ by positions of the neutral building units; the exact description of their positions with respect to the parameter i is given in Definition 2.5, though it is unfortunately quite unappealing and, seemingly, there is no getting around that). Anyway, we show that it is possible to build such 2-dimensional 3×3 layers by first constructing some 1-dimensional layers (of length 3) and then rely on them to obtain 2-dimensional 3×3 layers. Similarly, if $d = 4$, the 4-dimensional $4 \times 4 \times 4 \times 4$ hypercube is built by relying on 3-dimensional layers (which are $4 \times 4 \times 4$ cubes), which are in turn built by relying on 2-dimensional 4×4 layers etc. Lemma 2.7 plays the key role here: in it we demonstrate that all the considered layers indeed can be constructed, which is an inductive argument showing how to use smaller-dimensional layers to build larger-dimensional ones.

So, let us now move to the formalization of all this.

Definition 2.5. Let $\delta \in \{1, 2, \ldots, d\}$ and $i \in \{0, 1, \ldots, d-1\}$. A (δ, i)-*layer* is a δ-dimensional hypercube of side d, up to bumps and nicks, that satisfies the following two properties:

- it can be obtained by arranging a total of $(d-\delta)d^{\delta-1}$ neutral δ-hypercubes and $\delta d^{\delta-1}$ spikey δ-hypercubes;
- the structure can be placed in the integer grid, with centers of the constituting hypercubes at the coordinates $\{0, 1, \ldots, d-1\}^{\delta}$, in such a way that the constituting neutral δ-hypercubes are centered exactly at the coordinates (where the operations are performed modulo d)

$$\left\{ (c_1, c_2, \ldots, c_\delta) : \sum_{k=1}^{\delta} c_k \in \{i, i+1, i+2, \ldots, i+(d-\delta)-1\} \right\}.$$

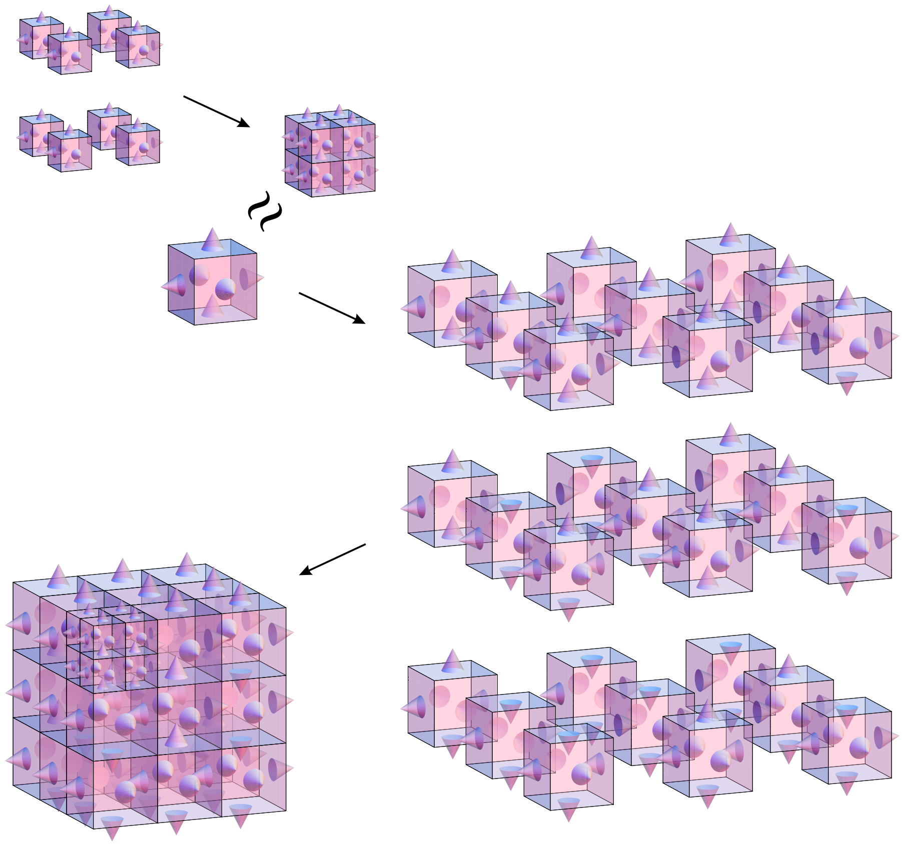

FIGURE 4. Overview of initial steps of the construction for $d = 3$.

Example 2.6. Figure 5 illustrates some layers for $d = 4$. The left-top part illustrates $(1, i)$-layers for $i = 0, 1, 2, 3$ (top to bottom). The left-bottom part illustrates $(2, i)$-layers for $i = 0, 1, 2, 3$ (left to right, top to bottom; the horizontal "blocks" are drawn with a little space between them in order to visualize how the building process runs). The part to the right illustrates a $(3, 2)$-layer (the x-axis goes from left to right, the y-axis from bottom to top, and the z-axis from front to back), where the greenish cubes are neutral, while the yellowish cubes are spikey of type 2, and all the other ones are spikey of type 1 (these two types will be introduced, and needed, in the proof of Lemma 2.7).

Although everything we shall need is just the existence of one (d, i)-layer, all (δ, i)-layers for a smaller δ are necessary for the inductive argument. This brings us to the following lemma.

Lemma 2.7. *A (δ, i)-layer exists for each δ and i, with $\delta \in \{1, 2, \ldots, d\}$ and $i \in \{0, 1, \ldots, d-1\}$.*

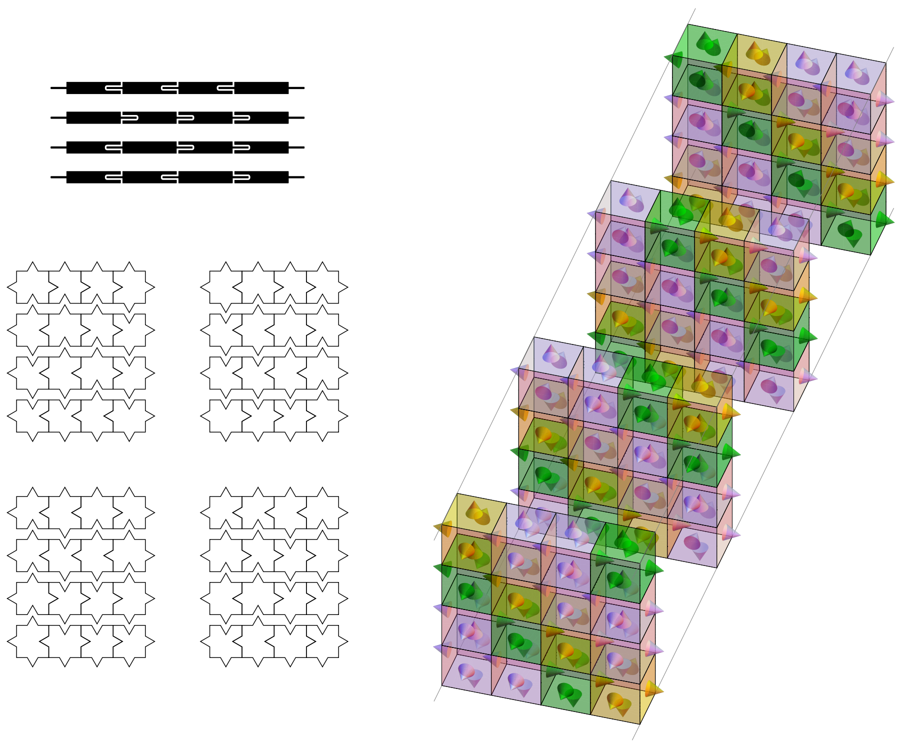

FIGURE 5. Various (δ, i)-layers.

Proof. We proceed by induction on δ.

Let first $\delta = 1$. We need to arrange a total of $d-1$ neutral 1-hypercubes and 1 spikey 1-hypercube. In order to make a $(1, i)$-layer, the neutral 1-hypercubes have to be centered (along the only existing coordinate axis) at the points $i, i+1, i+2, \dots, i+d-2$ (modulo d), which are all the points from 0 to $d-1$ with the exception of the point $i-1$. Putting the remaining spikey 1-hypercube in this point, and orienting the neutral hypercubes as $\begin{bmatrix} 1 & -1 \end{bmatrix}$ if they are centered at $0, 1, \dots, i-2$, and as $\begin{bmatrix} -1 & 1 \end{bmatrix}$ otherwise, gives precisely the $(1, i)$-layer that was looked for.

Now assume that, for some fixed δ, (δ, i)-layers exist for all i, $0 \leqslant i \leqslant d-1$, and let us prove that there exists a $(\delta+1, j)$-layer for any given j, $0 \leqslant j \leqslant d-1$. We shall describe how to arrange the available hypercubes in order to obtain a $(\delta+1, j)$-layer.

Consider the hypercubes centered at the points $(c_1, c_2, \dots, c_\delta, 0)$. Their corresponding matrices should be such that, if we erase the bottom row, the hypercubes represented by the resulting matrices build precisely a (δ, j)-layer (and we delay for a while the question of what should stand in the erased bottom row; it will be explained soon). Similarly, the hypercubes centered at the points of the form $(c_1, c_2, \dots, c_\delta, 1)$ should be such that, when bottom rows of the corresponding matrices are erased, the resulting hypercubes build a $(\delta, j-1)$-layer. In general, the hypercubes centered at the point $(c_1, c_2, \dots, c_\delta, c_{\delta+1})$, with bottom rows of their corresponding matrices erased, form a set of hypercubes that build a $(\delta, j-c_{\delta+1})$-layer (the subtraction is modulo d).

Consider now the hypercube centered at $(c_1, c_2, \dots, c_\delta, c_{\delta+1})$, and let us describe the bottow row, say $(b_{\delta+1,0}, b_{\delta+1,1})$, of its corresponding matrix. Let

$$t = \left(j - \delta - \sum_{k=1}^{\delta} c_k - 1\right) \bmod d. \tag{2.1}$$

Then:

$$(b_{\delta+1,0}, b_{\delta+1,1}) = \begin{cases} (1,-1), & \text{if } c_{\delta+1} < t; \\ (1,1), & \text{if } c_{\delta+1} = t; \\ (-1,1), & \text{if } c_{\delta+1} > t. \end{cases} \tag{2.2}$$

This completes the description of all $d^{\delta+1}$ hypercubes. We now have to prove three claims: i) every two hypercubes that have a common facet must have the corresponding facets marked in a matching way; ii) there is a total of $(d-\delta-1)d^\delta$ neutral $(\delta+1)$-hypercubes and a total of $(\delta+1)d^\delta$ spikey $(\delta+1)$-hypercubes; iii) the hypercubes are arranged precisely as required in the definition of a $(\delta+1, j)$-layer.

i) Recall that bumps and nicks along the first δ axes were determined based on some $(\delta, \dots)$-layers; because of them, matchings along these axes are correct. And from (2.2) it can be easily seen that matchings along the last axis are also correct.

ii) Let us first prove that each of the obtained hypercubes is indeed either neutral or spikey. Suppose that this is not the case for, say, the hypercube centered at $(c_1, c_2, \dots, c_\delta, c_{\delta+1})$. The only possible problem is if the bottom row of its corresponding matrix contains two 1s, and the same holds for one of the earlier rows. If the bottom row contains two 1s, then

$$c_{\delta+1} = t \text{ for } t \text{ given by (2.1).} \tag{2.3}$$

If one of the earlier rows contains two 1s, then the matrix with the bottom row erased represents a spikey δ-hypercube, but since that spikey δ-hypercube is a building element of a $(\delta, j-c_{\delta+1})$-layer, we have

$$\sum_{k=1}^{\delta} c_k \notin \{j-c_{\delta+1}, j-c_{\delta+1}+1, j-c_{\delta+1}+2, \dots, j-c_{\delta+1}+(d-\delta)-1\} \tag{2.4}$$

(of course, working modulo d). Now, note:

$$\begin{aligned} j - c_{\delta+1} + (d-\delta) - 1 &= j - t + (d-\delta) - 1 \\ &\equiv j - \left(j - \delta - \sum_{k=1}^{\delta} c_k - 1 \right) - \delta - 1 \\ &= \sum_{k=1}^{\delta} c_k \pmod{d}\,, \end{aligned} \tag{2.5}$$

which contradicts (2.4).

Therefore, it is enough to show that there is a total of $(\delta+1)d^{\delta}$ spikey $(\delta+1)$-hypercubes (then, by the previous paragraph, all the other ones are neutral, which matches what needs to be proved). We first count spikey $(\delta+1)$-hypercubes whose corresponding matrix contains two 1s in some non-bottom row (call them "type 1"). There are $\delta d^{\delta-1}$ such hypercubes centered at the points of the form $(c_1, c_2, \ldots, c_{\delta}, 0)$ (because that is how many spikey δ-hypercubes exist in a (δ, j)-layer). Similarly, whenever the rightmost coordinate is fixed, we have $\delta d^{\delta-1}$ such hypercubes, which makes altogether a total of δd^{δ} such hypercubes. We now count spikey $(\delta+1)$-hypercubes whose corresponding matrix contains two 1s in the bottom row (call them "type 2"). By (2.2), there is exactly one such hypercube whenever the first δ coordinates are fixed; this makes a total of d^{δ} such hypercubes. Summing the obtained values for type 1 and type 2, the total number of spikey $(\delta+1)$-hypercubes equals: $\delta d^{\delta} + d^{\delta} = (\delta+1)d^{\delta}$, which was to be proved.

iii) What is enough to prove, that is that each spikey $(\delta+1)$-hypercube is centered at some point $(c_1, c_2, \ldots, c_{\delta}, c_{\delta+1})$ where $\sum_{k=1}^{\delta+1} c_k$ is not congruent modulo d to any of the numbers $j, j+1, j+2, \ldots, j+(d-\delta-1)-1$. This is indeed enough, since this blocks a total of $(d-\delta-1)d^{\delta}$ possible positions, which means that all these positions have to be filled by neutral $(\delta+1)$-hypercubes, and then all the other positions have to be filled only by spikey ones (having in mind the total number of neutral ones and the total number of spikey ones). Therefore, let us prove the claim.

We shall use some lines of thought from ii). Assume first that a hypercube of type 1 is centered at $(c_1, c_2, \ldots, c_{\delta}, c_{\delta+1})$. Then we again reach (2.4), which immediately gives that $\sum_{k=1}^{\delta+1} c_k$ is not congruent to any of the "forbidden" values (actually even a little bit more, it is also not congruent to $j + (d-\delta) - 1$, though this value is not forbidden). And if we have a hypercube of type 2 at the same place, then we have (2.3), and now the calculation as in (2.5) gives $\sum_{k=1}^{\delta+1} c_k \equiv j + (d-\delta) - 1 \pmod{d}$, which is again an "allowed" value (actually, precisely the one that was "missed" with the type 1).

The proof is thus finished. □

The following lemma establishes a link between basic hypercubes and layers.

Lemma 2.8. *For each i, $i \in \{0, 1, \ldots, d-1\}$, a total of $(2d)^d$ basic hypercubes can be arranged to form a structure that is, when bumps and nicks are ignored, the same as a (d,i)-layer scaled by factor 2.*

Proof. By definition, a (d,i)-layer, for any i, is composed of d^d spikey hypercubes (and no neutral ones). We shall appeal to Definition 2.7 and Lemma 2.8 from [6]. Note that a sample of order 1 from Definition 2.7 there is a marked unit hypercube whose $d+1$ facets are marked by bumps and $d-1$ facets marked by nicks, where there are no two opposite facets both marked by nicks; in other words, a sample of order 1 is precisely what we call a spikey d-hypercube in this article. Therefore, by the mentioned Lemma 2.8, 2^d basic hypercubes can be stacked in such a way to form a structure that "behaves like" a spikey d-hypercube scaled by factor 2. Then, by Lemma 2.7 (from the present article), d^d such structures can be stacked in such a way to form a new structure that "behaves like" a (d,i)-layer. Note that this new structure consists of $(2d)^d$ basic hypercubes, which was to be proved. □

Finally, we have our main statement.

Theorem 2.9. *For any $d \in \mathbb{N}$, $d \geqslant 2$, the Heesch number of a basic hypercube in d dimensions equals $d-1$.*

Proof. By Lemma 2.4 we have that $d-1$ is an upper bound on the Heesch number of a d-dimensional basic hypercube. We shall see that $d-1$ is also a lower bound, for which we use Lemma 2.8. Namely, note that, for any i, a (d,i)-layer scaled by factor 2 is a hypercube of side $2d$; therefore, $(2d)^d$ basic hypercubes can be arranged to form such a hypercube (up to bumps and nicks). Then, trivially, $(2d-1)^d$ basic hypercubes can be arranged to form a hypercube of side $2d-1$, which means that a basic hypercube can be surrounded $d-1$ times by its isometric copies. Therefore, $d-1$ is also a lower bound on the Heesch number of a d-dimensional basic hypercube, which was to be proved. □

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