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ON IDENTIFYING VERTICES OF TOURNAMENT DIGRAPHS

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Abstract. An identifying code in a graph is a subset of its vertices where the neighbours' intersections with the subset are nonempty and different for every pair of vertices. After their introduction in 1998 by Karpovsky et al., the interest in this domain has never ended. This growing interest comes from, on the one hand, the theoretical aspect of this concept, and on the other hand, its applications, especially the indoor location and faulty processor network.

In this work, we study the identifying code on tournament digraphs which is probably the most studied class of digraphs. Hence, we give some minimum cardinality of special tournaments and show that only transitive tournaments can admit an r-identifying code when $r \geq 2$. We also obtain an upper bound for the quadratic residue tournament. Moreover, we study how to reach an optimal code when adding a vertex or inverting an arc in a transitive tournament.

1. Introduction

To model and solve the problem of locating faulty processors in a multiprocessor network, Karpovsky et al. proposed, in 1998, a new concept in [24], called an identifying code. After that, they extended the concept to several applications, such as locating and detecting danger or threats in indoor environments [27]. Recently, the same technique has been used as a model to detect the failure of high-power transformers in an electric power grid [5]. One can find up to five hundred manuscripts in an updated bibliography (see [25]) dealing with this concept and grouping numerous variants studied in several classes of graphs.

Let us start with some notations and definitions that we will introduce in this paper. Given $G = (V, E)$ a simple graph where $V(G)$ (resp. $E(G)$) is the set of vertices (resp. edges), we denote by $\Gamma(v)$ (resp. $\Gamma[v] = \Gamma(v) \cup \{v\}$) the neighbours (resp. closed neighbours) of the vertex v.

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A code in a graph is simply a subset of its vertices. The elements of a code are called codewords. A code C is a covering code if for all $v \in V$, $I(v) = \Gamma[v] \cap C \neq \emptyset$.

A separating code is a code such that for all pairs of distinct vertices (u, v), we have $I(u) \neq I(v)$. An identifying code has two properties, namely, covering and separating.

An r-identifying code is a distance generalization of an identifying code, such that we replace $\Gamma(x)$ by $\Gamma_r(x)$, which includes all the vertices having x at a distance at most r (also called a ball centered on x with radius r). Hence an r-identifying code C is such that all $I_r(u) = \Gamma_r(u) \cap C$ are different and nonempty. This set is called an *r-identifying set* of the vertex u. When $r = 1$, we say that C is an identifying code or simply a code if there is no ambiguity. This definition can be easly extended to digraphs, just by considering in-neighbours (or predecessors) instead of neighbours, which is denoted by $\Gamma^-(.)$.

We are interested in finding an identifying code with minimum cardinality and we call such a code optimal. For a given graph G, the optimal cardinality of an identifying code of G is denoted by $\gamma^{ID}(G)$.

Unfortunately, not all graphs admit an identifying code (a complete graph, for example), then we say that a graph admits an identifying code (or identifiable) if and only if it is twin-free. Two vertices u, v are called *twin* if they have the same closed neighbours, i.e., $\Gamma[v] = \Gamma[u]$. Moreover, we know that finding a minimum identifying code in a twin-free graph G is an NP-hard problem both in oriented and nonoriented graphs [12, 11].

Numerous generalizations and variants have been introduced to deal with many circumstances in practice, such as $(r, \leq l)$-identifying codes, robust identification and watching systems [25].

Furthermore, several classes of graphs were considered in studying the problem of finding an identifying code with the smallest cardinality, from the simplest ones, such as cycles, interval graphs, and trees, to the most sophisticated ones such as grids, a product of graphs, and Hamming graphs.

Digraphs are another class studied, but to our knowledge, few studies consider them in the digraph class. In the next section, we provide an overview of all works dealing with such a class. F. Foucaud et al. in [17] have classified all digraphs, which only admit their whole vertex set as an identifying code. Also, in [10] the authors give a linear algorithm to find a minimum identifying code in an oriented tree. The class of De Bruijn digraphs is another one studied by D. Boutin et al. in [8], where they show under which conditions it is r-identifiable (admitting an identifying code) or not. In his thesis, R. D. Skaggs (see [31]) gave some characterizations of graphs that reach bounds and studied how minor modifications on an edge or a vertex cause a change in the optimal identifying code.

In the papers of C. Balbuena et al. (see [3, 4]), the authors studied the conditions when digraphs can admit an $(1, \leq l)$-identifying code. They characterize all 2-in-regular digraphs admitting a $(1, \leq l)$-identifying code where

$l = 2, 3$ (a 2-in-regular digraph is a digraph where any vertex has its in-degree equal to 2). They also gave an algorithm to construct an identifying code in digraphs with a specific minimum in-degree and out-degree.

The same authors, in another recent work [2], gave us a characterization of 2-in-regular digraph admitting a $(1, \leq 2)$- or $(1, \leq 3)$-identifying code using algebraic-combinatorial argument. It consists of studying the identifying code problem from a spectral graph theory point of view. In 2018, N. Cohen et al. [13] considered an orientation of edges for a given undirected graph. The problem is to find an orientation of these edges, which leads to the smallest identifying code that can be found. These authors independently gave similar results with a specific orientation of the complete graph reaching the lower bound.

Recently, Foucaud et al. [15] investigated the extremal digraphs (those for which the whole set of vertices is a minimum code) in their work entitled *Extremal digraphs for open neighbourhood location-domination and identifying codes*. In this context, an identifying code C is defined using the open neighbours $\Gamma(\cdot)$ instead of closed neighbours $\Gamma[\cdot]$. By analyzing key properties of these digraphs, they gave new proofs of earlier results and also provided a characterization of a significant class of extremal digraphs.

In the same perspective, this paper deals with the problem of finding optimal identifying codes in directed graphs, especially tournament digraphs. By definition, a tournament denoted T is an orientation of edges in a complete graph.

This class of digraph appears as a model for the competition of n players in a round-robin tournament, where the vertices represent players and an edge $uv \in A(T)$ means that the player u beats the player v. Among applications that introduce a tournament digraph as a model, we have the voting theory and social choice theory [30, 21].

The study of domination problems is not overlooked in tournaments. This class of digraphs is investigated when studying various types of codes, including domination codes and dominating-locating codes, which are precursors to identifying codes.

Among the works that deal with tournaments, we have the one of Foucaud et al. [16], where, in addition to characterizing extremal digraphs admitting the whole set of vertices (or the whole minus one) as a minimum dominating locating code, they also improve an upper bound given in previous work. In particular, they showed that this bound is tight in the case of tournaments. This study was later extended in [6] by giving a generalization of the bound to the class of local tournaments for which the subgraphs induced by in or out-neighbours of every vertex are tournaments.

1.1. **Outline.** The next section presents some terminology and definitions needed in this work. In section 3, we study the existence of an r-identifying code in a tournament and show that only a transitive one can admit an

r-identifying code for $r \geq 2$. However, an identifying code always exists in tournaments.

Section 4 provides tight bounds for identifying code on transitive and locally-transitive tournaments. Moreover, an upper bound is given for a doubly regular tournament.

The final section is devoted to studying transformation in the transitive tournament. More precisely, we search for how to reduce, as small as possible, the cardinality of an optimal code when adding a vertex with a specific orientation of the arcs. Also, inverting one or multiple disjoint arcs is considered.

2. Preliminaries

Before giving the main results of this paper, let us introduce an adaptation of an identifying code in a digraph and give some definitions.

2.1. **Identifying code in digraphs.** In a directed graph or simply digraph $G = (V, A)$, an arc $uv \in A$ is said to be *symmetric* if $vu \in A$. For any vertex v, the *in-neighbours* of v, denoted $\Gamma^-(v)$, is a set of all vertices u such that $uv \in A$. The cardinality of $\Gamma^-(v)$ is called the *in-degree* of v and it is denoted by $d^-(v)$. Similarly, we define $\Gamma^+(v)$ and $d^+(v)$, the *out-neighbours* and the *out-degree* of v, regarding vertex u such that $vu \in A$. Closed in-neighbours (resp. out-neighbours) of v, denoted $\Gamma^-[v]$ (resp. $\Gamma^+[v]$), is $\Gamma^-[v] = \Gamma^-(v) \cup \{v\}$ (resp. $\Gamma^+[v] = \Gamma^+(v) \cup \{v\}$). Hence, an identifying code is a subset C of vertices such that for every vertex v its in-neighbours in C is unique and nonempty, i.e. $\Gamma^-[v] \cap C = I^-(v) \neq \emptyset$ and for every pair of distinct vertices u, v we have $I^-(u) \neq I^-(v)$.

Another way to define an identifying code is by considering the symmetric difference of closed in-neighbours in C, denoted $\Delta_{u,v}$, between two distinct vertices u, v. Hence, this difference must not be empty in addition to the first condition ($I^-(v) \neq \emptyset$ for every vertex v).

The distance generalization of an identifying code can be defined simply by replacing the ball of radius r with the semiball centred on v, denoted $\Gamma_r^-(v)$. This semiball includes all vertices having the vertex v at a distance at most r (v is included). Thus, we call an r-identifying code every subset C such that $I_r^-(v) = \Gamma_r^-[v] \cap C$ are different and nonempty for every vertex v.

A digraph is called r-identifiable (also means admitting an r-identifying code) if and only if it is twin-free, i.e., there is not a pair of vertices (u, v) such that $\Gamma_r^-[u] = \Gamma_r^-[v]$. We can obviously see that if the digraph is asymmetric (did not contain a symmetric arc), then it is 1-identifiable or simply identifiable.

2.2. **Tournament digraphs.** The tournament digraphs are one of the most studied classes of digraphs. We have a different characterization of tournaments according to the score or in-neighbours (respectively out-neighbours) of vertices. For more definitions and results dealing with tournaments, one can see [26, 19]. We begin with an alternative definition of the tournament.

Definition 2.1. *A* tournament *is a directed graph* $T = (V, A)$ *such that, for each pair* u, v *of distinct vertices, we have either* uv *or* vu *as an arc, but not both.*

We say that a vertex u dominates another vertex v if uv is an arc. So an out-degree (also called *score*) of a vertex v in T is the number of vertices that v dominates. For each tournament we associate what is commonly called *score sequence*, an ordered n-tuple $(s_1, s_2, \ldots, s_n)$, where s_i is the out-degree of vertex v_i, $1 \leq i \leq n$ and $s_1 \leq s_2 \leq \cdots \leq s_n$. Note that we can have different tournaments for the same score sequence. In the same way, we can define the in-degree sequence as a nondecreasing sequence of positive integers, where every integer in this sequence is the number of vertices that dominate one vertex of the tournament. In the following, we define a tournament that is characterized only by its out-degree sequence:

Definition 2.2. *A* transitive tournament *is a tournament having a score (out-degree) sequence equal to* $(0, 1, 2, \ldots, n-1)$.

Recall that the transitive property in a digraph is such that for every triplet u, v, w of distinct vertices of T, we have:

$$uv \in A \text{ and } vw \in A \Rightarrow uw \in A.$$

We will see next that local transitivity can help find the minimum identifying code.

Definition 2.3. *A tournament is* locally-transitive *if the subgraph* T^- *induced by the in-neighbours and the subgraph* T^+ *induced by the out-neighbours of every vertex are transitive.*

Definition 2.4. *When we have* $d^+(v) = d^-(v)$ *for any vertex* v *of a tournament, we call it a* regular tournament*.*

Remark: A regular tournament has an odd number of vertices.

Combining the above properties, namely transitive and regular, we obtain another class of tournaments known as a carousel tournament or a regular locally-transitive tournament. By using the well-organized structure of this tournament, we give an optimal identifying code (see proposition 4.4).

Let us give some properties of this tournament that are important in our study. It is known that a tournament is locally-transitive if and only if it is cone-free (see [19]). A cone-free circuit is a circuit of length 3 with another vertex as sink or source (see [14]). The second property, reported in the following, is that there is one up to isomorphism of a regular locally-transitive tournament of odd order (see [9], [14]).

Proposition 2.6 (Corollary 1.4,[14])**.** *For every* $k \in \mathbb{N}$*, there is exactly one up to isomorphism regular locally-transitive tournament* T *of order* $n =$

$2k+1$ *and it is given by*

$$V(T) = \{0, 1, \ldots, 2k\}; \text{ and}$$
$$A(T) = \{(x, x+i) \bmod (2k+1) : i \in \{1, 2, \ldots, k\}\}$$

This uniqueness helps us to provide obviously an optimal identifying code. The problem in the even case is the two possible configurations for which the tournament is always locally-transitive (see Figure 1).

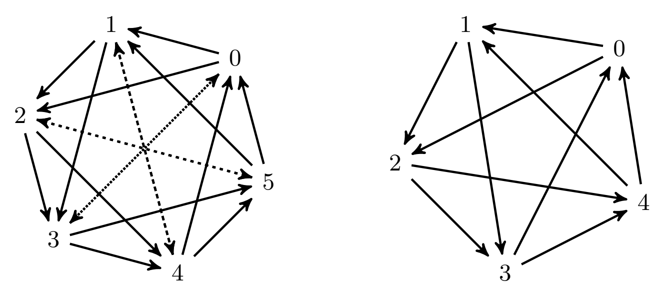

FIGURE 1. The tournament on the right is a carousel tournament whose representation is unique. In contrast, the one on the left admits several drawings given by the two possible orientations of the dotted arcs, which also lead to a locally-transitive tournament.

Definition 2.7. *A* doubly-regular tournament *is a regular tournament such that each pair of distinct vertices* u, v *dominates the same number of vertices. In other words, for all pairs of vertices* u, v *we have* $|\Gamma^+(u) \cap \Gamma^+(v)| = k$*, where* k *is an integer.*

Remark: If a doubly-regular tournament is of order n, then the value of k in the above definition is necessarily $\frac{n-3}{4}$.

A quadratic residue tournament, also known as a Paley tournament, is a specific tournament that has interesting combinatorial properties like strong regularity. For further details, we refer the reader to [18]. Its construction is based on the concept of quadratic residues in a finite field. A quadratic residue modulo n, in number theory (see [20, 29] for more details), is every integer a such that the following equation

$$x^2 \equiv a \bmod n$$

admit a solution x with $0 < x < n$. If the equation has no solution, we call a a quadratic nonresidue. Let us denote by respectively R and N the quadratic residue set and quadratic nonresidue set. When n is a prime integer then we have $|R| = |N| = \frac{n-1}{2} = 2k+1$. Below, we give a formal definition of this tournament:

Definition 2.9. *A quadratic residue tournament, also called a Paley digraph, of order* $n = 4k+3$ *with* n *a prime integer, is a tournament where* *the vertex set is* $\{0, 1, \ldots, n-1\}$*, and* ij *(respectively* ji*) is an arc if* $j - i$ *is a quadratic residue (respectively quadratic nonresidue).*

Note that the tournament is well-defined even in the case where $j - i < 0$ (see [1]). We know that the existence of a doubly regular tournament of order n is equivalent to the existence of a skew Hadamard matrix of order $n + 1$ (see [28]).

3. Identifiable tournaments

After the preliminaries given in the previous section, let us recall that only twin-free graphs are identifiable (admit an identifying code).

The following straight-forward result is reported independently in [13].

Proposition 3.1. *Any tournament admits an identifying code.*

Proof. It is easy to see that the result follows from the fact that the symmetric difference is nonempty for every pair of vertices because the tournament is asymmetric. □

If every tournament admits an identifying code, this is different for every $r \geq 2$. Thus, before giving the result, we need the following Lemma:

Lemma 3.2. *Let* T *be a tournament, and let* u *be a vertex of minimum out-degree in* T*. Let* $r \geq 2$ *be an integer. Then, for every* v *such that* uv *is an arc of* T*, every vertex* x *that* r*-separates* u *and* v *is such that* $d(x, u) = r$ *and* $d(x, v) = r + 1$*.*

Proof. If x r-separates u and v, then x must lie in the symmetric difference $\Gamma_r^-[u]\Delta\Gamma_r^-[v]$. Since $d(u, v) = 1$, then two cases may occur:

- we have $d(x, u) = r$ and $d(x, v) = r + 1$
- we have $d(x, u) \geq r + 1$ and $d(x, v) = r$

Now, we will show that the latter case is impossible. Indeed, let x be a vertex such that $d(x, u) \geq r + 1$ and $d(x, v) = r$. Since the graph T is a tournament, then every out-neighbour of x must also be an out-neighbour of u (else we would have $d(x, u) \leq 2$, which contradicts $d(x, u) \geq r + 1 \geq 3$). We must also have ux arc of T (else we would have $d(x, u) = 1$). Hence, the number of out-neighbours of x would be strictly less than that of u, which contradicts the fact that u has a minimum out-degree. □

Theorem 3.3. *Let* T *be a tournament admitting an* r*-identifying code, with* $r \geq 2$*. Then* T *is transitive.*

Proof. By way of contradiction, let T be a tournament admitting an r-identifying code with $r \geq 2$, and let us assume T is not transitive. If T admits a vertex x having no out-neighbours, then let us consider the tournament T_1 obtained from T by removing x and all arcs entering into x. Clearly, if T is not transitive, then neither is T_1. In addition, if C is an r-identifying code of T, then $C \setminus \{x\}$ is an r-identifying code of T_1. Thus, we can remove such vertices having no out-neighbours recursively until we reach a tournament T_k ($k > 1$) which is not transitive, admits an r-identifying code, and such that every vertex x of T_k has at least one out-neighbour. Such a tournament T_k must exist, else it would mean that T itself is transitive.

Now, let us label each arc uv of T_k with the label $d^-(u)d^-(v)$. Let us consider a maximum arc uv with respect to the lexicographical order on the labels of the arcs. Since T' admits an r-identifying code, then there exists a vertex x that r-separates u and v. Since we considered the lexicographical order, then, in particular, u has a maximum in-degree and necessarily the minimum out-degree in T_k. Lemma 3.2 then applies to u, and we then get that $d(x, u) = r$ and $d(x, v) = r + 1$. Now, let us consider any in-neighbour z of v: z must also be an in-neighbour of x, else we would have $d(x, v) \leq 2$, which contradicts $d(x, v) = r + 1 \geq 3$. This applies in particular to u, and u is an in-neighbour of x. This shows that $\Gamma^-(v) \subseteq \Gamma^-(x)$. In addition, we must have $v \in \Gamma^-(x)$, thus $d^-(x) > d^-(v)$. The contradiction comes from the fact that ux is an arc, and then ux has a label strictly greater than the one of uv, a contradiction. □

For the case of a regular tournament, we give an alternative short proof. Hence :

Proposition 3.4. *Let T be a regular tournament. Then T has no r-identifying code for all $r \geq 2$.*

Proof. To prove this, it suffices to show that $\forall\, v \in V(T), \Gamma_2^-[v] = V$. Let v be any vertex in $V(T)$ then it has exactly $\frac{n-1}{2}$ in-neighbours and the same number as out-neighbours. We want to show that every vertex $w \in \Gamma^+(v)$, the vertex v is at distance of at most 2 from w.

Indeed, since $w \in \Gamma^+(v)$ and $d^+ = \frac{n-1}{2}$ then at most $\frac{n-1}{2} - 1$ of its out-neighbour belong to $\Gamma^+(v)$, hence there is at least one out-neighbour in $\Gamma^-(v)$. In other words, there is a path of length two from any vertex in $\Gamma^+(v)$ to v. By this, we conclude the proof. □

4. Optimal identifying code

We now present two general bounds on the minimum identifying code:

Theorem 4.1. *Let G be a directed graph on n vertices having an identifying code. Then, we have*

$$\lceil \log_2(n+1) \rceil \;\leq\; \gamma^{\mathrm{ID}}(G) \;\leq\; n,$$

and the bounds are tight.

The lower bound is given in the seminal work of Karpovsky for every graph, whereas Skaggs [31], by proposition 5.9, gave a class of digraphs reaching the bound. Restricting this to the class studied in this work, we can give a construction of tournaments on n vertices such that $\gamma^{\mathrm{ID}}(T) = \lceil \log_2(n+1) \rceil$.

Proposition 4.2. *The lower bound is still valid and tight for the class of tournament digraphs.*

Proof. Start with a directed graph H without twins having $\lceil \log_2(n+1) \rceil$ vertices. The vertex set of H will be the code. Then add, for any subset $S \subseteq V(H)$ for which there is no vertex v having S as in-neighbours, a vertex v_S having S as in-neighbours. Add any arcs between vertices not in H and any arcs from vertices not in H to vertices in H. □

In the recent work of Cohen et al. (see [13]), the same construction is given independently. In [31, Theorem 5.8], the authors gave a graph reaching the upper bound. Another class of digraphs that reached this bound is given in [17]. The proof was based on a theorem of Bondy on distinguishing induced subsets. Note that this result is related to general directed graphs that may contain symmetric arcs. As a corollary, if a graph admits one symmetric arc and admits a code, then it is such that $\gamma^{\mathrm{ID}}(G) < |V|$. If we restrict ourselves to tournaments, we get the following:

Corollary 4.3. *A tournament T on n vertices is such that $\gamma^{\mathrm{ID}}(T) = n$ if and only if T is transitive.*

4.1. **Optimal code for some classes of tournaments.** In this section, we find optimal identifying codes in some tournaments. In addition, we give an upper bound for the Paley tournament which is also doubly regular.

Recall that the construction given in the proof of proposition 4.2 leads to a tournament admitting an optimal code of cardinality $\lceil \log_2(n+1) \rceil$. In addition, from corollary 4.3 we know that only a transitive tournament admits a whole set of vertices as an optimal identifying code.

The well-organized structure of a locally-transitive tournament, when it is also a regular one (also called a carousel tournament), allows us to give the following result:

Proposition 4.4. *Let T be a regular locally-transitive tournament with an odd number of vertices. Then we have $\gamma^{\mathrm{ID}}(T) = \frac{n+1}{2}$.*

Proof. Let T be a regular locally-transitive tournament of order n, with n odd. For every vertex v_i, $i = 1, \dots, n$ let us denote its in-neighbours (resp. out-neighbours) by $\Gamma^-(v_i) = \{v_k | k = i - j \bmod n, 1 \leq j \leq (n-1)/2\}$ (resp. $\Gamma^+(v_i) = \{v_k | k = i + j \bmod n, 1 \leq j \leq (n-1)/2\}$). We know from proposition 2.6 that the structure of this tournament is unique, and the set of its vertices can be cyclically ordered. Furthermore, if $x \in \Gamma^-(v_i)$ then $\Gamma^-(x)$ is the union of terminal interval of $\Gamma^+(v_i)$ and initial interval of $\Gamma^-(v_i)$. The proof of the last claim is omitted because it is very similar to a case where $x \in \Gamma^+(v_i)$, see [9]. Hence, we get $\Gamma^-[v_i] \Delta \Gamma^-[v_{i+1}] = \{v_{i-\frac{n-1}{2}}, v_{i+1}\}$ for every consecutive vertices in the cyclic order. Let us consider the auxiliary nonoriented graph H, having $V(G)$ as a vertex set, and such that xy is an edge if and only if there exist two consecutive vertices u, v in the cyclic order of T such that $\{x, y\} = \Gamma^-[u] \Delta \Gamma^-[v]$. Clearly, any identifying code of T must be a transversal of H (recall that a transversal is a subset K of vertices such that, for any edge uv, we have $K \cap \{u, v\} \neq \emptyset$). Moreover, any vertex u of H belongs to exactly two edges.

The construction given above leads us to a cycle of the form $v_1, v_{1-p}, v_2, v_{2-p}, \ldots, v_n, v_{n-p}, v_1$ of length n, where $p = \frac{n-1}{2}$. Hence, any transversal of H has at least $\frac{n+1}{2}$ vertices, thus $\gamma^{\mathrm{ID}}(T) \geq \frac{n+1}{2}$. Now, let us show that $\gamma^{\mathrm{ID}}(T) = \frac{n+1}{2}$. Finally, it suffices to construct an identifying code of T of cardinality $\frac{n+1}{2}$. It is easy to check that $\Gamma^+[v_i]$ is an identifying code of T, having the desired cardinality, which terminates the proof. □

To our knowledge, in the case of locally-transitive even order tournaments, few results exist apart from that of Brouwer [9]. It is even harder to determine an identifying code than in the previous case because there are two possible directions for certain arcs (see figure 1). Hence, this requires first looking at the structure of this type of tournament.

Proposition 4.5. *Let T be a regular locally-transitive tournament with an even number of vertices. Then we have $\gamma^{\mathrm{ID}}(T) = \frac{n}{2}$.*

Proof. Let T be a locally-transitive tournament of order $n = 2k$ ($k \geq 3$ is an integer). We say that T is semiregular if for each vertex $v_i \in T$ we have $|d^-(v_i) - d^+(v_i)| = 1$, i.e. for every vertex v_i we have $d^-(v_i) = k$ or $k - 1$.

Let us label $v_1, v_2, \ldots, v_{2k}$ its vertices. For every vertex v_i, it is known that the subtournament induced by $\Gamma^-[v_i]$ and $\Gamma^+[v_i]$ are both transitive, and every subtournament of order 4 is also transitive, i.e., the tournament is cone-free.

Let v_1 be any vertex of T. We claim that $v_1, v_2, \ldots, v_k, \ldots, v_{2k}, v_1$ is a hamiltonian directed cycle or simply circuit.

By the structure described above, we have the path from v_{k+2} to v_k. Hence, two cases are distinguished:

First, if $v_{k+1} \in \Gamma^+(v_1)$ then v_{k+1} have k in-neighbours by the transitivity of $\Gamma^+(v_1)$. Since $d^-(v_{k+1}) = k$ the remain vertices, i.e. $v_{k+2}, v_{k+3}, \ldots, v_{n-1}$, are out-neighbours by semiregularity.

Second, when $v_{k+1} \in \Gamma^-(v_1)$ we have $|\Gamma^-(v_1)| = |\Gamma^+(v_{k+1})| = k$ then the remain $k - 1$ vertices must be the in-neighbours of v_{k+1} due to the semi-regularity of T. Hence, in the two cases we have the arcs $v_k v_{k+1}$ and $v_{k+1} v_{k+2}$. Thus, we obtain the claimed circuit.

Moreover, we observe in the tournament's structure that all vertices on the circuit from v_1 to v_{k+1} (resp. v_{k+1} and v_{2k}) are in-neighbours (resp. out-neighbours) of v_{k+1}.

Ignoring the orientation of the arc between v_i and v_{i+k}, we can show that the structure described above is valid for all pairs of "opposite" (v_i, v_{i+k}) on the circuit, where $i \in 1, \ldots, k$.

We do this by induction. Let v_i be any vertex on the hamiltonian circuit. For $i = 1$, it is done by the two previous claims. By the induction hypothesis, suppose this is true for every vertex v_i where $i \in \{1, \ldots, k -$

$1\}$. Hence, we already know that the induced subtournaments by the sets $\{v_{i+2}, v_{i+3}, \ldots, v_{i+k}\}$ and $\{v_{i+k+2}, v_{i+k+3}, \ldots, v_{2k}\}$ are transitive. It remains to show that v_{i+1} and v_{i+k+1} are each out-neighbours of the $k-1$ vertices that precede it. By way of contradiction, assume that there is a vertex x between v_{i+k+1} and v_i (on the circuit) such that $v_{i+1}x$ is an arc. Then, the three vertices v_{i+1}, v_{i+k} and v_{i+k+1} are in-neighbours of x. Moreover, we have the arc v_{i+1}, v_{i+k+1} by transitivity. Therefore, the vertex v_{i+1} has $k+1$ out-neighbours, a contradiction.

Now, for every identifying code in this tournament, we observe that

(4.1) for every pair (v_i, v_{i+k}) at least one of them must belong to C.

Indeed, without loss of generality let $v_i v_{i+k}$ an arc. by considering the arc between the vertices v_{i-1} and v_{i+k-1}, then two cases are distinguished:

- $v_{i-1}v_{i+k-1}$ is an arc then $\Delta_{i,i-1} = \{v_i, v_{i+k}\}$.
- $v_{i+k-1}v_{i-1}$ is an arc then, $\Delta_{i+k-1,i+k} = \{v_{i+k}\}$.

Hence, in both cases, at least one vertex must belong to the code. Furthermore, this is true for the $n/2$ pairs of opposite vertices in the circuit. Then, we deduce that

$$|C| \geq \frac{n}{2}$$

To conclude, we can easily check that the sets $\{v_i | i \text{ is even}\}$ are an identifying code reaching the bound. □

The key to establishing the upper bound is the use of two properties, namely: the fact that the tournament is doubly-regular (see [32]) and the fact that the number of common in-neighbours and out-neighbours of any two distinct vertices is equal (Corollary 3.4 in [23]).

Theorem 4.6. *If T is a quadratic residue tournament on $n \equiv 3 \bmod 4$ vertices, then*

$$\gamma^{\mathrm{ID}}(T) \leq \frac{n+1}{2}$$

Proof. Let n be an odd prime integer such that $n \equiv 3 \bmod 4$, i.e. $n = 4k+3$. By definition, we know that $|\Gamma^-(u)| = |\Gamma^+(u)| = 2k+1$ for every vertex u. In addition, we have the fact that the common in-neighbours (denoted $\bigcap_{uv}^-$) and common out-neighbours (denoted $\bigcap_{uv}^+$) have the same cardinality that is equal to k for every pair of distinct vertices u and v (Corollary 3.4 in [23]). To prove the upper bound, we claim that every subset $C \subset V$ of cardinality $\frac{n+1}{2} = 2k+2$ is an identifying code. First, we prove that C is a covering code. Second, we prove it is also a separating one.

By construction, we have the fact that every vertex in C is covered by itself. For the remaining vertices, we proceed by contradiction. Suppose that C is not a covering code; thus, at least one vertex in $V \setminus C$ is not covered. If v is one of these vertices, then we must have $C \subseteq \Gamma^+(v)$, which is equivalent to saying that the number of out-neighbours of v is at least $\frac{n+1}{2}$, which contradicts the fact that T is regular.

Now, it remains to show that C is also a separating code. Without loss of generality, let u and v be two vertices such that uv is an arc. To show the separation property of the code, we should prove that the symmetric difference between the identifying sets of any pair of vertices is nonempty. Let us denote this difference by Δ_{uv}, hence if C is not a separating code, then we must have $C \cap \Delta_{uv} = \emptyset$. Thus, C must be a subset of $\bigcap_{uv}^{-} \cup \bigcap_{uv}^{+}$, but this is impossible because the cardinality of the set C is simply greater than the union of the common neighbours, which concludes the proof. □

Unfortunately, we think that the above bound is far from being tight. Indeed, Table 1 gives some examples.

n	7	11	19	23	31	43
upper bound	4	6	10	12	16	22
$\gamma^{\text{ID}}(T_n)$	3	5	5	5	6	7
general lower bound	3	4	5	5	5	6

Table 1. This table shows the gap between the found upper bound and the optimal code for a doubly regular tournament of order up to 43 vertices.

As Table 1 shows, the bound we give is not tight. On the other hand, the optimal cardinality for a small value of n (obtained by a brute force algorithm) is very close to the general lower bound. Hence, the given upper bound must be improved. Moreover, we think that finding a lower bound will give us a better approximation of the optimal value.

5. Transformations in transitive tournament

This section aims to consider a few transformations in a transitive tournament, such as inverting an arc or adding a vertex, and look at how the cardinality of an optimal identifying code can be derived from the original tournament. We can call such a tournament *quasitransitive* in the sense that the deletion of one of its vertices gives us a transitive subtournament. First, we claim that adding a vertex to a transitive tournament can reduce the cardinality of the code by at most half. Second, we claim that the inverting of an arc can reduce the cardinality of an optimal code by at most two.

Proposition 5.1. *Let T be a transitive tournament on n vertices, and T' the tournament obtained by adding a single vertex to T. Then we have $\gamma^{\text{ID}}(T') \geq \lfloor \frac{n}{2} \rfloor + 1$, and the bound is tight.*

Proof. Let us first show that $\gamma^{\text{ID}}(T') \geq \lfloor \frac{n}{2} \rfloor + 1$. Let us label $v_1, \dots, v_n$ the vertices of T, in a way such that, for all i, vertex v_i has in-degree $i-1$, and $v_i v_j$ is an arc whenever $i < j$. Since T is transitive, this can always be done. Now, let us add x to T, and let us consider C an identifying code of the new tournament $T' = T \cup \{x\}$.

We claim that, for each i, at least one of the vertices v_i, v_{i+1} must belong to C. Indeed, by way of contradiction, assume that none of v_i, v_{i+1} belong to C. Notice that $I^-(v_i) \setminus \{x\} = I^-(v_{i+1}) \setminus \{x\}$, hence we must have $x \in C$ in order to separate v_i from v_{i+1}. Two cases follow:

a): xv_i is an arc and $v_{i+1}x$ is an arc. In this case, we have $i > 1$, else v_{i+1} would not be covered. Let us consider now the vertex v_{i-1}. If x, v_{i-1} is an arc, then v_{i-1} and v_i are not separated. If $v_{i-1}x$ is an arc, then v_{i-1} and v_{i+1} are not separated, a contradiction.

b): v_ix is an arc and xv_{i+1} is an arc. Also in this case, we have $i > 1$, else v_i would not be covered. Let us consider now the vertex v_{i-1}. If xv_{i-1} is an arc, then v_{i-1} and v_{i+1} are not separated. If $v_{i-1}x$ is an arc, then v_{i-1} and v_i are not separated, a contradiction.

This shows that:

(5.1) For all i, at least one the vertices v_i, v_{i+1} must belong to C.

Let us now consider an identifying code C of T'. If there are two vertices v_i, v_{i+1} that both belong to C, then (5.1) implies $|C| \geq \lfloor \frac{n}{2} \rfloor + 1$. If for all i, exactly one the vertices v_i, v_{i+1} belongs to C, then we must have x in C in order to separate v_{i-1} from v_i, or to cover v_{i-1}. Thus (5.1) implies $|C| \geq \lfloor \frac{n}{2} \rfloor + 1$.

Now let us show that the bound is tight. Let us consider the tournament T' such that, if i is odd, we have xv_i an arc, else we have v_ix an arc. It is easy to see that $C = \{x\} \cup \{v_i \mid i \text{ even }\}$ is an identifying code of T', of cardinality $\lfloor \frac{n}{2} \rfloor + 1$. □

In the second part of this section, we study the influence of inverting an arc or multiple disjoint arcs on the optimal identifying code C in a transitive tournament T. Assume that v_iv_{i+k} is the arc that will be inverted, and if there is no ambiguity, we denote each vertex v_i simply by its index i. We will denote T', the resulting tournament, and C' its optimal identifying code.

An important observation in the inversion of one arc is that only two vertices have their in-neighbours altered in T'. That is, all vertices keep their closed in-neighbours the same as in T except for the extremities of the inverted arc where we have $\Gamma^-[v_i] = \{v_1, v_2, \dots, v_i, v_{i+k}\}$ and $\Gamma^-[v_{i+k}] = \{v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_{i+k}\}$.

Lemma 5.2. *If v_j and v_{j+1} the consecutive vertices in T' then at least one of them must belong to C' for all $j \in \{1, \dots, n-1\}$.*

Proof. Let us denote S_1, S_2 and S_3 respectively the following subsets $\{1, 2, \dots, i-2\}$, $\{i+1, i+2, \dots, i+k-2\}$ and $\{i+k+1, i+k+2, \dots, n-1\}$. We must remember that sometimes we can have one or more of these sets empty. For example, when $i = 1$ and $i + k = n$, we have only S_2, which is not empty.

First, we have $\Delta_{j,j+1} = \{j+1\}$ for all $j \in S_1 \cup S_2 \cup S_3$ thus $j + 1$ must belong to C'.

Let us verify it for the remaining two consecutive pairs of vertices, namely $(i, i{+}1)$ and $(i{+}k, i{+}k{+}1)$. Indeed, if no vertex of the first pair is a codeword, then

- $i+1$ is not covered in the case where $i = 1$ (S_1 is empty) or,
- the pair $(i-1, i+1)$ is not separated because $\Delta_{i-1,i+1} = \{i, i+1\}$.

For the second pair, because we have $\Delta_{i+k-1,i+k+1} = \{i+k, i+k+1\}$ then $i+k$ and $i+k+1$ can not be both not in C' when $S_3 \neq \emptyset$.

Hence in all cases, at least one of the consecutive vertices in T' must belong to the code. □

As a consequence of this last result, we have all the vertices of S_1, S_2 and S_3 except the first one in S_2 and S_3 must be a codeword. Hence, a straightforward claim is that only the vertices $i, i+1, i+k$, and $i+k+1$ may not be a codeword. Finally, we deduce that the resulting tournament T' by inverting one arc in T admit a code C' such that $|C'| \geq |C| - 2$.

Theorem 5.3. *Let T' be the resulting tournament when we invert an arc in a transitive tournament on $n \geq 3$ vertices. Then we have*

$$n - 2 \leq \gamma^{\mathrm{ID}}(T') \leq n.$$

Proof. Recall that we denote ij ($j = i + k$, k a positive integer) the arc that will be inverted. It is easy to see that when the vertices i and j are consecutive ($k = 1$), the tournament is still transitive. Hence $|C'| = n$, by proposition 4.2.

When $n = 3$, we get an oriented cycle if we invert arc 13, otherwise; the tournament is still transitive. Hence, it is obvious to check that the set $C = \{1, 2\}$ is a minimum identifying code.

If $n = 4$, then by Theorem 4.1, any minimum identifying code must have at least three codewords. Thus, we can check that the set $\{1, 3, 4\}$ is a minimum one.

Now, assume that $n \geq 5$. Hence, two cases arise:

Case 1 Let us first consider the case where $k = 2$. Hence we show that, at most, one vertex may not belong to C':

If i is not a codeword, then $i+1$ must be by Lemma 5.2. Moreover, if $i+k$ or $i+k+1$ is not in C' (not both due to the Lemma 5.2) then we have $\Delta_{i+k-1,i+k} \cap C' = \{i, i+k\} \cap C' = \emptyset$ or $\Delta_{i+k,i+k+1} \cap C' = \{i, i+k+1\} \cap C' = \emptyset$ respectively. Hence, the vertices $i+1, i+k$ and $i+k+1$ must be codewords.

Now, if $i+1$ is not a codeword then the vertices $i, i+k$ and $i{+}k{+}1$ must be codewords because $\Delta_{i,i+1} = \{i{+}1, i{+}k\}, \Delta_{i,i+k+1} = \{i{+}1, i{+}k{+}1\}$ also the vertex i must belong to the code, by Lemma 5.2.

Similarly, we can show that, at most, one vertex among $i, i+1, i+k, i+k+1$ may not be a codeword when $i+k$ or $i+k+1$ is not a codeword. Hence, in all cases, we have at most one vertex, which may not be a codeword. To conclude, we can check that the code $V(T) \setminus \{i\}$ is an identifying code.

CASE 2 Now, we assume that $k \geq 3$. First, by Lemma 5.2, it is easy to see that, at most, two vertices may not be codewords. Because $\Delta_{i-1,i+1} = \{i, i+1\}, \Delta_{i,i+1} = \{i+1, i+k\}, \Delta_{i+k-1,i+k} = \{i, i+k\}, \Delta_{i+k-1,i+k+1} = \{i+k, i+k+1\}$ and $\Delta_{i+k,i+k+1} = \{i, i+k+1\}$, then precisely only the vertices $i+1$ and $i+k+1$ may not be a codeword. Hence, we have $\gamma^{\text{ID}}(T') \geq n-2$. This bound can be met with equality when $i+k < n$ by taking as code $V \setminus \{i+1, i+k+1\}$. However, if $i+k = n$, then we observe that in the case where this last is not a codeword, then both i and $i+1$ must be in order to:

- cover the vertex i and separate them when $i = 1$,
- separate two pairs $(i, i+1)$ and $(i+k-1, i+k)$ because $\Delta_{i,i+1} = \{i+1, i+k\}$ and $\Delta_{i+k-1,i+k} = \{i, i+k\}$ respectively when $i > 1$.

Furthermore, if either i or $i+1$ is not a codeword (but not both, by Lemma 5.2), then $i+k$ must be otherwise $\Delta_{i,i+1} \cap C' = \{i+1, i+k\} \cap C' = \emptyset$ or $\Delta_{i+k-1,i+k} \cap C' = \{i, i+k\} \cap C' = \emptyset$

Hence, in all cases, one vertex may not be a codeword. To conclude, we can take the set $V \setminus \{i+1\}$ as the optimal code, which reaches the bound.

□

As an observation, the above result can be extended to the inversion of multiple arcs. In what follows, we will consider the inversion of multiple arcs that do not intersect and are not too close or simply disjoint. Indeed, we say that any two consecutive arcs $i, i+k$ and $j, j+k'$ are disjoint if we have $j-(i+k) \geq 3$. In this case, it is easy to see that the results already mentioned are still valid. However, this is not true if $1 \leq j-(i+k) \leq 2$ because two pairs of consecutive vertices, namely $(j, j+1)$ and $(j-1, j)$, may not be codewords when $j+k'$ and i belong to the code. This comes from, respectively, the fact that :

- $\Delta_{j,j+1} = \{j+1, j+k'\}$ and $\Delta_{j-1,j+1} = \{j, j+1, i\}$, if $j-(i+k) = 1$
- $\Delta_{j-1,j} = \{j, j+k'\}$ and $\Delta_{i+k,j-1} = \{j-1, i\}$, if $j-(i+k) = 2$

Moreover, the subtournament induced by vertices between the extremities of the inverted arc is not transitive when the inverted arcs intersect, i.e., $i < j < i+k$. However, note that the statement of Lemma 5.2 holds only when $j-(i+k) > 2$.

Let p_1 (resp. p_2) be the number of arcs for which their extremities are at a distance of two (resp. at least three). Then we have the following tight bounds:

Corollary 5.4. *Let T' be a tournament obtained from a transitive one by inverting p_1 and p_2 arcs where its extremities are at distance two and at least three, respectively. Then we have*

$$n - 2p_2 - p_1 \leq \gamma^{\text{ID}}(T') \leq n - 2p_2 - p_1 + 1$$

Proof. Recall that many disjoint arcs will be inverted. Let $(i, i+k)$ and $(j, j+k')$ be any consecutive inverted arcs and let us denote $T_{i,j-1}$ the subtournament induced by the set of vertices from i to $j-1$ in T.

Observe that every vertex $v_p \notin T_{i,j-1}$ such that $p < i$ (resp. $p > j-1$) is an in-neighbour (resp. out-neighbour) of all vertices of $T_{i,j-1}$. This allows us to consider such a subtournament as a whole transitive tournament with one inverted arc. Hence, by Theorem 5.3, the minimum possible cardinality is the number of vertices of the subtournament minus one if the extremities are at distance 2. On the other hand, if the extremities are at a distance of at least 3, then the minimum possible is the whole set of vertices minus two, except one, for which the second extremity matches with the last vertex of the tournament, where at most one vertex may not be a codeword. This concludes the proof. □

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References

1. N. Alon and J. H. Spencer, *The probabilistic method*, Wiley Publishing, 2004.
2. C. Balbuena, C. Dalfó, and B. Martínez-Barona, *Characterizing identifying codes from the spectrum of a graph or digraph*, Linear Algebra and its Applications **570** (2019), 138–147.
3. ———, *Sufficient conditions for a digraph to admit a $(1, \leq l)$-identifying code*, Discussiones Mathematicae Graph Theory (2019), 1–20.
4. ———, *Identifying codes in line digraphs*, Applied Mathematics and Computation **383** (2020), 125357.
5. K. Basu, M. Padhee, S. Roy, A. Pal, A. Sen, M. Rhodes, and B. Keel, *Health monitoring of critical power system equipments using identifying codes*, Critical Information Infrastructures Security (E. Luiijf, I. Žutautaitė, and B. M. Hämmerli, eds.), Springer International Publishing, 2019, pp. 29–41.
6. T. Bellitto, C. Brosse, B. Lévêque, and A. Parreau, *Locating-dominating sets in local tournaments*, Discrete Applied Mathematics **337** (2023), 14–24.
7. N. Bousquet, Q. Deschamps, T. Lehtilä, and A. Parreau, *Locating-dominating sets: From graphs to oriented graphs*, Discrete Mathematics **346** (2023), no. 1, 113–124.
8. D. Boutin, V. Horan, and M. Pelto, *Identifying codes on directed de bruijn graphs*, Discrete Applied Mathematics **262** (2019), 29–41.
9. Andries E. Brouwer, *The enumeration of locally transitive tournaments*, Mathematisch Centrum Amsterdam. Afdeling Zuivere Wiskunde. ZW, no. ZW 138/80, Stichting Math. Centrum, 1980.
10. I. Charon, S. Gravier, O. Hudry, A. Lobstein, M. Mollard, and J. Moncel, *A linear algorithm for minimum 1-identifying codes in oriented trees*, Discrete Applied Mathematics **154** (2006), no. 8, 1246–1253.
11. I. Charon, O. Hudry, and A. Lobstein, *Identifying and locating- dominating codes: NP-completeness results for directed graphs*, IEEE Transactions on Information Theory **48** (2002), no. 8, 2192–2200.
12. ———, *Minimizing the size of an identifying or locating-dominating code in a graph is NP-hard*, Theorectical Computer Science **290** (2003), no. 3, 2109–2120.

13. N. Cohen and F. Havet, *On the minimum size of an identifying code over all orientations of a graph*, The Electronic Journal of Combinatorics **25** (2018), no. 1.
14. L. N. Coregliano, *Quasi-carousel tournaments*, Journal of Graph Theory **88** (2018), no. 1, 192–210.
15. F. Foucaud, N. Ghareghani, and P. Sharifani, *Extremal digraphs for open neighbourhood location-domination and identifying codes*, Discrete Applied Mathematics **347** (2024), 62–74.
16. F. Foucaud, S. Heydarshahi, and A. Parreau, *Domination and location in twin-free digraphs*, Discrete Applied Mathematics **284** (2020), 42–52.
17. F. Foucaud, R. Naserasr, and A. Parreau, *Characterizing extremal digraphs for identifying codes and extremal cases of bondy's theorem on induced subsets*, Graphs and Combinatorics **29** (2013), no. 3, 463–473.
18. C. Godsil and G. Royle, *Algebraic graph theory*, Springer Science & Business Media, 2001.
19. J. L. Gross, J. Yellen, and P. Zhang, *Handbook of graph theory*, CRC PRESS, 2004.
20. J. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, Oxford University Press, 1968.
21. O. Hudry, *Voting procedures, complexity of*, pp. 9942–9965, Springer New York, 2009.
22. L. K. Jorgensen, *Normally regular digraphs*, the electronic journal of combinatorics **22** (2015), no. 4.
23. L. K. Jørgensen, G. A. Jones, M. H. Klin, and S. Y. Song, *Normally regular digraphs, association schemes and related combinatorial structures*, Séminaire Lotharingien de Combinatoire **71** (2014) (English).
24. M. G. Karpovsky, K. Chakrabarty, and L. B. Levitin, *On a new class of codes for identifying vertices in graphs*, IEEE Transactions on Information Theory **44** (1998), no. 2.
25. A. Lobstein, *Maintained and updated bibliography*, `https://dragazo.github.io/bibdom/main.pdf`, Accessed: 2025-02-22.
26. J. W. Moon, *Topics on tournaments*, Athena series, Holt, Rinehart and Winston, 1968.
27. S. Ray, D. Starobinski, A. Trachtenberg, and R. Ungrangsi, *Robust location detection with sensor networks*, IEEE Journal on selected areas in communications **22** (2004), no. 6, 1016–1025.
28. K. B. Reid and E. Brown, *Doubly regular tournaments are equivalent to skew hadamard matrices*, Journal of Combinatorial Theory, Series A **12** (1972), no. 3, 332–338.
29. Kenneth H. Rosen, *Elementary number theory and its applications*, Addison Wesley, 2005.
30. T. Schwartz, *Cyclic tournaments and cooperative majority voting: A solution*, Social Choice and Welfare **7** (1990), no. 1, 19–29.
31. R. D. Skaggs, *Identifying vertices in graphs and digraphs*, Ph.D. thesis, University of South Africa,Pretoria, Fibruary 2007.
32. D. J. Stewart, *Domination and matrix properties in tournaments and generalized tournaments*, Ph.D. thesis, University of Colorado, USA, 2005.
33. R. Ungrangsi, A. Trachtenberg, and D. Starobinski, *An implementation of indoor location detection systems based on identifying codes*, Intelligence in Communication Systems, Lecture Notes in Computer Science, vol. 3283, Springer Berlin Heidelberg, 2004, pp. 175–189.

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