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A GENERALIZATION OF THE BERAHA–KAHANE–WEISS THEOREM WITH GRAPH POLYNOMIAL APPLICATIONS

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Abstract. The beautiful Beraha–Kahane–Weiss (BKW) theorem has found many applications within graph theory, allowing for the determination of the limits of zeros of graph polynomials in a wide range of settings such as chromatic polynomials, network reliability, and generating polynomials related to independence and domination. However, the proof only provides solutions for linear recurrence relations of polynomials whose characteristic polynomials have simple zeros. Here we extend the class of functions to which the BKW theorem can be applied, and provide some applications in combinatorics.

1. Introduction

There are many instances where *graph polynomials* arise. For example, the well-known *chromatic polynomial* $\chi(G, x)$ (see, for example, [9]) counts the number of proper x-colourings of the vertex set of G, when x is a non-negative integer. The *(all-terminal) reliability polynomial* $\mathrm{Rel}(G, p)$ is the function whose value at $p \in [0, 1]$ is the probability that the graph is connected, given that the vertices are always operational, but the edges are independently operational with probability p (this model of reliability models the robustness of the network to random failures [7]). There are many other related polynomials, including *two-terminal* [7] and *strongly connected* [3] reliability polynomials (the latter for directed graphs). Moreover, there are polynomials that are used in various graphical sequences. For example, the *independence polynomial* $i(G, x)$, is the generating function for the number of independent sets of each cardinality in the graph G; likewise, the *domination polynomial* serves a similar function for dominating sets of a graph. In all cases, there has been a significant amount of interest in the

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zeros (or roots) of such polynomials. The addiction of the Four Colour Problem (which can be succinctly stated that 4 is never the zero of a chromatic polynomial of a planar graph) led Tutte and others (in both the mathematics and theoretical physics arenas) to investigate the location of zeros of chromatic polynomials in the complex plane. An initial study of the zeros of reliability polynomials led to the conjecture in 1992 [2] that they all lie in the closed unit disk centered at $z = 1$, and although the conjecture is now known to be false, the counterexamples found in [13] have reliability zeros lying outside the unit disk by the slimmest of margins. The zeros of each of chromatic, independence and domination polynomials have each been shown to be dense in the complex plane [5, 6, 14] (while, of course, this has not been shown for reliability polynomials).

Many graph polynomials for a family of graphs satisfy a fixed-term recurrence

$$P_{n+k}(x) = -\sum_{i=1}^{k} f_i(x) P_{n+k-i}(x), \tag{1.1}$$

where the f_i's are polynomials in x. Such a recurrence can be solved using the usual method for linear recurrences to derive an explicit formula,

$$P_n(x) = \sum_{i=1}^{k} \alpha_i(x)(\lambda_i(x))^n, \tag{1.2}$$

where the λ_i's are the zeros of the characteristic equation of the recursive relation (1.1). Beraha, Kahane and Weiss proved a beautiful result concerning the limits of the zeros of such polynomials. To be more precise, z is a *limit of zeros* of the sequence of polynomials $P_1, P_2, \ldots$ if there is a sequence $z_1, z_2, \ldots$ of complex numbers such that $P_n(z_n) = 0$ and $\lim_{n\to\infty} z_n = z$. Then the Beraha–Kahane–Weiss (BKW) Theorem is stated as follows:

Theorem 1.1. *Suppose that $P_1, P_2, \ldots$ satisfies (1.1) and the following two* nondegeneracy conditions*:*

- *the polynomials P_n do not satisfy a lower order recurrence than that in (1.1), and*
- *there are no distinct i and j for which $\lambda_i = \omega\lambda_j$ for some ω of unit modulus.*

Then $z \in \mathbb{C}$ is a limit of zeros of the P_n if and only if the λ_i can be reordered such that one of the following holds:

i. $|\lambda_k(z)| > |\lambda_i(z)|$ for all $i \neq k$ and $\alpha_k(z) = 0$, or
ii. for some $l \geqslant 2$, $|\lambda_1(z)| = |\lambda_2(z)| = \cdots = |\lambda_l(z)| > |\lambda_j(z)|$ for all $j > l$.

The BKW theorem has been applied to great effect for a number of graph polynomials (including Sokal's proof of the surprising result that chromatic zeros are dense in the complex plane [14]). Moreover, often, instead of a recurrence, the BKW Theorem can be applied to any family of polynomials that has an expression of the form in (1.2) and reverse the process and uncover the underlying recurrence (see [4]). In fact, Beraha, Kahane and Weiss' proof of their result is really of limits of zeros of sequences of polynomials whose form is given in (1.2).

However, the statement of the BKW theorem is not quite accurate. Specifically, the zeros λ_i of the characteristic polynomial of the recurrence (1.1) may be repeated. Indeed, Beraha, Kahane and Weiss note this [1], stating that in such a case (1.2) "is modified in the usual way, e.g., if $\lambda_1(z) = \lambda_2(z) \neq \lambda_j(z)$ for $j > 2$, the term $\alpha_1\lambda_1^n + \alpha_2\lambda_2^n$ is replaced by $\alpha_1\lambda_1^n + n\alpha_2\lambda_2^{n-1}$." And yet, despite the explicit mention in [1], a closer look reveals that repeated λ_i's are not covered by the theorem and proof. There are a number of instances of formulas for graph polynomials that satisfy a recurrence with exactly such repeated zeros — equivalently, the explicit formula (1.2) has the α_i a polynomial function of both z and n. For example, in [3], the limits of zeros of the family of polynomials

$$f_n = p^{2n}(p^3-4p^2+3p+1)^n+2np^2(1-p)^3(p^3+p^4-p^5)^{n-1}+p^{2n}-(p^3+p^4-p^5)^n$$

play a key role in proving that the zeros of *strongly connected reliability polynomials* (a natural generalization of all-terminal reliability to directed graphs) are dense in the complex plane. The BKW theorem was applied there, under the understanding that it held for recursive families of polynomials where the zeros of the associated characteristic polynomial had repeated roots (as indicated by the presence of n in one of the coefficients).

It is exactly the omission of the general repeated zeros case that motivated this work, where we revisit the statement and proof of the BKW theorem and extend it to such cases. Subsequently, we shall apply the result to a variety of graph polynomials (and justify the use of the BKW theorem in previous applications, including the density of roots of strongly connected reliability). We expect that our generalization will have applications elsewhere.

2. Extending the Beraha–Kahane–Weiss Theorem

In [1] there is no hint at what the extension might be when the roots of the characteristic equation are repeated. In the next theorem, we state our extension to the BKW Theorem that includes the case where the zeros of the characteristic equation λ_i's of the recursive relation (1.1) are not necessarily distinct but can be repeated zeros of arbitrary order. It is straightforward to see that this is equivalent to the condition that the coefficients α_i's in (1.2) can be functions of both n and x.

Theorem 2.1. *Let $\{P_n(x)\}$ be a sequence of analytic functions of the form*

$$P_n(x) = \sum_{i=1}^{k} \alpha_i(n;x)(\lambda_i(x))^n, \tag{2.1}$$

where $\lambda_i(x)$ are analytic and nonzero, $\lambda_i(x) \neq \omega\lambda_j(x)$ for any $\omega \in \mathbb{C}$ of unit modulus, and $\alpha_i(n;x)$ have the form

$$\alpha_i(n;x) = n^{d_i}p_{i,d_i}(x) + n^{d_i-1}p_{i,d_{i-1}}(x) + \cdots + np_{i,1}(x) + p_{i,0}(x) \tag{2.2}$$

where d_i is the degree of $\alpha_i(n;x)$, the coefficient functions $p_{i,j}$ are analytic, and p_{i,d_i} are nonzero.

Then $z \in \mathbb{C}$ is a limit of zeros of the family $\{P_n(x)\}$ if $\lambda_i(x)$ can be reordered such that either of the following conditions hold.

i. $|\lambda_k(z)| > |\lambda_i(z)|$ for all $i \neq k$ and $p_{k,d_k}(z) = 0$.

ii. for some $l \geqslant 2$, $|\lambda_1(z)| = |\lambda_2(z)| = \cdots = |\lambda_l(z)| > |\lambda_j(z)|$ for all $j > l$ and there exists at least one i such that $1 \leqslant i \leqslant l$ and $p_{i,d_i}(z) \neq 0$.

Remark: In the generalization Theorem 2.1 stated above, the sufficient condition for z to be a limit of zeros is not that the full coefficient function $\alpha_k(n;x)$ must equal zero at $x = z$ as in the original Beraha–Kahane–Weiss Theorem, but only the leading term $p_{k,d_k}(z)$ of $\alpha_k(n;x)$ be 0 at $x = z$.

Proof of Theorem 2.1. We in general follow the proof of [1], but with the additional wrinkles afforded by the two-variable generality of the coefficients. The Beraha–Kahane–Weiss Theorem was proved using Rouché's Theorem, and we shall make repeated use of this well-known result in the following form: Suppose that two functions $f(z)$ and $g(z)$ are analytic inside and on a simple closed curve C. If $|f(z)| > |g(z)|$ at each point z on C, then $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside C.

Part (i): By assumption $|\lambda_k(z)| > |\lambda_i(z)|$ for $i \neq k$ and $p_{k,d_k}(z) = 0$; we prove that z is a limit of zeros. To do so, we base our proof on the approach taken for the corresponding case in [1]. We will show that for every $\epsilon > 0$ there is a sufficiently large N such that for all $n \geqslant N$, $P_n(z_n) = 0$ for some $z_n \in D_\epsilon(z) = \{x : |x - z| < \epsilon\}$. Let C denote the boundary of $D_\epsilon(z)$.

Since the zeros of the nonzero analytic function p_{k,d_k} are isolated, by taking ϵ sufficiently small, we can assume that $|p_{k,d_k}(x)| > 0$ for $x \in C$; moreover, by the dominant condition on $\lambda_k(z)$, we can assume as well that for all $i \neq k$, $|\lambda_i(z)/\lambda_k(z)| \leqslant \rho < 1$ on C as well. Define

$$\overline{p}_{k,d_k}(n;x) = \frac{n^{d_k-1}p_{k,d_k-1}(x) + \cdots + np_{k,1}(x) + p_{k,0}(x)}{n^{d_k}}.$$

Then

$$\left(p_{k,d_k}(x) + \overline{p}_{k,d_k}(n;x)\right) n^{d_k} = \alpha_k(n;x).$$

Next define

$$w_n(x) = -\Bigg(\overline{p}_{k,d_k}(n;x) + \frac{\alpha_1(n;x)\lambda_1(x)^n}{n^{d_k}\lambda_k(x)^n} + \frac{\alpha_2(n;x)\lambda_2(x)^n}{n^{d_k}\lambda_k(x)^n} + \cdots + \frac{\alpha_{k-1}(n;x)\lambda_{k-1}(x)^n}{n^{d_k}\lambda_k(x)^n}\Bigg);$$

this yields the relationship

$$P_n(x) = (p_{k,d_k}(x) - w_n(x))\, n^{d_k}\lambda_k(x)^n.$$

For x on the boundary C and for all $i \neq k$, we have (i) $|\overline{p}_{k,d_k}(n;x)| \to 0$ as $n \to \infty$, (ii) $|\lambda_i(x)|/|\lambda_k(x)| < 1$ and (iii) $\alpha_i(n;x)$ is polynomial in n. Therefore, we conclude that $|w_n(x)|$ can be made sufficiently small on C for n large. Specifically, there exists $N > 0$ such that for all $n \geqslant N$, $|w_n(x)| < |p_{k,d_k}(x)|$ for $x \in C$.

By Rouché's Theorem, for $n \geqslant N$, $p_{k,d_k}(x) - w_n(x)$ and $p_{k,d_k}(x)$ have the same number of zeros in $D_\epsilon(z)$. Finally, since $p_{k,d_k}(z) = 0$, for all $n \geqslant N$, there exists at least one $z_n \in D_\epsilon(z)$ such that $p_{k,d_k}(z_n) - w_n(z_n) = 0$, which implies that $P_n(z_n) = 0$.

Part (ii): For this part, we present the general construction of the proof which is a winding number argument. We include the justification for the winding number along two of the four boundaries and refer the reader to the proof of the original BKW theorem [1] for the technical details (which applies for our extended case) for the winding number of the remaining two boundaries.

As in [1], in order to help with the exposition, we present the proof for $l = 3$ which is sufficiently general.

Let U be a disk about z and define $\mu(x) = \lambda_1(x)/\lambda_2(x)$. The assumption that $\lambda_1(x) \neq \omega\lambda_2(x)$ for any $\omega \in \mathbb{C}$ of unit modulus implies that μ is not constant and that $\mu'(z) \neq 0$ (since the derivative of a quotient of functions being zero would imply $\lambda_1(x)$ would be proportional to $\lambda_2(x)$ and $|\mu(z)| = 1$ would imply the modulus of the proportionality constant violates the assumption). Thus, for U sufficiently small, μ is invertible from U onto a neighborhood V of $\omega = \mu(z)$. By the assumption that $|\lambda_1(z)| = |\lambda_2(z)|$, we have $|\omega| = 1$.

By symmetry, we can assume that $p_{1,d_1}(z) \neq 0$, and hence for all sufficiently large n, $\alpha_1(n;x) \neq 0$ on U. If ν denotes the inverse of μ, then

$$
\begin{aligned}
\Pi_n(x) &= \frac{P_n(\nu(x))}{\alpha_1(n;\nu(x))\lambda_2(\nu(x))^n} \\
&= x^n + \frac{\alpha_2(n;\nu(x))}{\alpha_1(n;\nu(x))} + \frac{\alpha_3(n;\nu(x))}{\alpha_1(n;\nu(x))}\left(\frac{\lambda_3(\nu(x))}{\lambda_2(\nu(x))}\right)^n \\
&\quad + \sum_{j=4}^{k} \frac{\alpha_j(n;\nu(x))}{\alpha_1(n;\nu(x))}\left(\frac{\lambda_j(\nu(x))}{\lambda_2(\nu(x))}\right)^n
\end{aligned}
$$

By taking a smaller (noncircular) neighborhood in U, we can assume that for some $r_0 > 1$ and $\theta_0 > 0$, the image of

$$V = N_{r_0,\theta_0}(\omega) = \{re^{i\theta}\omega : r_0^{-1} < r < r_0, -\theta_0 < \theta < \theta_0\}$$

under ν is a subset of U. Let $C = C_1 + C_2 + C_3 + C_4$ denote the boundary curve of $N_{r_0,\theta_0}(\omega)$ (see Figure 1), where $|x| = r_0$ on C_1, $|x| = r_0^{-1}$ on C_3, $x = re^{i\theta_0}\omega$ on C_2, and $x = re^{-i\theta_0}\omega$ on C_4. We traverse around C in the counterclockwise direction.

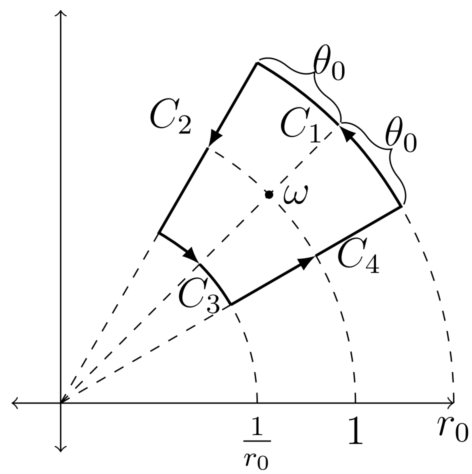

FIGURE 1. The set $N_{r_0,\theta_0}(\omega)$ with counterclockwise oriented boundary $C = C_1 + C_2 + C_3 + C_4$

Since α_i are all polynomial in n, the relationship of the λ_i, and the modulus of x on C_1 is greater than 1, the x^n term dominates Π_n for large n on C_1 and the image of Π_n of C_1 winds around 0 on the order of $n\theta_0$ times. On the other hand, along C_3 the modulus is less than 1 and thus the $\frac{\alpha_2(n;v(x))}{\alpha_1(n;v(x))}$ term dominates and since $\frac{\alpha_2(n;v(x))}{\alpha_1(n;v(x))}$ is close to $\frac{\alpha_2(n;z)}{\alpha_1(n;z)}$, for n large, the image of Π_n of C_3 does not wind around 0.

Lastly, along C_2 and C_4, the argument of x is fixed and it was shown in [1] that the winding numbers of Π_n along C_2 and C_4 are bounded below by a constant. Therefore, we conclude that, for all n sufficiently large, the image of Π_n of the boundary C of the set V winds around 0 a positive number of times. This implies that for all n sufficiently large, Π_n has a zero in V which implies that P_n has a zero in U. □

We next prove a partial converse of Theorem 2.1 for the case $k = 3$ which again is sufficiently general. The proof of this result applies Lemma 4.1 in [1] which we state below for reference.

Lemma 2.3. *Suppose that $\gamma_n, \delta_n, \mu_n, \sigma_n, \lambda_j > 0, \gamma_n, \delta_n \leqslant M, n = 1, 2, \ldots,$ $j = 1, 2$ and that $\mu_n \to \lambda_1, \sigma_n \to \lambda_2$. Then*

$$\limsup_{n\to\infty}(\gamma_n\mu_n^n + \delta_n\sigma_n^n)^{1/n} \leqslant \max(\lambda_1, \lambda_2).$$

Theorem 2.4. *Suppose that $z \in \mathbb{C}$ is a limit of zeros z_n of*

$$P_n(x) = \sum_{i=1}^{3} \alpha_i(n;x)(\lambda_i(x))^n, \tag{2.3}$$

where $\lambda_i(x)$ are analytic and nonzero, $\lambda_i(x) \neq \omega\lambda_j(x)$ for any $\omega \in \mathbb{C}$ of unit modulus, and $\alpha_i(n;x)$ have the form

$$\alpha_i(n;x) = n^{d_i} p_{i,d_i}(x) + n^{d_i-1} p_{i,d_{i-1}}(x) + \cdots + np_{i,1}(x) + p_{i,0}(x)$$

where d_i is the degree of $\alpha_i(n;x)$ and the coefficient functions $p_{i,j}$ are analytic.

If $|\lambda_3(z)| > |\lambda_1(z)|, |\lambda_2(z)|$, then $p_{3,d_3}(z) = 0$.

Proof. The dominant root assumption implies that

$$|\alpha_3(n;z_n)|^{1/n}|\lambda_3(z_n)| = |\alpha_1(n;z_n)\lambda_1(z_n)^n + \alpha_2(n;z_n)\lambda_2^n|^{1/n}. \tag{2.4}$$

If $p_{3,d_3}(z) \neq 0$ then the left hand side of the above equation converges to $|\lambda_3(z)|$ when $n \to \infty$. The application of Lemma 2.3 to (2.4) requires that $|\alpha_1(n;z_n)|, |\alpha_2(n;z_n)| \leqslant M$ for $n = 1, 2, \ldots$. This is satisfied in the original case where α_i does not depend on n but our generalization assumes α_i are polynomials in n. In order to apply Lemma 2.3 to the generalized case, we rewrite

$$|\alpha_1(n;z_n)||\lambda_1(z_n)|^n = \\ n^{d_1}\left|p_{1,d_1}(z_n) + \frac{1}{n}p_{1,d_1-1}(z_n) + \cdots + \frac{1}{n^{d_1}}p_{1,0}(z_n)\right||\lambda_1(z_n)|^n$$

and we bound n^{d_1} by ϵ_1^n where $\epsilon_1 > 1$ for n sufficiently large. We do the same for the $\alpha_2(n;z_n)$ term and now since $\left|p_{1,d_1}(z_n) + \frac{1}{n}p_{1,d_1-1}(z_n) + \cdots + \frac{1}{n^{d_1}}p_{1,0}(z_n)\right|$ (and the analogous term for $\alpha_2(n;z_n)$) is bounded, we can apply Lemma 2.3 and we get

$$|\lambda_3(z)| \leqslant \max\{\epsilon_1|\lambda_1(z)|, \epsilon_2|\lambda_2(z)|\}$$

for any $\epsilon_1, \epsilon_2 > 1$, which contradicts the assumption that $|\lambda_3(z)| > |\lambda_1(z)|, |\lambda_2(z)|$. □

For the $k = 2$ case, the above argument greatly simplifies to the following. Without loss of generality, assume that $|\lambda_1(z)| > |\lambda_2(z)|$. Then we have

$$\alpha_1(n;z_n)\lambda_1(z_n)^n = -\alpha_2(n;z_n)\lambda_2(z_n)^n.$$

which implies that

$$|\alpha_1(n;z_n)|^{1/n}|\lambda_1(z_n)| = |\alpha_2(n;z_n)|^{1/n}|\lambda_2(z_n)|. \tag{2.5}$$

If $p_{1,d_1}(z) \neq 0$ and $p_{2,d_2}(z) \neq 0$ then close to z, both $\alpha_1(n;z_n)$ and $\alpha_2(n;z_n)$ are bounded away from 0, and hence their n–th zeros go to 1. It follows by taking limits in (2.5) that $|\lambda_1(z)| = |\lambda_2(z)|$, a contradiction.

Concerning Theorem 2.1, we remark that while the form of the coefficient functions α_i in (2.2) as polynomials in n cover the applications included in

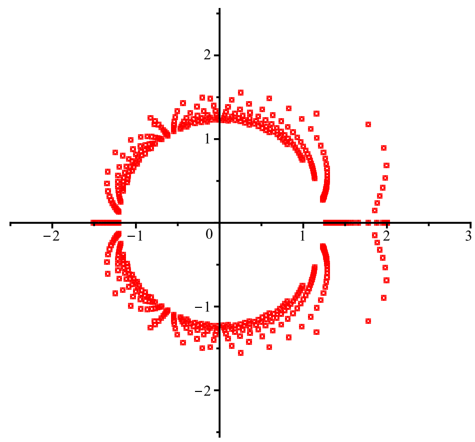

FIGURE 2. Zeros of $f_n(x) = x^{n+1} - 2x^n + x^2 + n^2$ for $2 \leqslant n \leqslant 30$.

this work, they are sufficient but not necessary assumptions for the conclusion of Theorem 2.1. For example, for part (i), the assumption on the α_i could be generalized to

$$|\alpha_i(n;x)| = o\left(\left|\frac{\lambda_i(x)}{\lambda_k(x)}\right|^n\right)$$

(where o denotes the limit as $n \to \infty$). For part (ii), the generalization extends to

$$\left|\frac{\alpha_i(n;x)}{\alpha_1(n;x)}\right| = o(|x|^n).$$

We illustrate Theorem 2.1 with a couple of examples of sequences of polynomials (in the next section, we give applications of Theorem 2.1 to various graph polynomials). Consider the sequence of polynomials

$$f_n(x) = x^{n+1} - 2x^n + x^2 + n^2 = (x-2)x^n + (n^2 + x^2) \cdot 1^n$$

By Theorem 2.1, the limits of zeros of f_n are the unit circle $|z| = 1$ centred at the origin and the isolated point $z = 2$ (see Figure 2). On the other hand, for

$$g_n(x) = x^{n+1} - 2x^n + n^2x^2 + 5nx + 1 = (x-2)x^n + (n^2x^2 + 5nx + 1) \cdot 1^n,$$

the limits of zeros of g_n are the unit circle centred at the origin and the isolated points $z = 2$ and $z = 0$ (see Figure 2). As stated in Remark 2.2, the isolated limit of zeros $z = 0$ is derived from the zero of the leading term n^2x^2 of $n^2x^2 + 5nx + 1$.

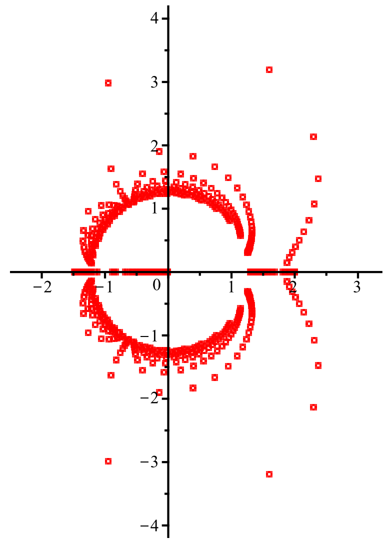

FIGURE 3. Zeros of $g_n(x) = x^{n+1} - 2x^n + n^2x^2 + 5nx + 1$ for $2 \leqslant n \leqslant 30$.

3. Graph Polynomial Applications

We return to the example from the opening section (and from [3]),

$$\begin{aligned} f_n &= p^{2n}(p^3 - 4p^2 + 3p + 1)^n + 2np^2(1-p)^3(p^3 + p^4 - p^5)^{n-1} + p^{2n} \\ &\quad -(p^3 + p^4 - p^5)^n \\ &= \sum_{i=1}^{3} \alpha_i \lambda_i^N, \end{aligned}$$

where $N = n - 1$,

$\alpha_1 = p^2(p^3 - 4p^2 + 3p + 1),\ \alpha_2 = -(p^3 + p^4 - p^5) + 2np^2(1-p)^3,\ \alpha_3 = p^2,$

and

$$\lambda_1 = p^2(p^3 - 4p^2 + 3p + 1),\ \lambda_2 = p^3 + p^4 - p^5,\ \lambda_3 = p^2.$$

In [3], the solutions to

$$|\lambda_1| = |\lambda_2|$$

were solved for, yielding (in part) a curve $\mathcal{C}$, given by

$$b = \sqrt{\frac{16a - 15a^2 - 8 + 6a^3 + \sqrt{532a^2 - 408a^3 - 292a + 112a^4 + 57}}{7 - 6a}}.$$

where $z = a + bi$. The curve $\mathcal{C}$ does not contain 0, and runs from 1 to $\frac{7}{6} + \sqrt{\frac{19}{18}}\, i$. On this curve (except at $z = 1$), it was shown that

$$|\lambda_1| = |\lambda_2| > |\lambda_3|.$$

Using the notation of Theorem 2.1, note that for all points on $\mathcal{C} - \{1\}$, $p_{2,\alpha_2} = 2p^2(1-p)^3$ is nonzero, so by Theorem 2.1, all points on $\mathcal{C} - \{1\}$ (and hence on $\mathcal{C}$) are limits of zeros of f_n. This justifies the claim in [3], and the remainder of the proof in [3] showing that the zeros of strongly connected reliability polynomials are dense in the complex plane proceeds exactly as provided there.

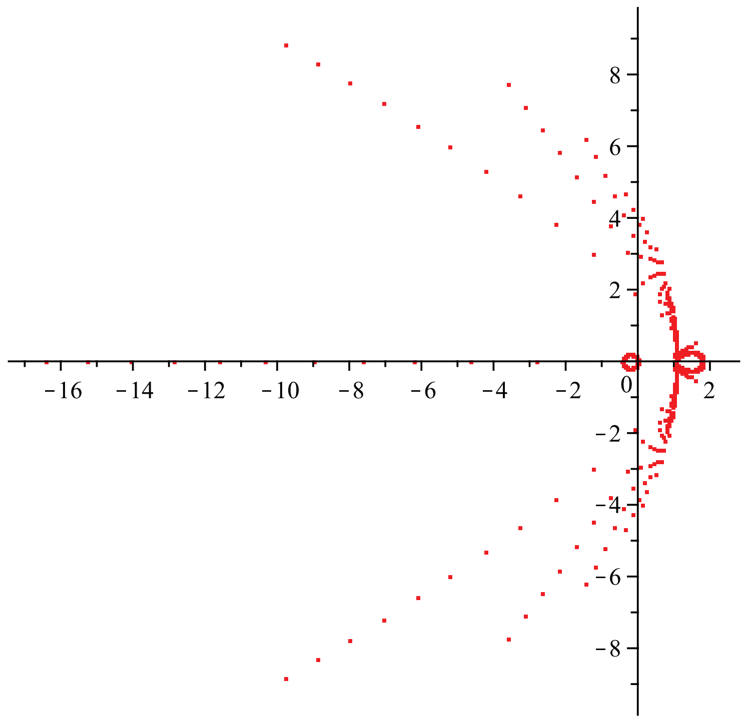

FIGURE 4. Zeros of $f_n = p^{2n}(p^3 - 4p^2 + 3p + 1)^n + 2np^2(1 - p)^3(p^3 + p^4 - p^5)^{n-1} + p^{2n} - (p^3 + p^4 - p^5)^n$ for $2 \leqslant n \leqslant 24$.

Finally, the impetus for our extension of the BKW Theorem began with a new graph polynomial stemming from the theory of minimal spanning trees of graphs with random edge lengths. The topic of random minimal spanning trees are well-studied (see [8] and the references therein) with the first major result by Frieze [11] that proved the convergence of the mean minimal spanning the of the complete graph with random edge weights distributed uniformly over the unit interval $[0, 1]$ as the number of vertices tends to infinity. While most of the theory in the field are asymptotic results, a significant preasymptotic exact formula was proved by Steele in [15].

In that work Steele derived an integral formula for the mean length of minimal spanning trees of G. Specifically, each edge of G is assigned independent random variables distributed uniformly over the unit interval $[0, 1]$.

For each assignment of random edges, the total length of minimal spanning trees is denoted by $L(G)$. The mean over all possible sets of random edge lengths was proven to have the form $\mathbb{E}[L(G)] = \int_0^1 S(G;t)\,dt$ where

$$S(G;t) = \frac{(1-t)}{t}\frac{T_x(G;1/t,1/(1-t))}{T(G;1/t,1/(1-t))} \tag{3.1}$$

and $T(G;x,y)$ is the well-known Tutte polynomial of the graph G. Here T_x denotes the partial derivative of $T(G;x,y)$ with respect to x. In [12], it was shown that $S(G;t)$ is a polynomial of degree at most equal to the number of edges of G which we refer to as the *Steele polynomial* of the graph G. Not only do the Steele polynomials yield the exact mean minimal spanning tree, Steele's result revealed another connection to the Tutte polynomial, which has many applications in graph theory. See [10] for a recent survey book on the Tutte polynomial. Additional properties of Steele polynomials, including information on the coefficients of the polynomials, appear in [12].

Steele polynomials are trivial for trees and cycles, so the next most interesting case is a *theta graph* $\theta_{a,b,c}$, consisting of two vertices joined by internally disjoint paths of lengths a, b and c. A straightforward calculation of the Tutte polynomial of the theta graph using the deletion-contraction algorithm (see [12]) yields the Steele polynomial for the graphs $\theta_{n-2,2,1}$ as

$$S(\theta_{n-2,2,1};t) = (n-1) - (n+1)t + t^3 + t^{n-1} + t^n - t^{n+1}. \tag{3.2}$$

which satisfies the following recurrence relation

$$P_n(t) = (t+2)P_{n-1}(t) - (2t+1)P_{n-2}(t) + tP_{n-3}(t).$$

The corresponding characteristic polynomial $\lambda^3 - (t+2)\lambda^2 + (2t+1)\lambda - t$ has a simple zero at $\lambda_1(t) = t$ and a repeated zero at $\lambda_2(t) = 1$ which (with appropriate initial conditions) yield the solution $S(\theta_{n-2,2,1};t)$ above. It is the repeated zero $\lambda_2(t) = 1$ that forces the Steele polynomials of these graphs beyond the scope of the original BKW Theorem (Theorem 1.1) and the investigation into the limits of zeros of these polynomials that prompted our work on the extension (Theorem 2.1).

However, the Steele polynomials $S(\theta_{n-2,2,1};t)$ defined in (3.2) can be expressed as

$$S(\theta_{n-2,2,1};t) = \alpha_1(n-1;t)(\lambda_1(t))^{n-1} + \alpha_2(n-1;t)(\lambda_2(t))^{n-1} \tag{3.3}$$

where , $\lambda_1(t) = t, \lambda_2(t) = 1, \alpha_1(n-1;t) = t - t^2 + 1$, and $\alpha_2(n-1;t) = n-1-(n+1)t+t^3$. Theorem 2.1 yields that the limits of zeros of $S(\theta_{n-2,2,1};t)$ are the unit circle $|t| = 1$ and the isolated limit of zeros at the golden ratio $\phi = (1+\sqrt{5})/2$.

References

1. S. Beraha, J. Kahane and N. Weiss, *Limits of zeroes of recursively defined families of polynomials*, in: Studies in Foundations and Combinatorics, Vol. 1, Advances in

Mathematics Supplementary Studies (G.-C. Rota, ed.), Academic Press, New York (1978), 213–232.
2. J. I. Brown and C.J. Colbourn, *Roots of the reliability polynomial*, SIAM J. Discrete Math., **5** (1992), 571–585.
3. J. I Brown and D. Cox, *The closure of the roots of strongly connected reliability polynomials is the entire complex plane*, Discrete Math. **309** (2009), 5043–5047.
4. J. I. Brown and C. A. Hickman, *On chromatic roots of large subdivisions of graphs*, Discrete Mathematics **242** (2002), 17–30.
5. J. I. Brown, C. A. Hickman, and R. J. Nowakowski, *On the location of roots of independence polynomials*, J. Algebraic Combin. **19** (2004), 273–282.
6. J. I. Brown and J. Tufts, *On the Roots of Domination Polynomials*, Graphs Combin. **30** (2014), 527–647.
7. C. J. Colbourn, *The Combinatorics of Network Reliability*, Oxford University Press, New York, 1987.
8. C. Cooper, A. Frieze, N. Ince, S. Janson and J. Spencer, *On the length of a random minimum spanning tree*, Combin., Prob., and Computing. **25** (2016), 89–107.
9. F. M. Dong, , K. M. Koh and K. L. Teo, *Chromatic Polynomials and Chromaticity of Graphs*, World Scientific, London, 2005.
10. J. A. Ellis-Monaghan and I. Moffatt, *Handbook of the Tutte polynomial and related topics*, CRC Press, 2022.
11. A. M. Frieze, *On the value of a random minimum spanning tree problem*, Discrete Applied Mathematics **10** (1985), 47–56.
12. J. Nishikawa, P. T. Otto, and C. Starr, Polynomial representation for the expected length of minimal spanning trees, Pi Mu Epsilon Journal **13** (2012), 357–365.
13. G. Royle and A.D. Sokal, *The Brown-Colbourn conjecture on the zeros of reliability polynomials is false*, J. Combin. Theory Ser. B **91** (2004), 345–360.
14. A.D. Sokal, *Chromatic roots are dense in the whole complex plane*, Probab. Combin. Comput. **13** (2004), 221–261.
15. J. M. Steele, *Minimum spanning trees for graphs with random edge lengths*, Mathematics and Computer Science **II** (2002), 223–245.

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