Contributions to Discrete Mathematics

Volume 19, Number 4, Pages 85–109
ISSN 1715-0868

BETA DISTRIBUTIONS WHOSE MOMENT SEQUENCES ARE RELATED TO INTEGER SEQUENCES LISTED IN THE OEIS

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Abstract. We recall some basic properties of the Beta distribution and some of its modifications. We identified around 20 of the moment sequences of Beta distributions as important integer sequences in the OEIS base of integer sequences. Among those identified are Catalan, Riordan, Motzkin, or 'super ballot numbers'. By applying a method of expansion of the ratio of densities of involved distributions we are able to obtain some known and many unknown relationships between e.g. Catalan numbers and other moment sequences of the Beta distributions.

1. Introduction

This paper intends to show that many well-known discrete sequences of numbers that are important in many, distant from the theory of probability, branches of mathematics are, in fact, closely connected with moment sequences of the beta distribution. By close connection we mean some simple operation like, for example, multiplication by a sequence of powers of some number or binomial transformation.

The idea of representing known integer sequences as moment sequences is becoming more and more popular in recent years. Nice arguments to follow this idea were recently presented in the paper of Sokal [10]. There is also a nice review of basic facts concerning moment sequence as well as some criteria both sufficient and necessary and only sufficient for a number sequence to be a moment sequence.

Received by the editors October 12, 2022, and in revised form February 28, 2023.

1991 *Mathematics Subject Classification.* Primary 05A19, 05A10; Secondary 44A60, 60E05.

Key words and phrases. Beta distribution, Catalan numbers, Motzkin numbers, Riodran numbers, super ballot numbers, moment sequences.

The author is grateful to the referees for their remarks and suggestions. The first one inspired the author to formulate and prove Theorem 2.4. The second one pointed out numerous inaccuracies, misspellings and inconsistencies in the previous version of the paper.

Some of these arguments and facts we repeat here below, however, to get the definition of the moment sequences and their basic properties as well as the properties of the set of such sequences we refer the reader to the paper of Sokal or to the appendix of the recently published, paper of Szabłowski [14].

As far as moments of the Beta distribution are concerned, it turns out that, for example, sequences of Catalan, Motzkin, Riordan or 'super ballot' numbers are such sequences. They are, in fact, moments of some modification the classical beta considered on the segment $[0,1]$. In order to indicate briefly what modifications we mean firstly we will recall the definition of Beta distribution and the modifications that we are going to consider.

We will be dealing with the moment sequences $\{m_n\}$ i.e. sequences that are defined by the following formula

$$m_n = \int x^n d\mu\left(x\right),$$

$n \geq 0$, where μ denotes a positive measure on the real line.

Recall, that a numerical sequence $\{m_n\}_{n\geq 0}$ with $m_0 = 1$, is a moment sequence of a probability distribution with infinite support iff all its Hankel matrices $H_n := [m_{i+j}]_{0\leq i+j\leq n}$ are positive definite, or equivalently, the values of all determinants $\{\det[H_n]\}_{n\geq 0}$ have positive values. Additionally, if the distribution whose moments are elements of the sequence $\{m_n\}$ has the support contained in the non-negative axis, then the moment sequence also satisfies the following condition: The following sequence

$$\left\{\det H_n^{'}\right\}_{n\geq 0}$$

also assumes non-negative values. Here matrix $H_n^{'}$ has $\{i,j\}$ −th entry equal to m_{1+i+j}, for $0 \leq i,j \leq n$.

Let us recall the general property of the moment sequences that will be of use in the sequel.

Proposition 1.1. *Suppose $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ are two moment sequences. Then $\{a_n b_n\}_{n\geq 0}$, $\left\{\sum_{j=0}^{n}\binom{n}{j}a_j b_{n-j}\right\}_{n\geq 0}$ and $\left\{\sum_{j=0}^{n}(-1)^j\binom{n}{j}a_j b_{n-j}\right\}_{n\geq 0}$ are also moment sequences.*

Proof. For the proof see, e.g., either [3] or [15]. □

Finally, by expanding the ratio of the densities, whose moments we are considering, and then integrating, we get for free the relationships between involved moments. The idea of expanding the ratio of the densities (the Radon-Nikodym derivative more generally) has been presented in [12] and later developed and generalized in [13]. It happens very often that the sequence of the moments has a deep combinatorial interpretation and by using this method of expansion, we can get relationships between these important combinatorial sequences that are usually difficult to prove by combinatorial means. For example, we can easily show that

$$\binom{2n}{n} = 4^n - 2\sum_{j=0}^{n-1} C_j 4^{n-1-j},$$

$$C_n = \frac{3}{2}\sum_{i\geq 0}\frac{1}{4^i}\frac{(2n+2i)!}{(i+n)!(n+i+2)!},$$

where $\{C_n\}_{n\geq 0}$ are the so-called Catalan numbers. Numbers $\{C_n\}$ constitute, as it turns out, the moment sequence of the distribution with the following density:

$$\frac{1}{2\pi}\sqrt{\frac{4-x}{x}},$$

where $0 < x < 4$.

We will go into detail and give more examples in the sequel. The paper is organized as follows. In the next Section 2, we present a definition of Beta distribution, its modifications and the basic properties of its moments. The following Section 3, is dedicated to the presentation of particular examples, mostly concerning cases when α and β are multiples of $1/2$. In this section we identify many moments of the beta distribution appearing in the OEIS. The last Section, 4, is devoted to the presentation of some (by no means all) possible expansions finite as well as infinite expansions of elements of one sequence in terms of the other.

2. Basic ingredients and properties

Let us recall the so-called beta distribution, i.e., the distribution with the density defined for $|x| < 1$ and $\alpha, \beta > 0$,

$$a(x;\alpha,\beta) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}, \tag{2.1}$$

where $B(\alpha, \beta)$ denotes the value of the well-known Beta function taken at α and β. It is well-known that the sequence of moments of this distribution is given by the formula:

$$M_n(0,\alpha,\beta)/4^n = \int_0^1 x^n a(x;\alpha,\beta)dx = \frac{\alpha^{(n)}}{(\alpha+\beta)^{(n)}}. \tag{2.2}$$

We use notation that appears to be redundant at first sight. The reason for this will be clear in the sequel.

In formula (2.2), we used the following notation. For $x \in \mathbb{C}$ let us denote:

$$(x)_{(n)} = x(x-1)\cdots(x-n+1). \tag{2.3}$$

This polynomial in x will be called falling factorial while the following polynomial

$$(x)^{(n)} = x(x+1)\cdots(x+n-1), \tag{2.4}$$

will be called rising factorial, (sometimes called Pochhammer polynomial, function or symbol). In both cases we set 1 when $n = 0$.

It is also well-known that

$$(x)_{(n)} = (-1)^n(-x)^{(n)}, \text{ and } (x)^{(n)} = (-1)^n(-x)_{(n)}. \tag{2.5}$$

By the binomial theorem, for all complex $|x| < 1$ we have:

$$(1-x)^\alpha = \sum_{j\geq 0}(-x)^j(\alpha)_{(j)}/j! = \sum_{j\geq 0} x^j(-\alpha)^{(j)}/j!. \tag{2.6}$$

The modifications of the beta distribution that we are going to consider, are the following. Namely, we will examine the distribution family as well as densities.

$$g(x;c,\alpha,\beta) = \frac{(x-c)^{\alpha-1}(4+c-x)^{\beta-1}}{4^{\alpha+\beta-1}B(\alpha,\beta)}, \tag{2.7}$$

supported on the segment $[c, 4+c]$, $c \in \mathbb{R}$. If we denote by X the random variable with the density $a(s;\alpha,\beta)$ and by Y the random variable with the density $g(x;c,\alpha,\beta)$, then we see that

$$Y = 4X + c.$$

Let us denote by $M_n(c,\alpha,\beta)$ the $n-$th moment of Y, i.e.,

$$M_n(c,\alpha,\beta) = \int_c^{4+c} x^n g(x;c,\alpha,\beta)dx. \tag{2.8}$$

Note that for all $\alpha,\beta > 0$ and $c \in \mathbb{R} : M_0(c,\alpha,\beta) = 1$.

We have the following Lemma:

Lemma 2.1. *i)* $\forall b,c \in \mathbb{R}$, $\alpha,\beta > 0$ *and* $n \in \mathbb{N}\cup\{0\}$ *we have:*

$$M_n(b,\alpha,\beta) = \sum_{j=0}^n \binom{n}{j} M_j(c,\alpha,\beta)(b-c)^{n-j}. \tag{2.9}$$

In particular, we get:

$$M_n(c,\alpha,\beta) = \sum_{j=0}^n \binom{n}{j} M_j(0,\alpha,\beta)c^{n-j}, \tag{2.10}$$

$$4^n\frac{\alpha^{(n)}}{(\alpha+\beta)^{(n)}} = M_n(0,\alpha,\beta) = \sum_{j=0}^n \binom{n}{j} M_j(c,\alpha,\beta)(-c)^{n-j}. \tag{2.11}$$

ii)

$$M_n(c,\alpha+1,\beta) = \frac{\alpha+\beta}{4\alpha}(M_{n+1}(c,\alpha,\beta) - cM_n(c,\alpha,\beta)),$$
$$M_n(c,\alpha,\beta+1) = \frac{\alpha+\beta}{4\beta}((4+c)M_n(c,\alpha,\beta) - M_{n+1}(c,\alpha,\beta)).$$

iii) If $c = 0$ we have also for $n \geq 1$

$$M_n(0, \alpha, \beta) = \frac{4\alpha}{\alpha + \beta} M_{n-1}(0, \alpha + 1, \beta), \tag{2.12}$$

$$M_n(0, \alpha, \beta) = \frac{\beta}{\alpha + \beta} \sum_{j \geq 0} M_{n+j}(0, \alpha, \beta + 1). \tag{2.13}$$

Proof. i) Let us denote by $Y_b = 4X + b = 4X + c + (b - c) = Y_c + (b - c)$, so we have:

$$\begin{aligned} M_n(b, \alpha, \beta) &= EY_b^n \\ &= \sum_{j=0}^{n} \binom{n}{j} EY_b^j (b - c)^{n-j} \\ &= \sum_{j=0}^{n} \binom{n}{j} M_j(c, \alpha, \beta)(b - c)^{n-j}. \end{aligned}$$

ii) We have:

$$\begin{aligned} M_n(c, \alpha + 1, \beta) &= \frac{\alpha + \beta}{4\alpha} \int_c^{4+c} (x - c) x^n g(x; c, \alpha, \beta) dx, \\ &= \frac{\alpha + \beta}{4\alpha} (M_{n+1}(c, \alpha, \beta) - cM_n(c, \alpha, \beta). \\ M_n(c, \alpha, \beta + 1) &= \frac{\alpha + \beta}{4\beta} \int_c^{4+c} (4 + c - x) x^n g(x; c, \alpha, \beta) dx, \\ &= \frac{\alpha + \beta}{4\beta} ((4 + c) M_n(c, \alpha, \beta) - M_{n+1}(c, \alpha, \beta)). \end{aligned}$$

iii) First of all notice that

$$B(\alpha + 1, \beta)/B(\alpha, \beta) = \frac{\alpha}{\alpha + \beta},$$

$g(x; 0, \alpha, \beta)/g(x; 0, \alpha + 1, \beta) = \frac{4\alpha}{(\alpha+\beta)x}$. Now,

$$M_n(0, \alpha, \beta) = \frac{4\alpha}{\alpha + \beta} M_{n-1}(0, \alpha + 1, \beta).$$

Similarly

$$\frac{g(x; 0, \alpha, \beta)}{g(x; 0, \alpha, \beta + 1)} = \frac{4\beta}{(\alpha + \beta)(4 - x)} = \frac{\beta}{\alpha + \beta} \sum_{j \geq 0} (x/4)^j,$$

and notice that the series is convergent and increasing for $x \in [0, 4)$. Thus, by the Lebesgue's monotone convergence theorem we get the second assertion. □

Remark: Notice also, that we have

$$\frac{M_n(b, \alpha, \beta)}{(b - c)^n} = \sum_{j=0}^{n} \binom{n}{j} (-1)^j \frac{M_j(c, \alpha, \beta)}{(c - b)^j}. \tag{2.14}$$

Thus, the sequence $\{M_n(b,\alpha,\beta)/(b-c)^n\}_{n\geq 0}$ is the so-called binomial transformation of the sequence $\{M_n(c,\alpha,\beta)/(c-b)^n\}_{n\geq 0}$. Conversely, notice that the sequence $\{M_n(c,\alpha,\beta)/(c-b)^n\}_{n\geq 0}$ is the binomial transform of the sequence

$$\{M_n(b,\alpha,\beta)/(b-c)^n\}_{n\geq 0}.$$

However, in the very important base of integer sequences known as the OEIS, there is a slightly different definition of the binomial transform.

Namely, given two sequences $\{a_n\}_{n\geq 0}$ and $\{b_n\}_n$, if we have for all $n\geq 0$:

$$b_n=\sum_{j=0}^{n}\binom{n}{j}a_j,$$

then, we say, that sequence $\{b_n\}$ is the binomial transform of the sequence $\{a_n\}$. Note, that then we also have for all $n\geq 0$:

$$a_n=\sum_{j=0}^{n}(-1)^j\binom{n}{j}b_j.$$

Then, according to the OEIS terminology sequence $\{a_n\}$ is the inverse binomial transform of the sequence $\{b_n\}$.

Hence, using this terminology, the sequence $\{M_n(b,\alpha,\beta)/(b-c)^n\}$, is the binomial transform of the sequence $\{M_n(c,\alpha,\beta)/(b-c)^n\}$.

Furthermore, it follows from these considerations that to get moments $M_n(c,\alpha,\beta)$ it is enough to calculate moments $M_n(0,\alpha,\beta)$ and then apply the appropriate binomial transformation.

Since many integer sequences in the OEIS are identified by their generating functions, we will also calculate generating functions of many of these integer sequences. Let us remark, that we need to have a generating function defined only on the small open interval around 0. What matters, are the coefficients of its Taylor expansion around zero. That is why all considered below generating functions will be considered for $x\in(-\delta,\delta)$, $\delta>0$. Very often δ will be equal to 1.

Lemma 2.3. *Let us consider the sequence of moments given by the formula (2.2) such that $\alpha+\beta$ is an integer, then*

i)

$$\begin{aligned}&g(t;\alpha,\beta)\\&=\sum_{j\geq 0}t^jM_j(0,\alpha,\beta)/4^j\\&=\begin{cases}1/(1-t)^{\alpha} & \text{if}\quad \alpha+\beta=1\\ (1-(1-t)^{1-\alpha})/(t(1-\alpha)) & \text{if}\quad \alpha+\beta=2\\ ((1-t)^{2-\alpha}+t(2-\alpha)-1)/(t^2(1-\alpha)(2-\alpha)) & \text{if}\quad \alpha+\beta=3\\ (6(1-t)^{3-\alpha}-(\alpha^2t^2-5\alpha t^2+6t^2+2\alpha t-6t+2))\\ \quad /\left(t^3(\alpha-1)(\alpha-2)(\alpha-3)\right) & \text{if}\quad \alpha+\beta=4\end{cases},\end{aligned}$$

for $t \in (-1, 1)$.

ii) If $g(x)$ *is a generating function of* $\{M_n(0,\alpha,\beta)/4^n\}$ *then* $\frac{(\alpha+\beta)}{x\alpha}(g(x)-1)$ *is a generating function of* $\{M_n(0,\alpha+1,\beta)/4^n\}$*, while* $\frac{(\alpha+\beta)}{4x\beta}(1-(1-4x)g(x))$ *is a generating function of* $\{M_n(0,\alpha,\beta+1)/4^n\}$.

iii) Let $g(x)$ *be a generating function of the sequence* $\{f_n\}_{n\geq 0}$ *i.e.,* $g(x) = \sum_{n\geq 0} f_n x^n$*, then* $\frac{1}{1-cx} g(\frac{x}{1-cx})$ *is the generating function of the sequence* $\left\{\sum_{j=0}^{n}\binom{n}{j} f_j c^{n-j}\right\}_{n\geq 0}$.

Proof. i) Recall, that $(k)^{(n)} = (n+k-1)!/(k-1)!$ for all nonnegative integers k, n such that $k+n \neq 0$. We get for $\alpha+\beta=1$:

$$\sum_{j\geq 0} t^j M_j(0,\alpha,\beta)/4^j = \sum_{j=0}^{\infty} t^j (\alpha)^{(j)}/j! = \sum_{j=0}^{\infty} t^j (-1)^j (-\alpha)_{(j)}/j! = \frac{1}{(1-t)^a},$$

by the binomial theorem (2.6). When $\alpha+\beta=2$, we have:

$$\begin{aligned}\sum_{j\geq 0} t^j M_j(0,\alpha,\beta)/4^j &= \sum_{j=0}^{\infty} t^j (\alpha)^{(j)}/(j+1)! \\ &= \frac{-1}{t(1-\alpha)} \sum_{j=0}^{\infty} (\alpha-1) t^{j+1} (\alpha)^{(j)}/(j+1)! \\ &= \frac{-1}{t(1-\alpha)} \sum_{j=0}^{\infty} t^{j+1} (\alpha-1)^{(j+1)}/(j+1).\end{aligned}$$

We act likewise when $\alpha+\beta=3$ and when $\alpha+\beta=4$.

ii) These assertions are based on the following observations.

$$\frac{a(x;\alpha+1,\beta)}{a(x;\alpha,\beta)} = x\frac{\alpha+\beta}{\alpha},$$

since $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, where $a(x;\alpha,\beta)$ is given by (2.1). Similarly

$$a(x;\alpha,\beta+1)/a(x;\alpha,\beta) = (1-x)\frac{\alpha+\beta}{\beta}.$$

Thus, we have $M_n(0,\alpha+1,\beta) = \frac{\alpha+\beta}{4\alpha} M_{n+1}(0,\alpha,\beta)$ and $M_n(0,\alpha,\beta+1) = \frac{\alpha+\beta}{\beta}(M_n(0,\alpha,\beta) - M_{n+1}(0,\alpha,\beta)/4)$.

iii) After applying ordinary change of the order of summation, we have:

$$\begin{aligned}\sum_{n\geq 0}^{\infty} x^n \sum_{j=0}^{n} \binom{n}{j} f_j c^{n-j} &= \sum_{j\geq 0} x^j f_j \sum_{n\geq j} (cx)^{n-j} \frac{(j+1)\cdots n}{(n-j)!} \\ &= \sum_{j\geq 0} x^j f_j \sum_{n\geq j} (cx)^{n-j} \frac{(j+1)\cdots n}{(n-j)!} \\ &= \sum_{j\geq 0} x^j f_j \sum_{k\geq 0} (cx)^k \frac{(j+1)^{(k)}}{k!}.\end{aligned}$$

Now we recall (2.5) and (2.6) and get:

$$\begin{aligned}\sum_{n\geq 0}^{\infty} x^n \sum_{j=0}^{n}\binom{n}{j} f_j c^{n-j} &= \sum_{j\geq 0} x^j f_j \sum_{k\geq 0}(cx)^k \frac{(-1)^k(-(j+1))_{(k)}}{k!} \\ &= \sum_{j\geq 0} x^j c_j (1-cx)^{-j-1} \\ &= \frac{1}{1-cx} g\left(\frac{x}{1-cx}\right).\end{aligned}$$

□

One of the referees posed the following question. For what triplets (c,α,β) the sequence $\{M_n(c,\alpha,\beta)\}_{n\geq 0}$ generates integers. Well, we will not answer this question fully, since it seems that a problem is more difficult than expected. We have, however the following partial result, exposing the possible complications in answering this question. Namely, we have:

Theorem 2.4. *Let $\alpha \in (0,1)$ be a rational number and let $\alpha = \frac{p}{r}$, where p and r are two positive integers relatively prime and let n be a positive integers. Then $\forall n \in \mathbb{N}$ the number*

(2.15)
$$\left(\frac{r}{4}\right)^n \prod_{j=1}^{k} d_j^{\sum_{m=1}^{\infty}\lfloor n/d_j^m\rfloor} M_n\left(0,\frac{p}{r},1-\frac{p}{r}\right) = \frac{\prod_{j=0}^{n}(jr+p)}{n!}\prod_{j=1}^{k} d_j^{\sum_{m=1}^{\infty}\lfloor n/d_j^m\rfloor},$$

is an integer. Here $r=\prod_{j=1}^{k} d_j^{\beta_j}$, is a prime decomposition of r.

Proof. Let us fix n. Then we have by (2.2):

$$r^n \frac{M_n\left(0,\frac{p}{r},1-\frac{p}{r}\right)}{4^n} = r^n \frac{(\alpha)^{(n)}}{n!}. \tag{2.16}$$

Now let us recall the so-called Chinese reminder Theorem stating that every two congruence equations

$$ax \equiv c, (\operatorname{mod} m_1)\ bx \equiv d\ (\operatorname{mod} m_2),$$

have unique solution $\operatorname{mod} m_1 m_2$ if and only if numbers m_1 and m_2 are relatively prime. Hence, taking $m_1 = r$ and $1 < n_1 \leq n$ relatively prime, we see that the set of two congruence equations:

$$x \equiv p,\ \operatorname{mod} r\ x \equiv 0,\ \operatorname{mod} n_1$$

has a unique solution $\operatorname{mod}(n_1 r)$. In other words, that among numbers $jr+p$, $j=1,\ldots,n$ at least one is divided by n_1. That means that the number (2.16) is a rational number that has the denominator composed by the numbers that are not relatively prime to r. Or in other words, are composed of powers of the divisors of r. Now, one has to count sums of powers of particular divisors less then or equal to n. Let us fix the divisor, let it be d_j. Then there are $\lfloor n/d_j\rfloor$ factors of the form kd_j, $k=1,\ldots,$ then there are $\left\lfloor n/d_j^2\right\rfloor$

factors of the form kd_j^2 and so on. Now since we multiply those factors (we had $n!$ before canceling out factor relatively prime to r) so we have

$$d_j^{\lfloor n/d_j \rfloor + \lfloor n/d_j^2 \rfloor + \dots}$$

in the denominator of (2.16). Now to get an integer out of this number we have to multiply it by the denominator. □

Remark: If one considers the more general situation like e.g., $M_n(0, p/r, k - p/r)/4^n$ then the situation is not that simple. Namely, in the denominator, there might appear prime factors that are different than the prime factors of the denominator of parameter α, i.e., r. For example, if we consider the sequence $\forall n \geq 1$:

$$6 \times 3^{\sum_{j=0}^{\infty} \lfloor n/3^j \rfloor} 2^{\lfloor n/2 \rfloor} \frac{M_n(0, 1/3, 6 - 1/3)}{4^n},$$

then the first few elements of this sequence are the following :
$1, 8/7, 3, 20/3, 26/3, 832/11, 3952/33, 1216/3, 45600/7$. Hence we have apart of divisors of 3 and 6 we have also 7, 11 and maybe others.

Remark: Notice, that in general, the integer sequence

$$\left\{ \left(\frac{r}{4}\right)^n \prod_{j=1}^{k} d_j^{\sum_{m=1}^{\infty} \lfloor n/d_j^m \rfloor} M_n(0, p/r, 1 - p/r) \right\}_{n \geq 0}$$

is not a moment sequence, even though $\{r^n\}$ and

$$\{M_n(0, p/r, 1 - p/r)/4^n\}_{n \geq 0}$$

are. This is because, in general,

$$\left\{ \prod_{j=1}^{k} d_j^{\sum_{m=1}^{\infty} \lfloor n/d_j^m \rfloor} \right\}_{n \geq 0}$$

is not a moment sequence. On the other hand sequences given by (2.15), increase our knowledge about some sequences presented in the OEIS. For example sequence (2.15), with $p/r = 1/3$ is listed as A004117 in the OEIS or with $p/r = 1/8$ as A181161 in the OEIS.

Now, we will calculate moments $M_n(0, \alpha, \beta)$ for all values $\alpha = n/2$ and $\beta = m/2$, where n and m are natural numbers.

3. First, partial results

Lemma 3.1. *i) For natural n and non-negative integer $i$we have:*

$$\begin{aligned} \left(i + \frac{1}{2}\right)^{(n)} &= \frac{(2i + 2n)!i!}{4^n (i + n)!(2i)!}, \\ \left(\frac{1}{2}\right)_{(n)} &= (-1)^{n-1} \frac{2(2n - 2)!}{4^n (n - 1)!}, \end{aligned}$$

setting 1 *for* $n=0$ *and for* $n\geq 1:$

$$\left(\frac{3}{2}\right)_{(n)}=\left\{\begin{array}{ccc} 3/2 & if & n=1 \\ (-1)^{n}\frac{12(2n-4)!}{4^{n}(n-2)!} & if & n>1 \end{array}\right. ,$$

$$\left(\frac{5}{2}\right)_{(n)}=\left\{\begin{array}{ccc} 5/2 & if & n=1 \\ 3/2 & if & n=2 \\ (-1)^{n-3}\frac{120(2n-6)!}{4^{n}(n-3)!} & if & n>2 \end{array}\right. .$$

ii) For $n\geq 0:$

$$M_{n}\left(0,i+\frac{1}{2},j+\frac{1}{2}\right)=\frac{\binom{2i+2n}{i+n}\binom{i+n}{n}}{\binom{2i}{i}\binom{i+j+n}{n}}=\frac{(2n+2i)!i!(i+j)!}{(i+n)!(2i)!(i+j+n)!},$$

hence in particular:

iiA) $M_{n}(0,i+1/2,1/2)=\binom{2n+2i}{n+i}/\binom{2i}{i}$,
iiB) $M_{n}(0,i+1/2,3/2)=C_{i+n}/C_{i}$,
iiC) $M_{n}(0,i+1/2,5/2)=((i+2)!i!(2n+2i)!)/((2i)!(n+i)!(n+i+2)!)$
iiD) $M_{n}(0,i+1/2,7/2)=((i+3)!i!(2n+2i)!)/((2i)!(i+n)!(n+i+3)!)$,
iiE) $M_{n}(0,i+1/2,9/2)=(i!(i+4)!(2n+2i)!)/((2i)!(i+n)!(n+i+4)!)$,
iiF) $M_{n}(0,1/2,j+1/2)=(j!(2n)!)/(n!(n+j)!)$,
iiG) $M_{n}(0,3/2,j+1/2)=((j+1)!(2n+1)!)/(n!(n+j+1)!)$,
iiH) $M_{n}(0,5/2,j+1/2)=((j+2)!(2n+3)!)/((n+1)!(n+2+j)!)$,
iiI) $M_{n}(0,7/2,j+1/2)=((j+3)!(2n+5)!)/(60(n+2)!(n+j+3)!)$,
iiJ) $M_{n}(0,9/2,j+1/2)=(4!(4+j)!(2n+8)!)/(8!(n+4)!(n+4+j)!)$.

iii) For $n\geq 0:$

$$M_{n}(0,i+1/2,j)=4^{n}\frac{(2i+2j)!i!}{(2i)!(i+j)!}\frac{(2i+2n)!(i+j+n)!}{(i+n)!(2i+2j+2n)!},$$

hence in particular we get:

iiiA) $M_{n}(0,1/2,1)=4^{n}/(2n+1)$,
iiiB)$M_{n}(0,3/2,1)=3\times 4^{n}/(2n+3)$,
iiiC) $M_{n}(1/2,2)=3\times 4^{n}/((2n+1)(2n+3))$,
iiiD) $M_{n}(0,3/2,2)=4^{n}15/((2n+3)(2n+5))$.

iv) For $n\geq 0:$

$$M_{n}(0,i,j+1/2)=4^{2n}\frac{(i+n-1)!(i+j+n)!(2i+2j)!}{(2i+2j+2n)!(i-1)!(i+j)!},$$

hence in particular we have:

ivA) $M_{n}(0,1,1/2)=4^{2n}(n!)^{2}/(2n+1)!$,
ivB) $M_{n}(0,1,3/2)=4^{2n}12n!(n+2)!/(2n+4)!$,
ivC) $M_{n}(0,2,1/2)=4^{2n}12(n+1)!(n+2)!/(2n+4)!$,
ivD) $M_{n}(0,2,3/2)=4^{2n}120(n+1)!(n+3)!/(2n+6)!$,
ivE) $M_{n}(0,3,1/2)=60\times 4^{n}(n+2)!(n+3)!/(2n+6)!$,
ivF) $M_{n}(0,4,1/2)=4^{2n}280(n+4)!(n+3)!/(2n+8)!$.

Proof. i) Following definitions given by (2.3) and (2.4) we get $(i+1/2)^{(n)}$ $=(i+1/2)(i+3/2)\cdots(i+1/2+n-1)=(2n+2i-1)!!/(2^{(n)}(2i-1)!!)$.

Now, it is elementary to check that $(i+1/2)^{(n)} = \frac{(2i+2n)!i!}{4^n(i+n)!(2i)!}$. Generally, we have $(i+1/2)_{(n)} = (-1)^n(-i-1/2)^{(n)}$, hence we could use this formula to get $(1/2)_{(n)}$, $(3/2)_{(n)}$ and $(5/2)_{(n)}$ but it seems that it might be easier to check these formulae directly. ii) Applying assertion i) twice we get ii). iii) and iv) we apply assertion i) once but in the case of iii) in the numerator and in the case of iv) in the denominator. v) Is direct application of (2.10) and (2.11). □

Some sequences of integers, that will appear in the sequel, are identifiable by their generating functions in the OEIS, hence we need to calculate also the generating functions of the moments of the Jacobi distributions that will appear in the sequel. In fact, there are two ways of calculating the generating functions of moment sequences. The first one, so to say, direct, uses a formula (2.2) and the other uses the integral representation of the moment sequence. The Lemma 2.3, above lists some of the generating functions and presents some of the ways to transform them.

Thus, as a corollary we have:

Corollary 3.2. *Let us denote by* $G(x;c,\alpha,\beta) = \sum_{k\geq 0} x^k M_k(c,\alpha,\beta)$. *Then we have for* $c \in \mathbb{R}$, $\alpha,\beta > 0$ $G(0,c,\alpha,\beta) = 1$ *while for* $x \neq 0$ *and* $|x| < 1/4$ *we get:*

a) $G(x;0,1/2,1/2) = 1/\sqrt{1-4x}$,

b) $G(x;0,3/2,1/2) = (1-\sqrt{1-4x})/(2x\sqrt{1-4x})$,

c) $G(x;0,1/2,3/2) = \left(1-\sqrt{1-4x}\right)/(2x)$,

d) $G(x;0,1/2,5/2) = \left((1-4x)^{3/2}+6x-1\right)/\left(6x^2\right)$,

e) $G(x;0,3/2,3/2) = \left(1-2x-\sqrt{1-4x}\right)/\left(2x^2\right)$,

f) $G(x;0,5/2,1/2) = \left(1-\sqrt{1-4x}-2x\sqrt{1-4x}\right)/\left(6x^2\sqrt{1-4x}\right)$,

g) $G(x;0,1/2,7/2) = \left(1-(1-4x)^{5/2}-10x+30x^2-20x^3\right)/\left(10x^4\right)$,

h) $G(x;0,3/2,5/2) = \left((1-4x)\sqrt{1-4x}-1+6x-6x^2\right)/\left(4x^3\right)$,

i) $G(x;0,5/2,3/2) = \left(-1+(1-4x)^{1/2}+2x+2x^2\right)/\left(4x^3\right)$,

j) $G(x;0,7/2,1/2) = \left(1-(1-4x)^{1/2}(1+2x+6x^2)\right)/\left(20x^3\sqrt{1-4x}\right)$,

In the identification of particular moment sequences in the OEIS below, we will use the following terminology: we say that a sequence A "is equal to $l-s(p_1,p_2\ldots)$ sequence B" if A is obtained from B by omitting the first elements that are equal to $p_1,p_2,\ldots$. Similarly, we say that sequence A is "$r-s(p_1,p_2,\ldots)$ sequence B" if B is obtained from A by adding its beginning numbers $p_1,p_2,\ldots$. Finally, we say that B is obtained from A by sign-change, briefly "*sc*", if the even elements of both sequences are the same while odd elements have different signs by the same absolute values.

Remark: In particular

$$\sum_{k\geq 0} C_n/4^n = 2. \tag{3.1}$$

Since $C_n \cong 4^n/n^{3/2}$, the series above is convergent. Its sum can be found easily by passing to the limit $x \to 1/4$ in assertion c of Corollary 3.2.

Remark: a) $M_n(0,1/2,1/2) = \binom{2n}{n}$, sequence A000984 in the OEIS. It is called the sequence of "Central binomial",

b) $M_n(0,1/2,3/2) = \binom{2n}{n}/(n+1)$, sequence A000108 in the OEIS. It is called the sequence of "Catalan numbers",

c) $M_n(0,3/2,1/2) = \binom{2n+1}{n+1} = \binom{2n+2}{n+1}/2$, sequence A001700 in the OEIS,

d) $M_n(0,1/2,5/2) = (2(2n)!)\,/\,(n!(n+2)!)$, $\frac{1}{3}\times$ "super ballot numbers" i.e., $\frac{1}{3}\times$ of the sequence A007054 in the OEIS,

e) $M_n(0,3/2,3/2) = \binom{2n+2}{n+1}/(n+2)$, $l-s(1)$ Catalan numbers, i.e., $l-s(1)$ i.e., sequence A000108 in the OEIS,

f) $M_n(0,5/2,1/2) = \binom{2n+4}{n+2}/6$, $\frac{1}{3}$ $l-s(1)$ sequence A001700 in the OEIS,

g) $M_n(0,1/2,7/2) = (6(2n+2)!)\,/\,((n+1)!(n+4)!)$,$\frac{1}{10}\times r-s(10)$ super ballot numbers i.e., $\frac{1}{10}\times r-s(10)$ of the sequence A007272 in the OEIS,

h) $M_n(0,3/2,5/2) = (3(2n+2)!)\,/\,((n+1)!(n+3)!)\,,\frac{1}{2}\times r-s(3)$ super ballot numbers more precisely $\frac{1}{2}\times r-s(3\ )$ sequence A007054 in the OEIS,

i) $M_n(0,5/2,3/2) = \binom{2n+4}{n+2}/\left(2\left(n+3\right)\right),\frac{1}{2}\times r-s(1,1)$ Catalan numbers,

j) $M_n(0,7/2,1/2) = ((2n+6)!)\,/\,(20(n+3)!(n+3)!)$, $\frac{1}{10}\times r-s(1,3)$ sequence A001700 in the OEIS.

Corollary 3.5. *1) $4^nM_n(-3/4,1/2,1/2)$ is the sequence A322248 in the OEIS, since the g.f. of this sequence is $1/\sqrt{(1-13x)(1+3x)}$ which can be obtained by the formula given in assertion iii) of Lemma 2.3.*

2) $4^nM_n(-7/4,1/2,1/2)$ is the sequence A098441 in the OEIS, since the g.f. of this sequence is $1/\sqrt{1-2x-63x^2}$ which can be obtained by the formula given in assertion iii) of Lemma 2.3.

3) $2^nM_n(-3/2,1/2,1/2)$ is the sequence A084605 in the OEIS, since the g.f. of this sequence is $1/\sqrt{1-2x-15x^2}$ which can be obtained by the formula given in assertion iii) of Lemma 2.3.

4) $2^nM_n(-3/2,1/2,1/2)$ is the sequence A084605 in the OEIS, since the g.f. of this sequence is $1/\sqrt{1-2x-15x^2}$ which can be obtained by the formula given in assertion iii) of Lemma 2.3.

5) $M_n(-1,1/2,1/2)$ is the sequence A002426 "central trinomial coefficient" in the OEIS, by assertion i) of Lemma 2.1.

6) $2^nM_n(-1/2,1/2,1/2)$ is the sequence A322242 in the OEIS, since the g.f. of this sequence is $1/\sqrt{1-6x-7x^2}$ which can be obtained by the formula given in assertion iii) of Lemma 2.3.

7) $4^nM_n(-1/4,1/2,1/2)$ is not listed in the OEIS. The g.f. of this sequence is $1/\sqrt{1-14x-15x^2}$ which can be obtained by the formula given in assertion iii) of Lemma 2.3. Visibly this sequence is closely related to sequence A098441 in the OEIS by appropriate binomial transform (Lemma 2.1).

8) $4^n M_n(1/4, 1/2, 1/2)$ *is not listed in the OEIS. The g.f. of this sequence is* $1/\sqrt{1-18x+17x^2}$ *which can be obtained by the formula given in assertion iii) of Lemma 2.3. Visibly this sequence is closely related to sequence A098441 in the OEIS by appropriate binomial transform (Lemma 2.1).*

9) $2^n M_n(1/2, 1/2, 1/2)$ *is the sequence A084771 in the OEIS, since the g.f. of this sequence is* $1/\sqrt{1-10x+9x^2}$ *which can be obtained by the formula given in assertion iii) of Lemma 2.3.*

10) $M_n(1, 1/2, 1/2)$ *is the sequence A026375 in the OEIS, since the g.f. of this sequence is* $1/\sqrt{1-6x+5x^2}$ *which can be obtained by the formula given in assertion iii) of Lemma 2.3.*

11) $4^n M_n(5/4, 1/2, 1/2)$ *is not listed in the OEIS. The g.f. of this sequence is* $1/\sqrt{1-26x+105x^2}$ *which can be obtained by the formula given in assertion iii) of Lemma 2.3. Visibly this sequence is closely related to sequence A098441 in the OEIS by appropriate binomial transform (Lemma 2.1).*

12) $2^n M_n(3/2, 1/2, 1/2)$ *is the sequence A248168 in the OEIS, since the g.f. of this sequence is* $1/\sqrt{1-14x+33x^2}$ *which can be obtained by the formula in assertion iii) of Lemma 2.3. given in assertion iii) of Lemma 2.3.*

13) $M_n(2, 1/2, 1/2)$ *is the sequence A081671 in the OEIS, since the g.f. of this sequence is* $1/\sqrt{1-8x+12x^2}$ *which can be obtained by the formula given in assertion iii) of Lemma 2.3.*

14) $M_n(2, 3/2, 1/2)$ *is the* $l-s(1)$ *sequence A005573 in the OEIS since the g.f. of this sequence is* $\left(\sqrt{1-2x}-\sqrt{1-6x}\right)/(2x\sqrt{1-6x})$ *which can be obtained by the formula given in assertion iii) of Lemma 2.3.*

15) $2^n M_n(5/2, 1/2, 1/2)$ *is not listed in the OEIS. The g.f. of this sequence is* $1/\sqrt{1-18x+65x^2}$ *that can be obtained by the formula given in assertion iii) of Lemma 2.3. Visibly this sequence is closely related to sequence A084771 in the OEIS by appropriate binomial transform (Lemma 2.1).*

Now, we examine some examples connected with Catalan numbers.

Corollary 3.6. *1)* $M_n(-1, 1/2, 3/2)$ *"Riordan numbers" i.e., sequence A005043 in the OEIS. This is because by Lemma 2.3 its generating function is equal to* $(1-\sqrt{(1-3x)/(1+x)})/(2x)$ *as given in an unnumbered formula on page 87 of* [4].

2) $M_n(-1, 3/2, 3/2)$ *sequence A001006 in the OEIS, the so-called Motzkin numbers. This is so since its g.f. is* $f(x) = (1-x-\sqrt{1-2x-3x^2})/(2x^2)$, *that satisfies the following equation* $x^2 f^2 + (x-1)f + 1 = 0$ *as given in Wikipedia.*

3) $2^n M_n(-1/2, 1/2, 3/2)$ *is the sequence A337168 in the OEIS, since the g.f. of this sequence is* $A(x) = \left(-1+\sqrt{(1-7x)/(1+x)}\right)/(4x)$ *that can be obtained by the formula given in assertion iii) of Lemma 2.3. (There is a small misprint in the formula for the g.f. in the OEIS (there is* 8 *instead of*

7). *However by a simple check one can see that* $A(x)$ *satisfies the following identity* $A(x) = 1/(x+1) + 2xA(x)^2$*).*

4) $2^n M_n(-3/2, 1/2, 3/2)$ *is not listed in the OEIS, it's g.f. is*

$$\left(-1+\sqrt{(1-5x)/(1+3x)}\right)/(4x)$$

which can be obtained by the formula given in assertion iii) of Lemma 2.3. Visibly this sequence is closely related to sequence A337168 in the OEIS by an appropriate binomial transform. Compare with Lemma 2.1.

5) $2^n M_n(1/2, 1/2, 3/2)$ *is the* $r-s(1)$ *sequence A162326 in the OEIS. This is so since the g.f. of* $2^n M_n(1/2, 1/2, 3/2)$ *is* $(1-\sqrt{(1-9x)/(1-x)})/(4y)$ *hence the g.f. of the sequence A162326 is* $1+x(1-\sqrt{(1-9x)/(1-x)})/(4y)$ $=(5-\sqrt{(1-9x)/(1-x)})/4$.

6) $M_n(1, 1/2, 3/2)$ *sequence A007317 in the OEIS defined as "the binomial transform of Catalan numbers" since we have Lemma 2.1 and the remark following it.*

7) $M_n(1, 3/2, 1/2)$ *is not listed in the OEIS, However by (2.10) we see that it is a binomial transform of the sequence A001700 of the OEIS.*

8) $2^n M_n(3/2, 1/2, 3/2)$ *sequence not listed in the OEIS. Its g.f. is equal to* $(1-\sqrt{(1-11x)/(1-3x)})/(4x)$*. Visibly this sequence is closely related to sequence A162326 in the OEIS by appropriate binomial transform. Compare Lemma 2.1.*

9) $M_n(2, 1/2, 3/2)$ *sequence A064613 in the OEIS defined as the "second binomial transform of Catalan numbers" its g.f. is*

$$(1-\sqrt{(1-6x)/(1-2x)})/(2x).$$

Now, we will present some assorted examples that seem to be important from the point of view of combinatorics.

Corollary 3.7. *1)* $M_n(1, 3/2, 3/2)$ *is the* $r-s(1)$ *sequence A002212 in the OEIS .*

2) $M_n(2, 3/2, 3/2)$ *sequence A005572 in the OEIS,*
since its g.f. is $(1-4x-\sqrt{1-8x+x^2})/(2x^2)$

3) $M_n(3, 3/2, 3/2)$ *sequence A182401 in the OEIS,*
since its g.f. is $(1-5x-\sqrt{1-10x+21x^2})/(2x^2)$

4) $M_n(1/2, 3/2, 3/2)$ *is the* $r-s(1)$ *sequence A059231 in the OEIS,*
since its g.f. is $f(x) = (1-5x-\sqrt{1-10x+9x^2})/(8x^2)$,
hence $1+xf(x) = (1+3x-\sqrt{1-10x+9x^2})/(8x)$.

Sometimes the description of the sequence in the OEIS is insufficient. Either the generating function is not given, or the formula for the n^{th} item in the sequence is missing. However, in some cases, one observes that the first several elements of the sequence $\{M_n(c, \alpha, \beta)\}$ for some values of parameters c, α, β agree with the elements of the sequence from the OEIS. Then there exists a strong supposition that these two sequences are identical. We will present a few of these suppositions in the form of conjectures, presented below.

Conjecture. *1.* $M_n(-1,3/2,1/2)$ *is the sequence A005773 of the OEIS. More precisely it is an inverse Binomial transform of the sequence A001700.*
2. $2^n M_n(-3/2,3/2,1/2)$ *is the sequence A151318 of the OEIS.*
3. $2M_n(-1,5/2,3/2)$ *is the sequence A005554 in the OEIS.*
4. $2M_n(1,5/2,3/2)$ *is the sequenceA045868 in the OEIS.*

Notice, that the distribution with the density $g(x;-2,\alpha,\alpha)$ is symmetric hence all odd moments of the form $M_n(-2,\alpha,\alpha)$ are equal to 0. Besides the support of this distribution is symmetric $[-2,2]$. All these reasons suggest that it should be in a special way. Namely, we will denote by

$$g(y;-2,\gamma,\delta) \overset{def}{=} b(y;\gamma,\delta) = \frac{(y+2)^{\gamma-1}(2-y)^{\delta-1}}{4^{\gamma+\delta-1}B(\gamma,\delta)},$$

$$S_n(\gamma,\delta) = \int_{-2}^{2} x^n b(x;\gamma,\delta)dx.$$

Remark: i) Notice, also, that we have for all $n \geq 0$, $\gamma,\delta > 0$:

$$S_n(\delta,\gamma) = (-1)^n S_n(\gamma,\delta).$$

This is so since we have for $\gamma,\delta > 0$ and $x \in [-2,2]$:

$$b(x;\gamma,\delta) = b(-x;\delta,\gamma).$$

Using Lemma 2.1, we have:

$$S_n(\gamma,\delta) = \sum_{j=0}^{n} \binom{n}{j} (-2)^{n-j} M_j(0,\gamma,\delta),$$

$$M_n(0,\gamma,\delta) = \sum_{j=0}^{n} \binom{n}{j} 2^{n-j} S_j(\gamma,\delta).$$

As far as the values of the moments S_n are concerned, we have:

Proposition 3.9. *a)* $S_n(1/2,1/2) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \binom{n}{n/2} & \text{if } n \text{ is even} \end{cases}$, *sequence A126869 in the OEIS,*

b) $S_n(1,1) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{2^n}{(n+1)} & \text{if } n \text{ is even} \end{cases}$,

c) $S_n(3/2,3/2) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ C_{n/2} & \text{if } n \text{ is even} \end{cases}$, *sequence A126120 in the OEIS,*

d) $S_n(2,2) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{3\times 2^n}{(n+1)(n+3)} & \text{if } n \text{ is even} \end{cases}$, *e)* $S_n(1/2,3/2) = (-1)^n\binom{n}{\lfloor n/2\rfloor}$*, sequence A126930 in the OEIS, since its g.f. is* $(1-\sqrt{(1-2x)/(1+2x)})/(2x)$*, while* $S_n(3/2,1/2) = \binom{n}{\lfloor n/2\rfloor}$*. Strangely, it has a different number in the OEIS (A001405),*

f)

$$S_n(1,2) = \begin{cases} \frac{-2^n}{n+2} & \text{if } n \text{ is odd} \\ \frac{2^n}{n+1} & \text{if } n \text{ is even} \end{cases}$$

$$S_n(2,1) = \begin{cases} \frac{2^n}{n+2} & \text{if } n \text{ is odd} \\ \frac{2^n}{n+1} & \text{if } n \text{ is even} \end{cases}$$

by Remark 3.8.

g) $2S_n(3/2,5/2)$ *is the* $r-s(1,1)$ *and sc sequence A089408 in the OEIS since the g.f. of* $2S_n(3/2,5/2)$ *is* $g(x) = (-6x^2+6x-1+(1-4x)^{3/2})/2x^3$ *by Lemma 2.3. Operation* $r-s(1,1)$ *changes this function to* $1-x+x^2g(x)$ $= (4x-1+(1-2x)\sqrt{1-4x^2})/(2x)$*. Now it remains to change* x *to* $-x$*. By Remark 3.8* $2S_n(5/2,3/2)$ *is* $r-s(1,1)$ *sequence A089408 in the OEIS, since following its description remains to change* x *to* $-x$ *in* $(4x-1+(1-2x)\sqrt{1-4x^2})/(2x)$*.*

Remark: Taking into account Corollary 3.62) identifying Motzkin numbers by their generating function, identity 2.9 with $b=-1$ and $c=-2$ and assertion c) of Proposition 3.9, we immediately arrive at the formula relating Motzkin numbers and Catalan numbers, namely:

$$Mo_n = \sum_{j=0}^{\lfloor n/2\rfloor}\binom{n}{2j}C_j, \tag{3.2}$$

where Mo_n denotes n^{th} Motzkin number. The formula above, relating Motzkin and Catalan numbers, appears in the solution of problem 4 in the Stanley's book [11].

4. Expansions

In this section, we are going to get some interesting identities involving the above-mentioned moments similar to the ones presented in the introduction. These identities will be obtained by simple expansions of the ratio of the densities of the involved beta distributions. To avoid unnecessary complications we will apply these expansions only in two cases. Namely, let us take real γ such that $\alpha-\gamma$ is an integer then we can expand the ratio of $b(x;\alpha,\beta)/b(x;\gamma,\delta)$ in the following series:

$$\frac{b(x;\alpha,\beta)}{b(x;\gamma,\delta)} = \frac{B(\gamma,\delta)4^{2\delta-\beta-1}}{B(\alpha,\beta)}\sum_{k\geq 0}\frac{x^{\alpha-\gamma+k}}{4^{k+\alpha-\gamma}}\frac{(\beta-\delta)_k}{k!}.$$

and consequently, we can relate moment sequences of the two distributions, one with the density $b(x;\alpha,\beta)$, and the other with the density $b(x;\alpha,\beta)$.

$$M_n(0,\alpha,\beta)=\frac{B(\gamma,\delta)4^{\gamma-\alpha+\delta-\beta}}{B(\alpha,\beta)}\sum_{k\geq 0}\frac{(\beta-\delta)_k}{k!4^{k+\alpha-\gamma}}M_{k+n+\alpha-\gamma}(0,\gamma,\delta). \tag{4.1}$$

We will also exploit the following similar trick, related to the expansion of the ratio $g(x;\alpha,\beta)/g\left(x;\gamma,\delta\right)$. Namely, we consider the following expansion:

$$b\left(x;\alpha,\beta\right)=\frac{B(\gamma,\delta)}{4^{\alpha-\gamma+\beta-\delta}B(\alpha,\beta)}b(x;\gamma,\delta)\sum_{k\geq 0}\frac{x^k}{4^k k!}c_k(\alpha-\gamma,\beta-\delta), \tag{4.2}$$

based on the fact that

$$g(x;\alpha,\beta)/g\left(x;\gamma,\delta\right)=\frac{B(\gamma,\delta)}{4^{\alpha-\gamma+\beta-\delta}B(\alpha,\beta)}(y+2)^{\alpha-\gamma}(2-y)^{\beta-\delta},$$

expansion (2.6) and the standard multiplication of power series. Thus we have:

$$c_k(\alpha-\gamma,\beta-\delta)=\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}(\alpha-\gamma)_{(j)}(\beta-\delta)_{(k-j)}. \tag{4.3}$$

(4.2) leads to the following expansion involving moments S_n.

$$S_n(\alpha,\beta)=\frac{B(\gamma,\delta)}{4^{\alpha-\gamma+\beta-\delta}B(\alpha,\beta)}\sum_{k\geq 0}\frac{c_k(\alpha-\gamma,\beta-\delta)}{4^k k!}S_{n+k}(\gamma,\delta). \tag{4.4}$$

As far as the question of convergence of the series (4.1) and (4.4) is concerned, we have the following remarks.

Proposition 4.1. *$\forall n\geq 0$:*

i) $|M_n(0,\alpha,\beta)|<4^n$, $|S_n(\alpha,\beta)|<2^n$,

ii) $|M_n(0,i+1/2,j+1/2)|=4^nO(1/n^{j+1/2})$,

iii)

$$\begin{aligned}|M_n(0,i+1/2,j)|&=\left|4^n\left((2n+2i)!i!(i+j)!\right)/\left((i+n)!(2i)!(i+j+n)!\right)\right|\\&=4^{2n}O(1/n^{j+1/2})\end{aligned}$$

iv)

$$\begin{aligned}&|M_n(0,i,j+1/2)|\\&=\left|4^{2n}\left((i+n-1)!(i+j+n)!(2i+2j)!\right)/\left((2i+2j+2n)!(i-1)!(i+j)!\right)\right|\\&=4^nO(1/n^{j+1/2}).\end{aligned}$$

Proof. i) This fact follows directly from the supports of the measures whose moments are considered and the fact that the measures concerned are absolutely continuous with respect to the Lebesgue measure, i.e., have densities. Namely, in the first case it is the segment $[0,4]$ while in the second case the segment $[-2,2]$. For ii), iii) and iv) we start with the fact that

$\binom{2n}{n}\frac{1}{4^n} \cong O(1/n^{1/2})$ as $n \to \infty$. Secondly, let us notice that $(n+k)!/n! = (n+1)\cdots(n+k) = O(n^k)$. Hence, for the case ii) we have

$$\left|\frac{(2n+2i)!i!(i+j)!}{(i+n)!(2i)!(i+j+n)!}\right| \cong \frac{i!(i+j)!}{(2i)!}\frac{4^{n+i}}{\sqrt{n+i}}\frac{(n+i)!}{(n+i+j+n)!} \\ \cong O(4^n/n^{j+1/2}),$$

iii)

$$\left|4^n\frac{(2n+2i)!i!(i+j)!}{(i+n)!(2i)!(i+j+n)!}\right| \cong 4^n\frac{i!(i+j)!}{(2i)!}\binom{2n+2i}{n+i}\frac{(n+i)!}{(i+j+n)!} \\ \cong 4^{2n}O(1/n^{j+1/2}),$$

iv)

$$\left|4^{2n}\frac{(i+n-1)!(i+j+n)!(2i+2j)!}{(2i+2j+2n)!(i-1)!(i+j)!}\right| \cong \\ 4^n\frac{(2i+2j)!}{(i-1)!(i+j)!}\left(4^n/\binom{2i+2j+2n}{i+j+n}\right)\frac{(i+n-1)!}{(i+j+n)!} \cong O(1/n^{j+1-1/2}).$$

□

Theorem 4.2. *For $n \geq 0$ we have:*

i)

$$C_n = 2\binom{2n}{n} - \frac{1}{2}\binom{2(n+1)}{n+1}. \tag{4.5}$$

ii)

$$C_{2n+1} = \sum_{i=0}^{n}\binom{2n}{2i}4^{n-i}C_i, \tag{4.6}$$

$$C_{2n+2} = 2\sum_{i=0}^{n}\binom{2n+1}{2i}4^{n-i}C_i \tag{4.7}$$

iii)

$$C_n = \frac{3}{2}\sum_{i\geq 0}\frac{1}{4^i}\frac{(2n+2i)!}{(i+n)!(n+i+2)!},$$

iv)

$$\frac{(n+1)!(n+2)!}{(2n+4)!} = \frac{1}{4^{n+2}}\sum_{j\geq 0}\binom{2j}{j}\frac{1}{4^j(n+j+2)}.$$

Or equivalently

$$\frac{4^{n+2}}{C_{n+2}} = \sum_{j\geq 0}\binom{2j}{j}\frac{1}{4^j}\frac{(n+2)(n+3)}{(2n+2j+2)}.$$

v)

$$\frac{1}{4^n}\binom{2n}{n} = 1 - \frac{1}{2}\sum_{j=0}^{n-1}\frac{C_j}{4^j},$$

vi)

$$\frac{n!n!}{(2n+1)!} = 4\sum_{j\geq 0}4^j\frac{(n+j)!(n+j+2)!}{(2n+2j+4)!}.$$

Or equivalently

$$\frac{4^n}{C_n} = \sum_{j\geq 0}\frac{4^{n+j+1}}{C_{n+j+1}}\frac{(2n+2)(n+1)}{(n+j+1)(2n+2j+3)(2n+2j+4)}.$$

Proof. i)We take $\alpha = 1/2$, $\beta = 3/2, \gamma = \delta = 1/2$. Then we apply (4.1). ii) First we note that $M_n(0,3/2,3/2) = C_{n+1}$, by Remark 3.4e), while $S_n(3/2,3/2) = 0$ for n odd and $C_{n/2}$ when n is even. Then we apply (2.11 and take $k = 2n$ getting directly (4.6). We get (4.7) likewise. iv) We take $\alpha = 1/2$, $\beta = 3/2$, $\gamma = 1/2$, $\delta = 5/2$. Then

$$\frac{a(x;\alpha,\beta)}{a(x;\gamma,\delta)} = \frac{3}{4-x} = \frac{3}{4}\sum_{k\geq 0}\frac{x^k}{4^k}.$$

Now recall that $M_n(0,1/2,3/2) = C_n$ and $M_n(0,1/2,5/2) = \frac{2(2n)!}{n!(n+2)!}$ by Remark 3.4 b) and d). Now it remains to apply (4.1). iv) We consider $\alpha = 2$, $\beta = 1/2$ and $\gamma = 3/2$ and $\delta = 1$. Then we apply (4.1) using expansion

$$\begin{aligned}\frac{g(x;0,2,1/2)}{g(x;0,3/2,1)} &= \frac{1}{2}\sqrt{\frac{x}{4-x}}\\ &= \frac{1}{4}\sum_{j\geq 0}(-1)^j(-1/2)_{(j)}\frac{x^{j+1/2}}{4^j j!}\\ &= \frac{1}{4}\sum_{j\geq 0}\frac{(2j)!}{4^j j!}\frac{x^{j+1/2}}{4^j j!}\\ &= \frac{1}{4}\sum_{j\geq 0}\binom{2j}{j}\frac{1}{4^{2j}}x^{j+1/2}.\end{aligned}$$

Now we notice that $\int_0^4 x^{n+1/2}g(x;0,3/2,1)dx = \frac{3}{2}\int_0^4 x^n g(x;0,2,1)dx = \frac{3}{2}4^n\frac{(2)^{(n)}}{(3)^{(n)}} = \frac{3}{2}4^n\frac{2(n+1)!}{(n+2)!} = \frac{3\times 4^n}{n+2}$. Hence

$$\begin{aligned}4^{2n}\frac{12(n+1)!(n+2)!}{(2n+4)!} &= \frac{1}{4}\sum_{j\geq 0}\binom{2j}{j}\frac{1}{4^{2j}}\int_0^4 x^{j+n+1/2}g(x;0,3/2,1)dx\\ &= \frac{1}{4}\sum_{j\geq 0}\binom{2j}{j}\frac{1}{4^{2j}}\frac{3\times 4^{n+j}}{(n+j+2)}.\end{aligned}$$

v) and vi) we apply assertion iii)(iiib) of Lemma 2.1 first for $\alpha = 3/2$, $\beta = 1/2$ and $\gamma = 3/2$ and $\delta = 3/2$ and secondly for $\alpha = 1$, $\beta = 1/2$ and $\gamma = 1$ and $\delta = 3/2$ then we use identity (3.1) in the case vi) and divide both sides by 4^{2n} in the case of vii). □

Remark: Assertion i) is well-known. It appears for example as exercise 4 on page 155 of [6]. It is also easy to get from the assertion i) the formula given in assertion i) of Theorem 3.1 in Stanley's book [11]. Namely, we have

$$\begin{aligned} C_n &= 2\binom{2n}{n} - \frac{1}{2}\binom{2(n+1)}{n+1} \\ &= \binom{2n}{n} + \binom{2n}{n} - \frac{1}{2}\binom{2(n+1)}{n+1} \\ &= \binom{2n}{n} + \frac{(2n)!}{n!n!} - \frac{(2n+1)!}{n!(n+1)!} \\ &= \binom{2n}{n} + \binom{2n}{n}\left(1 - \frac{2n+1}{n+1}\right) \\ &= \binom{2n}{n} - \binom{2n}{n-1}. \end{aligned}$$

Assertion ii) presents, in fact, two cases of the so-called Touchard's identity for odd and even cases.

Theorem 4.4. *We have:*

i) for all $n \geq 0$

$$\sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} 2^{n-2i} \binom{2i}{i} = \binom{2n}{n}, \tag{4.8}$$

ii) for all $n \geq 1$

$$\binom{2n}{n} = 2 \times 4^{n-1} - 2^{n-1} \sum_{j=1}^{\lfloor n/2 \rfloor} \binom{n}{2j} \sum_{s=0}^{j-1} \frac{C_s}{4^s}.$$

iii)

$$C_{n+1} = 2\binom{2n}{n} - \frac{1}{2}\sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} 2^{n-2j} \binom{2j+2}{j+1}.$$

iv)

$$C_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{k}{\lfloor k/2 \rfloor} 2^{n-k}.$$

v)

$$\frac{1}{2^{n-1}} \binom{n}{\lfloor n/2 \rfloor} = 2 - \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \frac{1}{4^j} C_j.$$

vi)

$$S_n(3/2,3/2) = 2\binom{n}{\lfloor n/2\rfloor} - \binom{n+1}{\lfloor (n+1)/2\rfloor}.$$

vii) Let us define $C_{-1} = -1/2$ *and* $d_0 = 1$ *and further let* $n > 0:$

$$d_n = \frac{n!}{2\times 4^{n-1}}\sum_{k=0}^{n}(-1)^{k-1}\binom{2k}{k}C_{n-k-1}.$$

Then

$$S_n(1,2) = \frac{\pi}{4}\sum_{k\geq 0}\frac{S_{n+k}(3/2,3/2)}{2^k k!}d_k.$$

In particular $\forall j \geq 0$

$$\frac{-2\times 2^{2j}}{2j+3} = \frac{\pi}{4}\sum_{k\geq 0}\frac{C_{j+k+1}}{2^{2k+1}(2k+1)!}d_{2k+1},$$

$$\frac{4^j}{2j+1} = \frac{\pi}{4}\sum_{k\geq 0}\frac{C_{j+k}}{4^k(2k)!}d_{2k}.$$

Proof. i) We use Proposition 3.9a) and (2.11). ii) We take $\gamma = \delta = 3/2$ and $\alpha = \beta = 1/2$. To use identity (4.4) we have to calculate only the coefficients $c_n(\alpha-\gamma,\beta-\delta) = c_n(-1,-1)$. One can easily check that

$$c_n(-1,-1) = \begin{cases} 0 & \text{if} \quad n \text{ is odd} \\ n! & \text{if} \quad n \text{ is even} \end{cases}.$$

Then we argue:

$$\begin{aligned}\binom{2n}{n} &= \frac{B(3/2,3/2)}{4^{2(1/2-3/2)}B(1/2,1/2)}\sum_{j=0}^{\lfloor n/2\rfloor}\binom{n}{2j}2^{n-2j}\sum_{s=0}^{\infty}\frac{C_{s+j}}{4^s}\\ &= \frac{1}{2}\sum_{j=0}^{\lfloor n/2\rfloor}\binom{n}{2j}2^{n-2j}4^j\sum_{s=0}^{\infty}\frac{C_{s+j}}{4^{s+j}}\\ &= 2^{n-1}\sum_{j=0}^{\lfloor n/2\rfloor}\binom{n}{2j}\left(2-\sum_{s=0}^{j-1}\frac{C_s}{4^s}\right).\end{aligned}$$

Further we make use of (3.1) and the following identity:

$$\sum_{j=0}^{\lfloor n/2\rfloor}\binom{n}{2j} = 2^{n-1}.$$

iii) We take $\gamma=\delta=1/2$ and $\alpha=\beta=3/2$. We will use Lemma 3.1iiB with $i=1$, then (2.11) obtaining

$$C_{n+1}=\sum_{j=0}^{n}\binom{n}{j}2^{n-j}S_{j}(3/2,3/2)$$
$$=\sum_{j=0}^{\lfloor n/2\rfloor}\binom{n}{2j}2^{n-2j}C_{j}.$$

Then we apply (4.5) and finally we use (4.8).

iv) let us take $\gamma=\delta=3/2$ then, using (4.4) and (3.1) and the following argument

$$\begin{aligned}&(-1)^{n}\binom{n}{\lfloor n/2\rfloor}\\
=&\int_{-2}^{2}x^{n}g(x;-2,1/2,3/2)dx\\
=&\int_{-2}^{2}x^{n}g(x;-2,3/2,3/2)\frac{1}{2+x}dx\\
=&\frac{1}{2}2^{n}(-1)^{n}\sum_{k=0}^{\infty}(-1)^{k+n}2^{-k-n}S_{n+k}(3/2,3/2)\\
=&(-1)^{n}2^{n-1}\\
&\times\left(\sum_{m=0}^{\infty}(-1)^{m}2^{-m}S_{m}(3/2,3/2)-\sum_{m=0}^{n-1}(-1)^{m}\frac{1}{2^{m}}S_{m}(3/2,3/2)\right).\end{aligned}$$

Consequently we have

$$\frac{1}{2^{n-1}}\binom{n}{\lfloor n/2\rfloor}=2-\sum_{j=0}^{\lfloor(n-1)/2\rfloor}\frac{1}{4^{j}}C_{j}.$$

v) We take $\alpha=\beta=3/2$ and $\gamma=3/2$ and $\delta=1/2$.

vi) We take $\delta=3/2$ and $\gamma=3/2$ and $\alpha=1$ and $\beta=2$. then we have

$$S_{n}(1,2)=\begin{cases}\frac{2^{n}}{n+1} & \text{if } n \text{ is even}\\ -\frac{2^{n}}{n+2} & \text{if } n \text{ is odd}\end{cases}$$
$$=\frac{1}{8}\int_{-2}^{2}y^{n}(2-y)dy.$$

We will be applying (4.4), hence we have to calculate coefficients $c_k(\alpha-\gamma,\beta-\delta)=c_k(-1/2,1/2)$. Now

$$c_s(-1/2,1/2)=\sum_{k=0}^{s}\binom{s}{k}(-1)^{s-k}(-1/2)_{(k)}(1/2)_{(s-k)},$$
$$=\sum_{k=0}^{s}\binom{s}{k}(-1)^{s-k}(-1)^k(1/2)^{(k)}(1/2)_{(s-k)}$$
$$=\sum_{k=0}^{s}\binom{s}{k}(-1)^k(1/2)^{(k)}(-1/2)^{(s-k)} \tag{4.9}$$

Now notice that if $k=0$ then $(1/2)^k=(1/2)_k=1$ while for $k>0$ we get

$$(1/2)^{(k)}=\prod_{m=0}^{k-1}(1/2+m)=\frac{(2k)!}{4^k k!},$$
$$(-1/2)^{(k)}=(-1)^k\prod_{m=0}^{k-1}(m-1/2)=-\frac{(2(k-1))!}{2\times 4^{k-1}(k-1)!}.$$

Hence

$$c_n(-1/2,1/2)$$
$$=(-1)^n\frac{(2n)!}{4^n n!}-\sum_{k=0}^{n-1}(-1)^k\frac{n!(2k)!(2(n-k-1))!}{k!(n-k)!4^k k!2\times 4^{n-k-1}(n-k-1)!}$$
$$=(-1)^n\frac{(2n)!}{4^n n!}-\frac{n!}{2\times 4^{n-1}}\sum_{k=0}^{n-1}(-1)^k\binom{2k}{k}C_{n-k-1}.$$

Notice, that if we define C_{-1} as $-1/2$ then the formula above, becomes more simple and has the more unified form:

$$c_n(-1/2,1/2)=\frac{n!}{2\times 4^{n-1}}\sum_{k=0}^{n}(-1)^{k+1}\binom{2k}{k}C_{n-k-1}.$$

Upon applying ((4.4), we get:

$$S_n(1,2)=\frac{\pi}{4}\sum_{k=0}^{\infty}\frac{S_{n+k}(3/2,3/2)}{2^k k!}c_k(-1/2,1/2),$$

i.e., assertion vii). □

Remark: Notice, that assertion v) of Theorem 4.4 is a generalization of the assertion v) of the Theorem 4.2.

Remark: Let us notice, that the sequence $\{d_n\}_{n\geq 0}$ defined in assertion viii) of Theorem 4.4 is also a moment sequence since it is originally defined by the formula (4.9). This is so since $\left\{(-1/2)^{(n)}/n!\right\}$ and $\left\{(1/2)^{(n)}/n!\right\}$ are both moment sequences by (2.2), $\{n!\}$ is the is the moment sequence of the distribution with the density $\exp(-x)$ on $\mathbb{R}^+$. Consequently, by Proposition 1.1 we deduce that $\{d_n\}$ is a moment sequence. Further we have for $|x|<1$;

$$\sum_{n\geq 0}\frac{x^n}{n!}d_n = \sum_{n\geq 0}\sum_{k=0}^{n}\frac{x^k}{k!}(-1)^k(1/2)^{(k)}\frac{x^{n-k}}{(n-k)!}(-1/2)^{(n-k)}$$
$$= \sum_{k=0}^{\infty}\frac{x^k}{k!}(-1)^k(1/2)^{(k)}\sum_{n\geq k}\frac{x^{n-k}}{(n-k)!}(-1/2)^{(n-k)}$$
$$= \sqrt{\frac{1-x}{1+x}}.$$

But $\sqrt{\frac{1-x}{1+x}} = \exp(\tanh^{-1}(-x))$ for x real and $|x|<1$. Hence we can identify sequence $\{d_n\}$ as *sc* version of the sequence A000246 in the OEIS.

It might be of interest to notice that the formula given in assertion viii) of Theorem 4.4 expresses d_n in terms of the Catalan numbers.

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