Contributions to Discrete Mathematics

Volume 20, Number 1, Pages 19–35
ISSN 1715-0868

ON THE DOUBLE COVERS OF A LINE GRAPH

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Abstract. Let $L(X)$ be the line graph of graph X. Let X'' be the Kronecker product of X by K_2. In this paper, we see that $L(X'')$ is a double cover of $L(X)$. We define the symmetric edge graph of X, denoted as $\gamma(X)$ which is also a double cover of $L(X)$. We study various properties of $\gamma(X)$ in relation to X and the relationship amongst the three double covers of $L(X)$ that are $L(X''), \gamma(X)$ and $L(X)''$. With the help of these double covers, we show that for any integer $k \geq 5$, there exist two equienergetic graphs of order $2k$ that are not cospectral.

1. Introduction

In this paper, we restrict ourselves to finite graphs with no self-loops and multiple edges. We denote the *cycle graph, the path graph, the complete graph* and the *star graph* on n vertices by C_n, P_n, K_n and $K_{1,n-1}$ respectively. A graph Y is a *covering graph* of a graph X if there is a map from the vertex set of Y to the vertex set of X such that the neighbourhood of a vertex v in Y is mapped bijectively onto the neighbourhood of $f(v)$ in X. If each vertex of X has exactly two preimages in Y then we say that Y is a *double cover* of X. An easy way to construct a double cover of a graph X is to take the *Kronecker product* of X by K_2 and it is denoted by X''. The Kronecker product $X_1 \times X_2$ of graphs X_1 and X_2 is a graph such that the vertex set is $V(X_1) \times V(X_2)$, vertices (x_1, x_2) and (x_1', x_2') are adjacent if and only if x_1 is adjacent to x_1' in X_1 and x_2 is adjacent to x_2' in X_2. In Section 3, we show that $L(X'')$ is a double cover of $L(X)$. The double cover for a graph X is not unique (see Example 1.1). Many researchers used covering of graphs in the construction of Ramanujan graphs (see [11]) and in the construction of pairs of cospectral but not isomorphic graphs. Additional information on covering of graphs can be found in [5, 14].

Received by the editors 76009 September 11, 2022, and in revised form March 14, 2023.
2000 *Mathematics Subject Classification.* 05C50, 05C99.
Key words and phrases. Line graph, edge adjacency matrix of graph, Ihara zeta function of a graph, Covering graph.
The authors would like to thank the handling editor and the anonymous reviewers for their careful reading of the manuscript.

Example 1.1. *In this example, we demonstrate the two non-isomorphic double covers of* K_4.

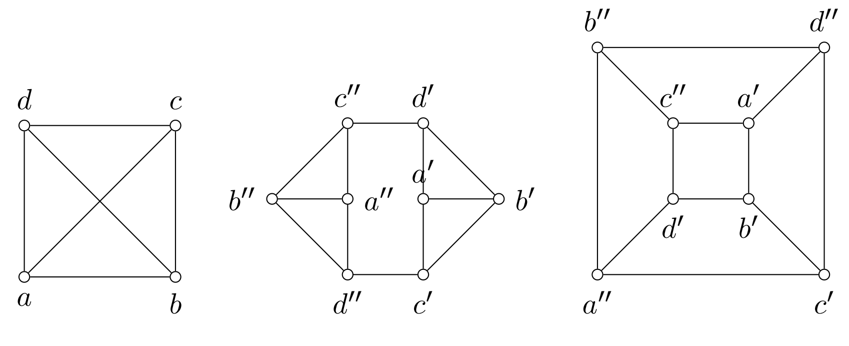


FIGURE 1

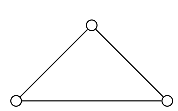
Let $X = (V, E)$ be a graph with $|V(X)| = n$, $|E(X)| = m$. We orient the edges arbitrarily and label them as $e_1, e_2, \ldots, e_m$ and also $e_{m+i} = e_i^{-1}$, $1 \leq i \leq m$, where e_k^{-1} denotes the edge e_k with the direction reversed. Then the *edge adjacency matrix* of X, denoted by $M(X)$ or simply M, is defined as

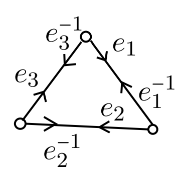
$$M_{ij} = \begin{cases} 1 & \text{if } t(e_i) = s(e_j) \text{ and } s(e_i) \neq t(e_j), \\ 0 & \text{otherwise.} \end{cases}$$

where $s(e_i)$ and $t(e_i)$ denote the starting and terminal vertex of e_i respectively.

Example 1.2. *The process of computation of matrix* $M(C_3)$ *is given below.*

X

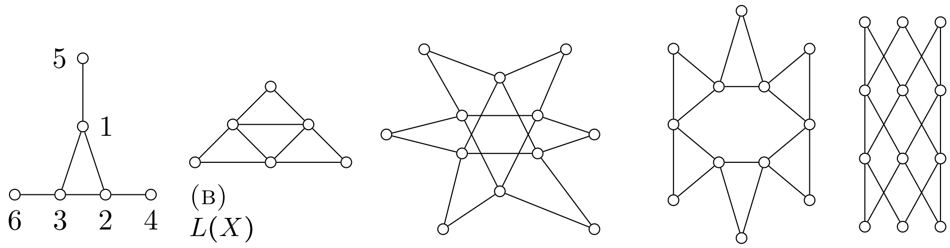




M

	e_1	e_2	e_3	e_1^{-1}	e_2^{-1}	e_3^{-1}
e_1	0	1	0	0	0	0
e_2	0	0	1	0	0	0
e_3	1	0	0	0	0	0
e_1^{-1}	0	0	0	0	0	1
e_2^{-1}	0	0	0	1	0	0
e_3^{-1}	0	0	0	0	1	0

It is interesting to see that $M + M^T$, where A^T denotes the transpose of A, is a symmetric matrix with entries 0 or 1. We call $M + M^T$ *symmetric edge adjacency matrix* of X, and the graph whose adjacency matrix is $M + M^T$ is called *symmetric edge graph* of X, denoted by $\gamma(X)$. We define $\gamma^k(X) = \gamma(\gamma^{k-1}(X))$, where $k \in \mathbb{N}$ with $\gamma^0(X) = X$. Later, we will see that $\gamma(X)$ is also a double cover of $L(X)$. In Figure 2, for a graph X we have given its line graph and the three non-isomorphic double covers of $L(X)$.



(A) X
(B) $L(X)$
(C) $\gamma(X)$
(D) $L(X'')$
(E) $L(X)''$

FIGURE 2

In the literature, a lot of work has been done on the properties of $L(X)$ in relation to X (see Chapter 8 of [6]). In Section 2, we study various properties of $\gamma(X)$ with respect to X. We provide a decomposition of $\gamma(X)$ in terms of crown graphs. With these three double covers of $L(X)$ in hand which are $L(X)'', L(X'')$ and $\gamma(X)$, we will study the relation amongst them in Section 3. In Theorem 3.3, we characterize all graphs X so that $\gamma(X) = L(X''), \gamma(X) = L(X)''$ and $L(X)'' = L(X'')$. In the rest of this section, we will discuss why the matrix M is important for the Ihara zeta function of a graph, the properties of the matrix M, and the symmetric edge graphs.

A path $P = e_1e_2\cdots e_t$, where e_i is an oriented edge, is said to *backtrack* if $e_{k+1} = e_k^{-1}$ for some $k \in \{1, 2, 3, \ldots, t-1\}$, *i.e.* it crosses the same edge twice in a row. A path P is said to have a *tail* if $e_t = e_1^{-1}$, *i.e.* the last edge of P is the reverse of the first edge. A closed path $C = e_1e_2\cdots e_t$ is said to be *prime* or *primitive* if it has no backtrack or tail and $C \neq D^f$ for some closed path D and $f > 1$. The *Ihara zeta function* of a graph X is defined to be

$$\zeta_X(u) = \prod_{[C]} \left(1 - u^{\ell(C)}\right)^{-1},$$

where the product is over the primes $[C]$ of X and $\ell(C)$ is the length of cycle C. The *fundamental group* $\pi_1(X, v)$ of a connected graph X is the free group consisting of all closed walks starting and ending at the vertex v together with the operation which concatenates walks. The rank r of the $\pi_1(X, v)$ is the number of elements in a minimal generating set of $\pi_1(X, v)$ which is also the number of edges left out of a spanning tree of X. The computation of Ihara zeta function using the definition is difficult except for the cycle graph. The following two results by Bass [2] and Hashimoto [7] simplified the evaluation of the Ihara zeta function for graphs that have a minimal degree of at least 2.

Theorem 1.3. [7] *Let $A(X)$ or A be the adjacency matrix of X and $Q(X)$ or Q be the diagonal matrix with j^{th} diagonal entry q_j such that $q_j + 1$ is the degree of the j^{th} vertex of X. Suppose that r is the rank of the fundamental* *group of X; $r-1=|E|-|V|$. Then*

$$\zeta_X(u)^{-1}=(1-u^2)^{r-1}det(I-Au+Qu^2). \tag{1.1}$$

The main purpose of introducing the matrix M can be seen in the following result.

Theorem 1.4. [2] *Let M be the edge adjacency matrix of a graph X. Then*

$$\zeta_X(u)^{-1}=det(I-Mu).$$

Now we will state a few properties of matrix M. Many of these have been discussed in the thesis of Horton [8] and one can also find them in the book by Terras [14].

$$M=\left[\begin{array}{c|c}\mathbb{A}&\mathbb{B}\\\hline\mathbb{C}&\mathbb{D}\end{array}\right], \tag{1.2}$$

where $\mathbb{A},\mathbb{B},\mathbb{C},\mathbb{D}$ are $m\times m$ matrices with the following properties:

1. $\mathbb{B}=\mathbb{B}^T$, $\mathbb{C}=\mathbb{C}^T$.
2. $\mathbb{D}=\mathbb{A}^T$.
3. The diagonals of $\mathbb{A},\mathbb{B},\mathbb{C}$ and $\mathbb{D}$ are zeros.
4. If $J=\begin{bmatrix}0&\mathbb{I}_m\\\mathbb{I}_m&0\end{bmatrix}$, where $\mathbb{I}_m$ denotes the identity matrix of order m then $M^T=JMJ$.
5. The i^{th} row sum of M is equal to $d_{t(e_i)}-1$, where d_v denotes the degree of vertex v.
6. The sum of the blocks of M, $\mathbb{A}+\mathbb{B}+\mathbb{C}+\mathbb{D}$ is the adjacency matrix of $L(X)$. Here one can note that the Hadamard product of any two matrices from $\{\mathbb{A},\mathbb{B},\mathbb{C},\mathbb{D}\}$ is the zero matrix.
7. Let M be the edge adjacency matrix of the graph X. Then $Tr(M^k)=N_k$, where N_k is the number of cycles of length k without backtracks and tails.

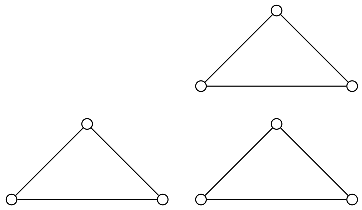
Now we provide two examples of $\gamma(X)$, from where one can note that γ does not preserve connectivity and $K_{1,3}$ is a tree but $\gamma(K_{1,3})$ is a cycle graph. After that, we shall state Theorem 1.6, which is essential for further discussion.

Example 1.5.
(1) *If $X=C_n$, then $\gamma(X)=2C_n$ and $\gamma^k(X)=2^kC_n$.*
(2) *If $X=K_{1,3}$, then $\gamma(X)=C_6$ and $\gamma^k(X)=2^{k-1}C_6$.*

Theorem 1.6. [4] *Let*

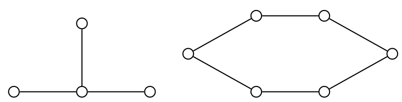
$$H=\begin{bmatrix}A'&B'\\B'&A'\end{bmatrix}$$

be a symmetric 2×2 block matrix, where A' and B' are square matrices of same order. Then the spectrum of H is the union of the spectra of $A'+B'$ and $A'-B'$.



	e_1	e_2	e_3	e_1^{-1}	e_2^{-1}	e_3^{-1}
e_1	0	1	1	0	0	0
e_2	1	0	1	0	0	0
e_3	1	1	0	0	0	0
e_1^{-1}	0	0	0	0	1	1
e_2^{-1}	0	0	0	1	0	1
e_3^{-1}	0	0	0	1	1	0

FIGURE 3. $C_3, \gamma(C_3)$ and $A(\gamma(C_3))$.



$$\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

FIGURE 4. $K_{1,3}, \gamma(K_{1,3})$ and $A(\gamma(K_{1,3}))$.

From Equation 1.2, we have

$$M + M^T = \left[\begin{array}{c|c} \mathbb{A}+\mathbb{D} & \mathbb{B}+\mathbb{C} \\ \hline \mathbb{B}+\mathbb{C} & \mathbb{A}+\mathbb{D} \end{array}\right]. \tag{1.3}$$

From Property 6 of M, we see that the row sum of Equation 1.3 is equal to $A(L(X))$, note that $\gamma(X)$ is the double cover of $L(X)$. By Theorem 1.6, we can see that the spectrum of $A(L(X))$ is contained in the spectrum of $A(\gamma(X))$. The following are a few immediate observations of the graph $\gamma(X)$.

(1) The number of vertices of $\gamma(X)$ is twice the number of edges of X.
(2) We have,

$$Tr\left((M + M^T)^2\right) = 2|E(\gamma(X))| = 2e^T Me,$$

where e denotes the column vector with all entries one and $J(M + M^T) = (M + M^T)J$.
(3) Note that

$$|E(\gamma(X))| = 2|E(L(X))| = \sum_{i=1}^{|V(X)|} d_i^2 - 2|E(X)|. \tag{1.4}$$

(4) It is easy to see that if X is Eulerian, then $\gamma(X)$ is Eulerian provided $\gamma(X)$ is connected which follows from the fact that if X is Eulerian then $L(X)$ is Eulerian (see Harary [6]). But if $\gamma(X)$ is Eulerian,

then X need not be Eulerian which is clear from Part 2 of Example 1.5.

(5) It is well known that if X is regular, then $L(X)$ is regular. This shows that the map γ maps regular graphs to regular graphs. Conversely, if $\gamma(X)$ is regular, then X is either a regular graph or a semi-regular bipartite graph. It can be seen from Lemma 6.2 in [13].

For further information on the matrix M and the Ihara zeta function, one can refer to [14]. For other results and proofs related to graph theory, we refer to [6, 12]. We recall once again that $L(X)$ and X'' denote the line graph and Kronecker double cover of X, respectively.

2. Properties of $\gamma(X)$

We begin this section by stating the famous Whitney theorem and then we present the analogous result for the γ function.

Theorem 2.1. [15] *Let X and Y be connected graphs with isomorphic line graphs. Then X and Y are isomorphic, unless one is K_3 and the other is $K_{1,3}$.*

Theorem 2.2. *Let X and Y be connected graphs. Then $\gamma(X)$ is isomorphic to $\gamma(Y)$ if and only if X is isomorphic to Y.*

Proof. Suppose that $\gamma(X)$ is isomorphic to $\gamma(Y)$, then by Property 6 of M we note that $L(X)$ is isomorphic to $L(Y)$. By Theorem 2.1 and Part 2 of Example 1.5, the result follows. □

Next, we prove that γ is additive with respect to the disjoint union.

Lemma 2.3. *Let X be a graph with connected components $X_1, X_2, \dots, X_k$ i.e., $X = X_1 \uplus X_2 \uplus \dots \uplus X_k$. Then*

$$\gamma(X_1 \uplus X_2 \uplus \dots \uplus X_k) \cong \gamma(X_1) \uplus \gamma(X_2) \uplus \dots \uplus \gamma(X_k).$$

Proof. We give the proof for $k = 2$ and the general case follows by induction on k. Let X_1, X_2 be graphs with m_1, m_2 edges, respectively. Then $A(\gamma(X_1 \uplus X_2))$ and $A(\gamma(X_1) \uplus \gamma(X_2))$ have the following block structures, respectively.

$$A(\gamma(X_1 \uplus X_2)) = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ B_1 & 0 & A_1 & 0 \\ 0 & B_2 & 0 & A_2 \end{pmatrix}$$

$$A(\gamma(X_1) \uplus \gamma(X_2)) = \begin{pmatrix} A(\gamma(X_1)) & 0 \\ 0 & A(\gamma(X_2)) \end{pmatrix} = \begin{pmatrix} A_1 & B_1 & 0 & 0 \\ B_1 & A_1 & 0 & 0 \\ 0 & 0 & A_2 & B_2 \\ 0 & 0 & B_2 & A_2 \end{pmatrix}.$$

It is easy to see that

$$P^T A(\gamma(X_1 \uplus X_2)) P = A(\gamma(X_1) \uplus \gamma(X_2)),$$

where $P = \begin{pmatrix} \mathbb{I}_{m_1} & 0 & 0 & 0 \\ 0 & 0 & \mathbb{I}_{m_2} & 0 \\ 0 & \mathbb{I}_{m_1} & 0 & 0 \\ 0 & 0 & 0 & \mathbb{I}_{m_2} \end{pmatrix}$ is a permutation matrix. □

We will see a few examples to observe the pattern of graphs under the γ function. For more examples, one can refer the Table 2.

Example 2.4. (1) *If $X = P_n$ then $\gamma^{n-1}(X)$ is a null graph. Table 1 shows the effect of repeated application of the γ function on the path graph.*

X	$\gamma(X)$	$\gamma^2(X)$	$\gamma^3(X)$

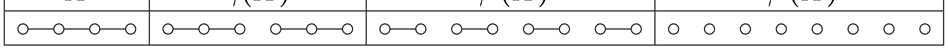


TABLE 1

(2) *If $X = K_{1,n}$, then $\gamma(X)$ is a crown graph on the $2n$ vertices. In particular, if $X = K_{1,4}$ then $\gamma(X)$ is a cube. Recall that a crown graph on $2n$ vertices is a graph with two sets of vertices $\{v_1, v_2, \dots, v_n\}$ and $\{v'_1, v'_2, \dots, v'_n\}$, with an edge from v_i to v'_j whenever $i \neq j$.*

(3) *If $X = K_{2,3}$, then $\gamma(X)$ is a 6-prism graph.*

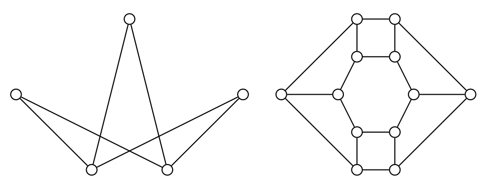


FIGURE 5. $K_{2,3}$ and $\gamma(K_{2,3})$.

The following results provide how the γ function preserves connectedness and bipartiteness. Unless specified otherwise, we assume that

$$A(\gamma(X)) = \left[\begin{array}{c|c} \mathbb{A}+\mathbb{D} & \mathbb{B}+\mathbb{C} \\ \hline \mathbb{B}+\mathbb{C} & \mathbb{A}+\mathbb{D} \end{array}\right],$$

and $A_0 = \mathbb{A} + \mathbb{D}, B_0 = \mathbb{B} + \mathbb{C}$.

Proposition 2.5. (1) *Let X be a connected graph. Then $\gamma(X)$ is connected if and only if X is not a cycle graph or a path graph. Moreover, $\gamma(X)$ cannot be a cycle graph unless $X = K_{1,3}$.*

(2) *Let $\gamma(X)$ be a connected graph, then $\gamma(X)$ has a cut edge if and only if X contains a pendant vertex which is adjacent to a vertex of degree two.*

(3) *Let X be a connected graph. Then X is bipartite if and only if $\gamma(X)$ is bipartite.*

Proof. Proof of Part 1. Let us suppose that $\gamma(X)$ is not a connected graph. Then $B + C = 0$ and hence $B, C = 0$. Thus we conclude that the degree of each vertex in X is at most 2. Since X is a connected graph, X is either a cycle graph or a path graph. From part 1 of Example 1.5 and 2.4 one can see that the converse also holds.

For the second part of the Proposition, let $\gamma(X)$ be a cycle graph on $2k$ ($k \neq 3$) vertices. From the structure of the adjacency matrix of a cycle graph, we see that when we add the four blocks of $A(C_{2k})$, we obtain $2A(C_k)$. On adding all the blocks of $A(\gamma(X))$, we get $2A(L(X))$. We deduce that $L(X)$ is a cycle graph on k vertices. However, we know from [6] that a connected graph is isomorphic to its line graph if and only if it is a cycle graph. This implies that X is a cycle graph on k vertices, which is a contradiction to Part 1 of Example 1.5. If $X = K_{1,3}$, then from Part 2 of Example 1.5 we have already seen that $\gamma(X)$ is C_6.

Proof of Part 2. Let $\gamma(X)$ have a cut edge and no pendant vertex. From the structure of $A(\gamma(X))$, it can be observed that $\gamma(X)$ has two copies of a graph each of whose adjacency matrix is A_0. Since $\gamma(X)$ is connected, the edges corresponding to the matrix B_0 connects the two copies of the graph given by A_0. As B_0 is symmetric, no edge given by the matrix B_0 can be a cut edge. Also, note that no edge in the two copies given by A_0 in $A(\gamma(X))$ can be a cut edge. Therefore $\gamma(X)$ has a pendant vertex which implies that X has a pendant vertex that is adjacent to a vertex of degree 2. The converse is easy to see as well.

Proof of Part 3. Suppose that X is bipartite with vertex partitions $\{v_1, v_2, \ldots, v_n\}$ and $\{v'_1, v'_2, \ldots, v'_k\}$. Choose an orientation in such a way that $e'_i s$ are the directed edges from v_i to v'_j for all $1 \leq i \leq n, 1 \leq j \leq k$. Then observe that

$$M = \left[\begin{array}{c|c} 0 & \mathbb{B} \\ \hline \mathbb{C} & 0 \end{array}\right]$$

which implies

$$M + M^T = \left[\begin{array}{c|c} 0 & B_0 \\ \hline B_0 & 0 \end{array}\right]. \tag{2.1}$$

Therefore, $\gamma(X)$ is bipartite. The converse is easy to see. □

From the proof of Part 3 of Proposition 2.5, one can see that if X is bipartite, the spectrum of $\gamma(X)$ is given by the union of spectra of $A(L(X))$ and $-A(L(X))$. It is possible to know the number of triangles in $\gamma^k(X)$, once we know the number of triangles in X from the following result.

Proposition 2.6. *Let t_i be the number of triangles in $\gamma^{i-1}(X)$, where $i \geq 1$. Then $t_i = 2^{i-1} t_1$.*

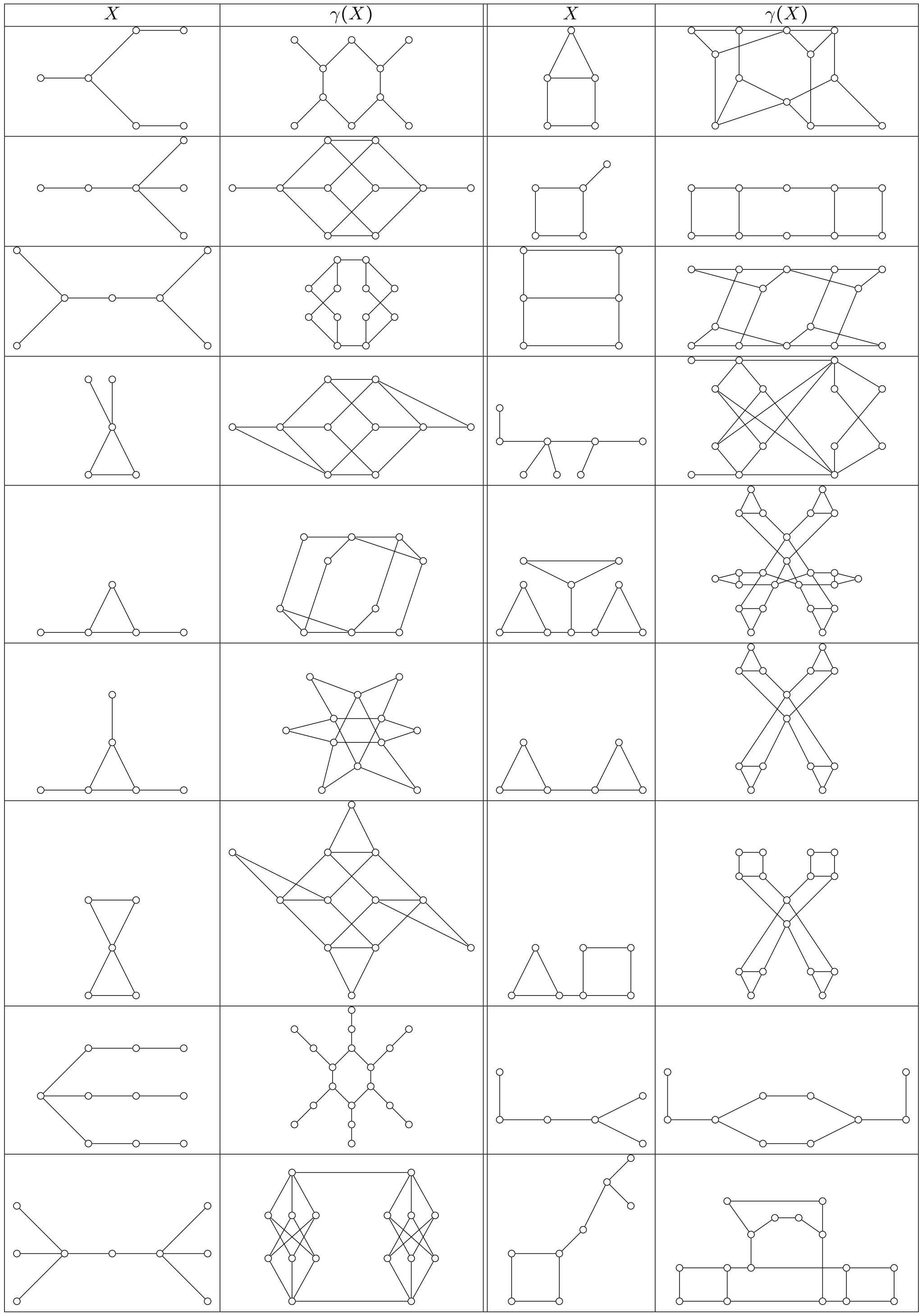


Table 2

Proof. We shall prove the result by induction on i. We begin by proving for $i = 2$. It is easy to see that

$$6t_2 = Tr((M + M^T)^3) = 2Tr(M^3) + 3Tr(M^2M^T) + 3Tr(M(M^T)^2).$$

We now claim that $Tr(M^2M^T) = Tr(M(M^T)^2) = 0$. Since M is a nonnegative matrix, $Tr(M^2M^T) = 0$ if and only if $(M^2M^T)_{ii} = 0$ for all i. We have

$$(M^2M^T)_{ii} = \sum_{k=1}^{2m}(M^2)_{ik}(M^T)_{ki} = \sum_{k=1}^{2m}\sum_{j=1}^{2m} M_{ij}M_{jk}M_{ik}.$$

If each of M_{ij}, M_{jk} and M_{ik} are nonzero, then $e_k = e_k^{-1}$. Consequently, X has multiple edges, which is a contradiction. Similarly, one can show that $(M(M^2)^T)_{ii} = 0$. Thus, $3t_2 = Tr(M^3)$. From Property 7 of M, we have another identity $t_2 = \frac{N_3}{3}$. Hence, the result follows from the fact that $t_1 = \frac{N_3}{6}$, as each vertex of a triangle can be an initial vertex and two directions. Assume that the result is true for all $i \le k-1$. Clearly, $t_k = 2t_{k-1}$. By the induction hypothesis, the proof is complete. □

Next, we will present a characterization of symmetric edge graphs analogous to that of line graphs, as given by Krausz in [10]. By the *star graph at the vertex* u in a graph X, denoted by $St(u)$, we mean a subgraph of X with $V(St(u)) = \{w \mid w \text{ is adjacent to } u\} \cup \{u\}$ and $E(St(u)) = \{e \mid u \text{ is incident with } e\}$. The approach used in the proof of Theorem 2.8 is motivated by the proof of Theorem 8.4 in [6].

Theorem 2.7. [10] *A graph is a line graph if and only if its edges can be partitioned into complete subgraphs with the property that no vertex lies in more than two of the subgraphs.*

Theorem 2.8. *A graph is a symmetric edge graph if and only if its edges can be partitioned into crown subgraphs in such a way that each vertex lies in at most two of the subgraphs.*

Proof. Let Y be the symmetric edge graph of X. Without loss of generality, X is connected. Let v be any vertex of X, then by Part 2 of Example 2.4 we see that $St(v)$ induces a crown subgraph of Y. The edges of Y are exactly in one of the subgraphs. For any $e \in E(X)$, there exists exactly two vertices $a, b \in V(X)$ such that $e \in St(a) \cap St(b)$, which shows that no vertex of Y is in more than two of the subgraphs.

Let $H_1, H_2, \ldots, H_n$ be the partition of the graph Y satisfying the hypothesis. We explain the construction of X from Y, where $Y = \gamma(X)$. Let $H = \{H_1, H_2, \ldots, H_n\}$, U be the set of vertices of Y which lies in only one of the partitions H_i. Also, note that $e_i \in U$ if and only if $e_i^{-1} \in U$. Let $U_1 \subset U$ such that U_1 contains half of the elements of U and either e_i or $e_i^{-1} \in U_1$. The vertices of X are given by $H \cup U_1$. Two vertices of X are adjacent if they have a nonempty intersection. □

Corollary 2.9. *Let X be a connected graph. Then $\gamma(X)$ is unicyclic if and only if X is a tree with $\Delta(X) = 3$, where $\Delta(X)$ denotes the maximum degree of X and there is exactly one vertex of degree three.*

Proof. Suppose that $\gamma(X)$ is unicyclic, which implies that X does not contain a cycle. By Theorem 2.8, it is clear that there does not exist a vertex in X with a degree greater than or equal to 4. If there exists more than one vertex of degree 3, then we get a contradiction to the hypothesis. The converse is easy to follow by Theorem 2.8. □

3. Double covers of line graph

Let X be a connected graph with n vertices and m edges. Let $\{v_1, v_2, \ldots, v_n\}$ be the vertex set of X. Let $V(X'') = \{v'_1, v'_2, \ldots, v'_n\} \cup \{v'_{n+1}, v'_{n+2}, \ldots, v'_{2n}\}$ be bipartition of X'' and $E(X'')$ be given by

$$\{e_1, e_2, \ldots, e_m, e_{m+1} = e_1^{-1}, e_{m+2} = e_2^{-1}, \ldots, e_{2m} = e_m^{-1}\}.$$

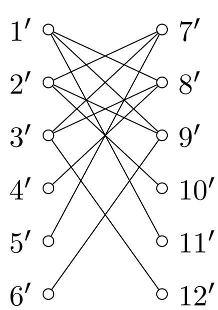
We define a map $\phi : V(X'') \mapsto V(X)$ such that $\phi(v'_i)$ and $\phi(v'_{n+i})$ are mapped to v_i for all $1 \le i \le n$. We label the edges of X'' such that e_k is an edge from v'_i to v'_{n+j} $(i \ne j)$ if and only if e_{m+k} is an edge from v'_j to v'_{n+i} $(i \ne j)$. We illustrate this labelling in Example 3.1. Recall that the adjacency matrix of a bipartite graph can be written as

$$\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix},$$

where B is called the *bi-adjacency matrix*.

Example 3.1. *In this example, we illustrate the labelling of X'', where X is given in Figure 2. We label the edges of X'' in the following manner:*

$$(1', 8') = e_1, (1', 9') = e_2, (1', 11') = e_3, (2', 9') = e_4, (2', 10') = e_5, (3', 12') = e_6,$$
$$(2', 7') = e_1^{-1}, (3', 7') = e_2^{-1}, (5', 7') = e_3^{-1}, (3', 8') = e_4^{-1}, (4', 8') = e_5^{-1},$$
$$(6', 9') = e_6^{-1}.$$



(A) X''

	7′	8′	9′	10′	11′	12′
1′	0	1	1	0	1	0
2′	1	0	1	1	0	0
3′	1	1	0	0	0	1
4′	0	1	0	0	0	0
5′	1	0	0	0	0	0
6′	0	0	1	0	0	0

(B) Biadjacency matrix of X''

Figure 6

The rows and columns of $A(L(X''))$ are indexed by $E(X'')$. It is easy to see that $A(L(X''))$ has the following structure

$$\begin{bmatrix} \mathbb{P} & \mathbb{Q} \\ \mathbb{Q}^{\mathbb{T}} & \mathbb{R} \end{bmatrix},$$

where $\mathbb{P}, \mathbb{Q}, \mathbb{R}$ are $m \times m$ matrices with the following properties:

(1) $\mathbb{P} = \mathbb{R}$. Since $\mathbb{P}_{ij} = 1$ implies that e_i is adjacent to e_j, the labelling defined above shows that e_{m+i} is adjacent to e_{m+j}.
(2) $\mathbb{Q} = \mathbb{Q}^T$. Since $\mathbb{Q}_{ij} = 1$ implies e_i is adjacent to e_{m+j}, the labelling defined above shows that e_j is adjacent to e_{m+i}.
(3) $\mathbb{P} + \mathbb{Q} = A(L(X))$. Note that if $\mathbb{P}_{ij} = 1$ then $\mathbb{Q}_{ij} = 0$ and vice-versa. If $(\mathbb{P} + \mathbb{Q})_{ij} = \mathbb{P}_{ij} + \mathbb{Q}_{ij} = 1$, then from the definition of covering graph we have $A(L(X))_{ij} = 1$.

We obtain $L(X'')$ is a double cover of $L(X)$. Also, from the point 3 mentioned above and Theorem 1.6 we see that the spectrum of $A(L(X))$ is contained in the spectrum of $A(L(X''))$. To proceed with the proof of Theorem 3.3, we need to define claw free graphs. Recall that a claw is another name for the complete bipartite graph $K_{1,3}$. In contrast, a claw-free graph is a graph in which no induced subgraph is a claw. It was proved by Beineke in [3] that the line graph of any graph is claw-free.

Proposition 3.2. *Let X be a connected graph. Then*

(1) *$L(X'')$ is disconnected if and only if X is bipartite.*
(2) *$2t' = t_2 + t_3$, where t', t_2, t_3 denotes the number of triangles in $L(X)$, $\gamma(X)$ and $L(X'')$, respectively.*

Proof. Proof of Part 1. The proof follows easily from the result proved in [9], which demonstrates that a Kronecker double cover of a graph X is connected if and only if X is connected and non-bipartite.

Proof of Part 2. We know from the definition of a line graph that

$$t' = t_1 + \sum_i \binom{d_i}{3}.$$

From Proposition 2.6, we know that $2t_1 = t_2$. Since X'' is bipartite, we have $t_3 = 2\sum_i \binom{d_i}{3}$. Hence $2t' = t_2 + t_3$. □

We are now interested to see the relationship among $\gamma(X), L(X'')$ and $L(X)''$ for a connected graph X. We begin with an example.

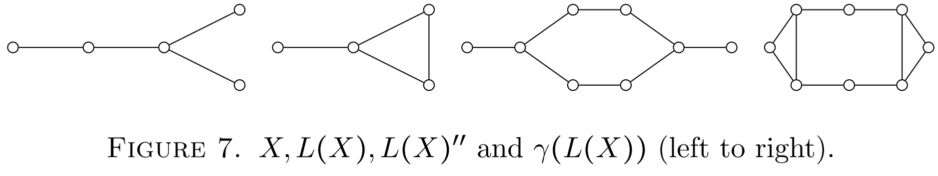


FIGURE 7. $X, L(X), L(X)''$ and $\gamma(L(X))$ (left to right).

From Figure 7, we see that $\gamma(X) = L(X)''$ and $\gamma(L(X)) = L(L(X)'')$, but it is not true in general, one can check with $X = C_3$. In the next theorem we characterize all those graphs which satisfy this property.

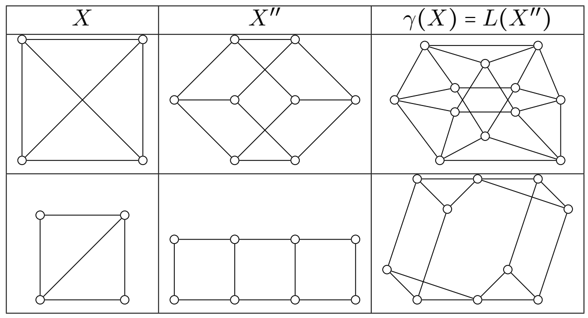


TABLE 3

Theorem 3.3. *Let X be a connected graph. Then*

(1) *$\gamma(X)$ is isomorphic to $L(X)''$ if and only if X is bipartite.*
(2) *$\gamma(X)$ is isomorphic to $L(X'')$ if and only if one of the following is true:*
  - *X is a path graph.*
  - *X is a cycle graph on even vertices.*
  - *$X = K_4, K_4 - \{e\}$ or a triangle with a pendant vertex.*

(3) *$L(X'')$ is isomorphic to $L(X)''$ if and only if X is either a cycle graph or a path graph.*

Proof. Proof of Part 1. If X is bipartite, then by Part 3 of Proposition 2.5, $\gamma(X)$ is bipartite which shows that

$$A(\gamma(X)) = A(L(X)'') = \begin{bmatrix} 0 & A(L(X)) \\ A(L(X)) & 0 \end{bmatrix}.$$

Proof of Part 2. In order to prove this, we first prove that $\gamma(X)$ is a line graph of some graph if and only if $|V(X)| \leq 4$ or X is either a cycle graph or a path graph.

Suppose that $\gamma(X)$ is a line graph of some graph. Clearly $\Delta(X) \leq 3$, since if any vertex v in X has a degree greater than or equal to 4, then by Theorem 2.8, v induces a crown graph on at least 8 vertices. Hence, $\gamma(X)$ cannot be a claw-free graph.

Case 1: If $\Delta(X) \leq 2$, then X is either a cycle graph or a path graph. From Part 1 of Example 1.5 and 2.4, it is clear that $\gamma(X)$ is a line graph of $2X$.

Case 2: Let $\Delta(X) = 3$ and $|V(X)| > 4$. Let v be a vertex of degree 3 and vertices adjacent to v be x, y, z. Since $|V(X)| > 4$, if we add a pendant edge on any of the vertices x, y, z, then the graph $\gamma(X)$ is not a claw-free graph, which is clear from Figure 7.

Conversely, if $X = C_n$ (or P_n), then $\gamma(X)$ is a line graph of two copies of C_n (or P_n). If $X = K_{1,3}$, then by Part 2 of Example 1.5 $\gamma(X)$ is C_6 which is a line graph of C_6. $\gamma(X)$ for other non-isomorphic graphs with $|V(X)| = 4$ are described in Table 3 and Figure 7.

Suppose that $\gamma(X) = L(X'')$. Then by the above statement, it can be noted that $|V(X)| \leq 4$ or $X = C_n$ or P_n. If $X = C_n$ and n is odd, then $L(X'') = C_{2n} \neq 2C_n = \gamma(X)$. If $X = C_n(n$ is even) or P_n, then $L(X'') = \gamma(X)$. For $X = K_4$ or $K_4 - \{e\}$ or a triangle with a pendant vertex, we can see from Table 3 and Figure 7 that $\gamma(X) = L(X'')$. If $X = K_{1,3}$ it can be seen that $L(X'') \neq \gamma(X)$. The converse part of the same is easy to follow.

Proof of Part 3. Assume that $L(X)'' = L(X'')$. From here it is clear that the degree of each vertex of X is less than or equal to two. Hence, X is either a cycle graph or a path graph. Conversely, if $X = C_k$ and k is odd then $X'' = C_{2k}$, $L(X'') = L(X)'' = C_{2k}$. If $X = C_k(k$ is even) or P_k then $X'' = 2X$, the result follows. □

We conclude from Theorem 3.3 that if X (not isomorphic to $K_4, K_4 - e, C_n$, or a triangle with a pendant vertex) is non-bipartite then $\gamma(X)$, $L(X)''$ and $L(X'')$ are three non-isomorphic double covers of $L(X)$. We have already seen that for a graph X, the spectrum of $A(L(X))$ is contained in the spectrum of $A(\gamma(X))$ and $A(L(X''))$. An immediate question arises about the remaining eigenvalues that is the eigenvalues given by $A_0 - B_0$ and $\mathbb{P} - \mathbb{Q}$. If X is bipartite, then $A_0 - B_0 = -A(L(X))$ and $\mathbb{P} - \mathbb{Q} = A(L(X))$. If X is non-bipartite we have Theorem 3.5. We shall discuss an example for further clarity.

Example 3.4. *Let X be the graph given in Figure 2. For a graph X'' we will continue to use the labelling defined in Example 3.1. The adjacency matrix corresponding to $L(X'')$ is equal to*

$$\begin{bmatrix} \mathbb{P} & \mathbb{Q} \\ \mathbb{Q} & \mathbb{P} \end{bmatrix}, \text{ where } \mathbb{P} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } \mathbb{Q} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

It is easy to see that $\mathbb{P} + \mathbb{Q} = A(L(X))$ and

$$\mathbb{P} - \mathbb{Q} = \begin{bmatrix} 0 & 1 & 1 & -1 & -1 & 0 \\ 1 & 0 & 1 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 \end{bmatrix}.$$

Now, we use the upper diagonal entries of matrix $\mathbb{P}-\mathbb{Q}$ *to assign an orientation to the graph* X *such that* $A_0 - B_0 = -(\mathbb{P}-\mathbb{Q})$. *For example:* $(\mathbb{P}-\mathbb{Q})_{14} = -1$. *From Example 3.1, we see that* e_1 *is an edge between* $1'$ *and* $2'+6'$, *and* e_4 *is an edge between* $2'$ *and* $3'+6'$. *Hence in* X, *we put* e_1 *from* 1 *to* 2 *and* e_4 *from* 2 *to* 3. *Similarly, we repeat the same process for all of the remaining upper diagonal entries in* $\mathbb{P}-\mathbb{Q}$ *and obtained the oriented graph given in Figure 8. It is easy to check that for the graph in Figure 8, we have* $A_0 + B_0 = A(L(X))$ *and* $A_0 - B_0 = -(\mathbb{P}-\mathbb{Q})$.

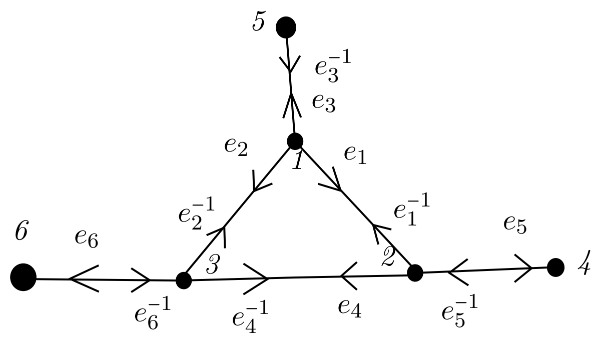


FIGURE 8

Now using the idea of Example 3.4, we prove that $A_0 - B_0 = -(\mathbb{P}-\mathbb{Q})$.

Theorem 3.5. *Let* X *be a connected graph. Then* $A_0 - B_0 = -(\mathbb{P}-\mathbb{Q})$.

Proof. If X is bipartite, then we are done. Suppose that X is non-bipartite. It is clear that $A_0 - B_0$ and $\mathbb{P}-\mathbb{Q}$ have zero entries at the same positions. Suppose that $(\mathbb{P}-\mathbb{Q})_{ij} = -1$. This implies that edge e_i is adjacent to e_{m+j} in X''. Let e_i be an edge between vertices v'_a and v'_{n+b}, and e_{m+j} be an edge between vertices v'_a and v'_{n+c}. In graph X, we label the edge from vertex v_b to v_a as e_i and the edge from v_a to v_c as e_j. This shows that $(A_0 - B_0)_{ij} = 1$. □

Recall that two graphs of the same order are called *equienergetic (resp., cospectral)* if they have the same energy (resp., spectrum). In [1] Balakrishnan showed that for any integer $k \geq 3$, there exist two equienergetic graphs of order $4k$ that are not cospectral. Let X be a graph on m edges where $m \geq 5$. Then from Theorem 3.5, we see that $\gamma(X)$ and $L(X'')$ are equienergetic graphs of order $2m$ that are not cospectral. We exclude the graphs given in Part 2 of Theorem 3.3. Using Theorem 3.5, we will provide a relation between the zeta function of $\gamma(X)$ and $L(X'')$ in Corollary 3.6. Consequently, we obtain that the zeta function of $L(X)$ divides the zeta function of $\gamma(X), L(X'')$ and $L(X)''$. The Kronecker product of matrices $A = [a_{ij}]$ and B is defined to be the partitioned matrix $[a_{ij}B]$ and is denoted by $A \otimes B$.

Corollary 3.6. *Let X be a connected graph with m edges. Then*

$$\zeta^{-1}_{\gamma(X)}(u) = \zeta^{-1}_{L(X)}(u)g(u),$$
$$\zeta^{-1}_{L(X'')}(u) = \zeta^{-1}_{L(X)}(u)g(-u),$$
$$\zeta^{-1}_{L(X)''}(u) = \zeta^{-1}_{L(X)}(u)\zeta^{-1}_{L(X)}(-u),$$

where $g(u) = (1-u^2)^{|E(L(X))|-|V(L(X))|}det(\mathbb{I}_m - (A_0 - B_0)u + Q(L(X))u^2)$.

Proof. Let

$$P = \begin{bmatrix} \mathbb{I}_m & \mathbb{I}_m \\ \mathbb{I}_m & -\mathbb{I}_m \end{bmatrix}.$$

Then

$$PA(\gamma(X))P^{-1} = \begin{bmatrix} C_0 & 0 \\ 0 & D_0 \end{bmatrix},$$

where $A_0 + B_0 = C_0$ and $D_0 = A_0 - B_0$ and $PQ(\gamma(X))P^{-1} = Q(L(X)) \otimes \mathbb{I}_2$. Let $s = |E(\gamma(X))| - |V(\gamma(X))|$. From Equation 1.1 we have,

$$\begin{aligned}
\zeta^{-1}_{\gamma(X)}(u) =& (1-u^2)^s det(\mathbb{I}_{2m} - A(\gamma(X))u + Q(\gamma(X))u^2) \\
=& (1-u^2)^s det(P(\mathbb{I}_{2m} - A(\gamma(X))u + Q(\gamma(X))u^2)P^{-1}) \\
=& (1-u^2)^s det(\mathbb{I}_{2m} - PA(\gamma(X))P^{-1}u + Q(\gamma(X))u^2) \\
=& (1-u^2)^s det(\mathbb{I}_m - C_0u + Q(L(X))u^2) \\
& \cdot det(I_m - D_0u + Q(L(X))u^2) \\
=& \zeta^{-1}_{L(X)}(u)g(u).
\end{aligned}$$

Similarly, we can see that

$$\begin{aligned}
\zeta^{-1}_{L(X'')}(u) =& \zeta^{-1}_{L(X)}(u)(1-u^2)^{|E(L(X))|-|V(L(X))|} \\
& \cdot det(\mathbb{I}_m - (\mathbb{P} - \mathbb{Q})u + Q(L(X))u^2).
\end{aligned}$$

By Theorem 3.5, we obtain $\zeta^{-1}_{L(X'')}(u) = \zeta^{-1}_{L(X)}(u)g(-u)$. As

$$A(L(X)'') = \begin{bmatrix} 0 & A(L(X)) \\ A(L(X)) & 0 \end{bmatrix},$$

we use the above technique which provides $\zeta^{-1}_{L(X)''}(u) = \zeta^{-1}_{L(X)}(u)\zeta^{-1}_{L(X)}(-u)$. This completes the proof. □

From the above corollary, we conclude that if X is bipartite, then $\zeta^{-1}_{\gamma(X)}(u) = \zeta^{-1}_{L(X)}(u)\zeta^{-1}_{L(X)}(-u)$ and $\zeta^{-1}_{L(X'')}(u) = (\zeta^{-1}_{L(X)}(u))^2$.

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