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ON THE BLOCKING NUMBERS OF SOME SPECIAL CONVEX BODIES

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ABSTRACT. In this paper, we study the blocking numbers of some special convex bodies. We determine the exact blocking number of a rhombic dodecahedra and a 3-dimensional cylinder H whose base is a 2-dimensional convex domain. We also estimate that the blocking number of the ℓ_p unit ball in $\mathbb{E}^3$ is at most 6, for $1 \leq p < +\infty$. In high dimensions, the blocking number of the ℓ_p unit ball in $\mathbb{E}^d$ is at most $2d$, for $\log_2 d < p < +\infty$.

1. INTRODUCTION

Let K be a convex body in $\mathbb{E}^d$ with boundary $\partial(K)$ and interior $\mathrm{int}(K)$. Denote by $\mathcal{K}^d$ and $\mathcal{C}^d$ the convex bodies and centrally symmetric convex bodies in $\mathbb{E}^d$, respectively. The Hadwiger covering number $\gamma(K)$ of K is the smallest number of translates of $\mathrm{int}(K)$ such that their union can cover K. In 1955, Levi [9] studied the Hadwiger covering number of an arbitrary convex domain K of $\mathbb{E}^2$ and proved

$$\gamma(K) = \begin{cases} 4 & \text{if } K \text{ is a parallelogram,} \\ 3 & \text{otherwise.} \end{cases} \tag{1.1}$$

In 1957, Hadwiger [8] made a conjecture:

Hadwiger's covering conjecture. For every d-dimensional convex body K,

$$\gamma(K) \leq 2^d,$$

where the equality holds if and only if K is a parallelepiped. This conjecture has been studied by many mathematicians. For example, Lassak [10] proved this conjecture for all centrally symmetric convex bodies in $\mathbb{E}^3$. In 1997, Rogers and Zong [11] obtained that $\gamma(K) \leq \binom{2d}{d}(d\log d + d\log\log d + 5d)$

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for all $K \in \mathcal{K}^d$ and $\gamma(K) \leq 2^d(d \log d + d \log\log d + 5d)$ for all $K \in \mathcal{C}^d$. However, the conjecture is still open for $d \geq 3$. For more details, see [3, 15].

For the purpose of studying Hadwiger's covering conjecture, Zong [16] first introduced the blocking number of a convex body. The blocking number $b(K)$ of a convex body K is the smallest number of non-overlapping translates $K + \mathbf{x}_i$ all of them touching K at its boundary and their union can block any other translate from touching K. Here, "block" means that if a new translate $K + \mathbf{y}$ touches the boundary of K, then it will insect one of the $\text{int}(K) + \mathbf{x}_i$. In this paper, non-overlapping means that their interiors have empty intersection. In 1993, Zong [16] studied the blocking number of an arbitrary 2-dimensional convex domain and proved that $b(K) = 4$. In 2018, Swanepoel [12] proved the 2-dimensional case in a new way by using the angular measure of Brass. In 1995, Zong [17] showed an inequality concerning the Hadwiger covering number and the blocking number as follows,

$$\gamma(C) \leq b(C) \tag{1.2}$$

for all $C \in \mathcal{C}^d$. Based on some facts, Zong [17] also posed the following conjecture:

Conjecture (Zong). *For every d-dimensional convex body K, we have*

$$2d \leq b(K) \leq 2^d,$$

where the equality holds in the upper bound if and only if K is a parallelopiped.

This number has attracted many mathematicians (see Böroczky [2], Brass, Moser, and Pach [3] and Zong [16, 17, 19, 20]) and some results have been obtained. In 2000, Dalla, Larman, Mani-Levitska and Zong [5] estimated the blocking numbers of d-dimensional unit cube I^d and ball B^d, they proved $b(I^d) = 2^d$, $b(B^3) = 6$ and $b(B^4) = 9$. They also gave a conditional lower bound of the blocking numbers of convex bodies. Using the Minkowski lemma, it was proved by Zong in [20] that

$$b(K) = b(D(K)) \tag{1.3}$$

for $K \in \mathcal{K}^d$, where $D(K)$ denotes the difference body of K and $D(K) = K - K = \{\mathbf{x} - \mathbf{x} : \forall\, \mathbf{x} \in K\}$. Thus, it reduces the problem of determining the blocking numbers of general convex bodies to centrally symmetric convex bodies. In 2009, Yu [13] determined that the blocking number of the cross-polytope in $\mathbb{E}^3$ is 6. He also estimated that the blocking number of the ℓ_p unit ball in $\mathbb{E}^3$ is at most 6, for $\frac{\ln 3}{\ln 2} < p < +\infty$. Moreover, Yu studied the blocking number of d-dimensional cylinder H whose base is a $(d-1)$-dimensional convex body K, he obtained a lower bound and an upper bound of its blocking number in terms of the Hadwiger covering number and the blocking number of K respectively. In 2009, Yu and Zong [14] studied various generalizations to the blocking number.

Let $a(K)$ denote the kissing number of K, the maximal number of non-overlapping translates of K all touching K at its boundary. Since a blocking configuration is a limited kissing configuration, then

$$b(K) \leq a(K)$$

for every $K \in \mathcal{K}^d$. Combining with (1.2), one can see that the blocking number serves as a bridge between the Hadwiger covering number and the kissing number. In [17, 19], Zong discovered a strange phenomena on the blocking number and kissing number which showed the complicity of the blocking number. Thus, to determine the blocking number of a convex body is challenging and meaningful. For more information about the kissing number, we refer to [4, 6, 7, 18].

The paper is organized as follows, in Section 2, we give some basic definitions and useful lemmas. In Section 3 and Section 4, we determine the exact blocking numbers of a rhombic dodecahedra and a 3-dimensional cylinder H. In Section 5, we estimate the blocking number of the ℓ_p unit ball.

2. Preliminaries

Let $C_{d,p} = \{(x_1, x_2, \ldots, x_d) : (|x_1|^p + |x_2|^p + \cdots + |x_d|^p)^{\frac{1}{p}} \leq 1\}$ $(p \geq 1)$ denote the ℓ_p norm unit ball. Let δ^C be the Minkowski-metric in $\mathbb{E}^d$ given by a centrally symmetric convex body C. In other words, denote by $C(\mathbf{z})$ the boundary point of C at direction $\mathbf{z}$,

$$\delta^C(\mathbf{x}, \mathbf{y}) = \begin{cases} \frac{\|\mathbf{x}-\mathbf{y}\|}{\|C(\mathbf{x}-\mathbf{y})\|} & \text{if } \mathbf{x} \neq \mathbf{y}, \\ 0 & \text{if } \mathbf{x} = \mathbf{y}, \end{cases}$$

where $\|\cdot\|$ indicates the Euclidean norm.

Definition 2.1. Let K be a convex body, and $X = \{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n\} \subset \mathbb{E}^d$. If $K + X$ are the non-overlapping translates of K, all of which touch K at its boundary and prevent any other translates $K + \mathbf{x}$ from touching K without overlapping $K + X$, then we call $K + X$ a blocking configuration of K.

To determine the exact blocking number of some special convex bodies, the following lemma is frequently demanded.

Lemma 2.2 ([17]). *Let C be a centrally symmetric convex body in $\mathbb{E}^d$ and $\mathbf{x}_1$, $\mathbf{x}_2$, ..., $\mathbf{x}_n \in \mathbb{E}^d$. Then we have $C + \mathbf{x}_1, C + \mathbf{x}_2, ..., C + \mathbf{x}_n$ is a blocking configuration of C if and only if*

$$\partial(2C) \subseteq \bigcup_{i=1}^{n} [\mathrm{int}(2C) + \mathbf{x}_i].$$

Furthermore, $b(C) \leq n$ holds.

By definition, we know $b(K)$ is the smallest cardinality of a discrete set X, such that $K + X$ is a blocking configuration. The blocking number is an invariant under affine transformation. Since C is a centrally symmetric convex body, the discrete subset $X = \{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n\}$ must be a subset of $\partial(2C)$.

Remark 2.3. For convenience, Lemma 2.2 can be reformulated as follows: Let $\frac{1}{2}C$ be a centrally symmetric convex body in $\mathbb{E}^d$ and $\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_n \in \mathbb{E}^d$. Then we have $\frac{1}{2}C + \mathbf{y}_1, \frac{1}{2}C + \mathbf{y}_2, ..., \frac{1}{2}C + \mathbf{y}_n$ is a blocking configuration of $\frac{1}{2}C$ if and only if $\partial(C) \subseteq \bigcup_{i=1}^{n}[\mathrm{int}(C) + \mathbf{y}_i]$. Furthermore, $b(\frac{1}{2}C) = b(C) \leq n$ holds.

3. Rhombic Dodecahedra

Firstly, we determine the exact blocking number of a rhombic dodecahedra.

Theorem 3.1. *Let P_1 be a rhombic dodecahedron, then $b(P_1) = 6$.*

Proof. For any $\mathbf{x}, \mathbf{y} \in \mathrm{int}(P_1)$, we have $\delta^{P_1}(\mathbf{x}, \mathbf{y}) < 2$. It is easy to find six vertices from the vertex set of P_1 such that the distance (with respect to δ^{P_1}) between every pair of which is 2. To cover the six vertices of P_1, we need at least six translates of $\mathrm{int}(P_1)$. Thus, $\gamma(P_1) \geq 6$. By (1.2), $b(P_1) \geq 6$.

On the other hand, given a rhombic dodecahedron P_1. Denote the fourteen vertices of P_1 by $A = (-1, 0, 0)$, $B = (0, 1, 0)$, $C = (1, 0, 0)$, $D = (0, -1, 0)$, $E = (0, 0, 1)$, $F = (0, 0, -1)$, $O = (-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$, $P = (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, $Q = (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$, $R = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, $P' = (-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$, $O' = (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$, $Q' = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$, $R' = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$. See Figure 1.

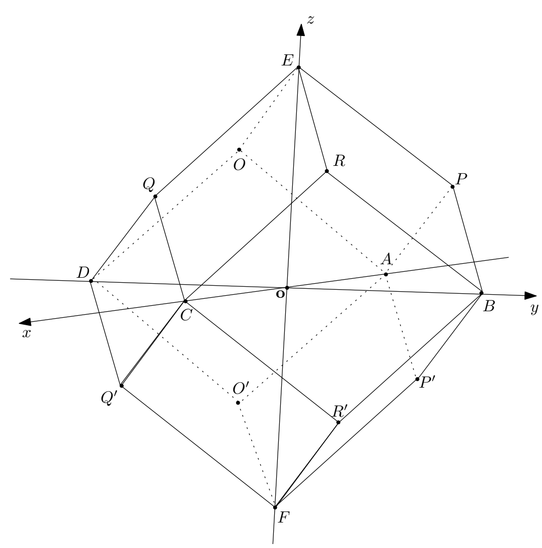

FIGURE 1. Rhombic Dodecahedron P_1.

By Lemma 2.2 and Remark 2.3, it is sufficient to find six points

$$\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_6,$$

on $\partial(P_1)$ satisfying

$$\partial(P_1) \subseteq \bigcup_{i=1}^{6} [\mathrm{int}(P_1) + \mathbf{u}_i]. \tag{3.1}$$

For the subset $V_{P_1} = \{A, B, C, D, E, F\}$, the distance (with respect to δ^{P_1}) between every pair of which is 2. By a simple observation, to prevent other translate $P_1 + \mathbf{u}$ from touching some vertex V in V_{P_1}, V should be in the relative interior of one face of $P_1 + \mathbf{u}$ in a blocking configuration. Otherwise, $P_1 + 2\mathbf{v}$ can touch P_1 at V, where $\mathbf{v}$ is the corresponding vector of V. Thus, the six translation vectors on $\partial(P_1)$ we select to make the equation (3.1) hold should not be in V_{P_1}.

If $\mathrm{int}(P_1) + \mathbf{u}$ contains some point X of V_{P_1} with vector $\mathbf{u} \in \partial(P_1)$, then we call $\mathbf{u}$ associated with X. We can also verify that the translation vector $\mathbf{u}$ lies on the boundary of a "cap"–the pyramid with an apex X.

Since the symmetry of P_1, considering half of the points in V_{P_1} is enough. Now take $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3 \in \partial(P_1)$, and they are associated with E, C, B, respectively. Denote them by $\mathbf{u}_1(E)$, $\mathbf{u}_2(C)$, $\mathbf{u}_3(B)$. Also, $\mathbf{u}_1$ lies on the boundary of the pyramid E-$POQR$, $\mathbf{u}_2$ lies on the boundary of the pyramid C-$QQ'R'R$, $\mathbf{u}_3$ lies on the boundary of the pyramid B-$PRR'P'$. Without loss of generality, we assume $\mathbf{u}_1 = (a_1, b_1, c_1)$ lies on the right half of the triangle EQR away from Q. Then

$$\begin{aligned} a_1 + c_1 - 1 &= 0, \\ \frac{1}{2} \le c_1 &< 1, \\ 0 \le b_1 &\le \frac{1}{2}, \\ 0 \le a_1 &\le \frac{1}{2}. \end{aligned}$$

We assume $\mathbf{u}_2 = (a_2, b_2, c_2)$ lies on the right half of the triangle $CQ'R'$ away from Q'. Then

$$\begin{aligned} a_2 - c_2 - 1 &= 0, \\ \frac{1}{2} \le a_2 &\le 1, \\ 0 \le b_2 &\le \frac{1}{2}, \\ -\frac{1}{2} \le c_2 &\le 0. \end{aligned}$$

We assume $\mathbf{u}_3 = (a_3, b_3, c_3)$ lies on the right half of the triangle $BP'P$ away from P'. Then

$$\begin{aligned} a_3 - b_3 + 1 &= 0, \\ -\frac{1}{2} &\leq a_3 \leq 0, \\ \frac{1}{2} &\leq b_3 \leq 1, \\ 0 &\leq c_3 \leq \frac{1}{2}. \end{aligned}$$

Hence six faces of P_1 must intersect $\text{int}(P_1) + \mathbf{u}_1$. See Figure 2. They are six faces of P_1, where the shadows represent the parts of $\partial(P_1)$ contained in $\text{int}(P_1) + \mathbf{u}_1$. Denote the six kinds of shadows by Γ_1, Γ_2, ..., Γ_6. To emphasize their connection to the vertex E, denote them by $\Gamma_1(E)$, $\Gamma_2(E)$, ..., $\Gamma_6(E)$, respectively. Similarly, we have six faces of P_1 must intersect $\text{int}(P_1) + \mathbf{u}_2$ and six faces of P_1 must intersect $\text{int}(P_1) + \mathbf{u}_3$.

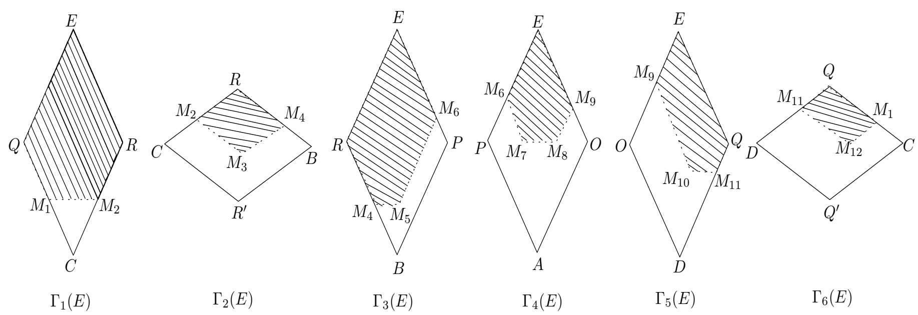

FIGURE 2. $\partial(P_1) \cap [\text{int}(P_1) + \mathbf{u}_1]$.

By the symmetry of P_1, we take $\mathbf{u}_4 = -\mathbf{u}_3$, $\mathbf{u}_5 = -\mathbf{u}_2$, $\mathbf{u}_6 = -\mathbf{u}_1$. Then $\mathbf{u}_4$ is associated with D, $\mathbf{u}_5$ is associated with A, $\mathbf{u}_6$ is associated with F. Denoted them by $\mathbf{u}_4(D)$, $\mathbf{u}_5(A)$, $\mathbf{u}_6(F)$. It is easy to know that $\Gamma_i(D)$ corresponds to $\Gamma_i(B)$, $\Gamma_i(A)$ corresponds to $\Gamma_i(C)$, $\Gamma_i(F)$ corresponds to $\Gamma_i(E)$ for $i = 1, ..., 6$.

We claim that every face of P_1 can be covered by the following covering system. See Figure 3 and Figure 4.

(1): The covering system of the face $EQCR$: $\Gamma_1(E)$, $\Gamma_4(C)$, $\Gamma_6(D)$. The covering condition is $|\overline{CN_9}| > |\overline{CM_1}|$, where $|\overline{CN_9}|$ represents the length of segment $\overline{CN_9}$. That is, $|b_2 - c_2 - 1| > |a_1 - c_1|$.

(2): The covering system of the face $CRBR'$: $\Gamma_2(E)$, $\Gamma_3(C)$, $\Gamma_4(B)$. Denote by T the intersection point of $\overline{N_5N_6}$ and $\overline{S_7S_8}$. The covering

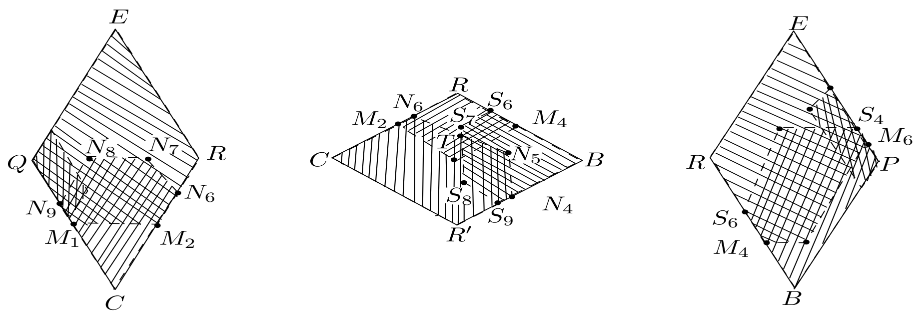

FIGURE 3. The covering system of faces $EQCR$, $CRBR'$, $ERBP$.

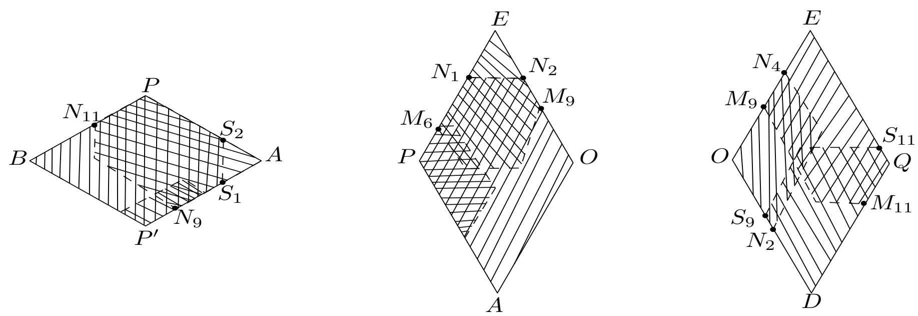

FIGURE 4. The covering system of faces $BPAP'$, $EPAO$, $EODQ$.

conditions are

$$\begin{aligned} &\mid \overline{RM_2} \mid > d(T, RB), \\ &\mid \overline{RM_4} \mid > d(T, RC), \\ &\mid \overline{R'N_4} \mid > \mid \overline{S_9R'} \mid, \end{aligned}$$

where $d(T, RB)$ denotes the Euclidean distance between the point T and the line RB. That is,

$$\begin{aligned} &\left| \frac{a_1 - c_1 + 1}{2} \right| > \frac{2\sqrt{2}}{3} \left| \frac{b_2 + c_2}{2} \right|, \\ &\left| \frac{c_1 - b_1 - 1}{2} \right| > \frac{2\sqrt{2}}{3} \left| \frac{b_2 + c_2}{2} \right|, \\ &|a_2 - b_1 - 1| > |a_3 - c_3| \,. \end{aligned}$$

(3): The covering system of the face $ERBP : \Gamma_3(E)$, $\Gamma_3(B)$, $\Gamma_6(A)$. The covering conditions are

$$\begin{aligned} &\mid \overline{RM_4} \mid > \mid \overline{S_6R} \mid, \\ &\mid \overline{EM_6} \mid > \mid \overline{ES_4} \mid \,. \end{aligned}$$

That is,

$$| c_1 - b_1 - 1 | > | a_3 + c_3 |,$$
$$| a_1 - b_1 - 1 | > | c_3 - b_3 | .$$

(4): The covering system of the face $BPAP'$: $\Gamma_1(B)$, $\Gamma_5(A)$, $\Gamma_6(F)$. The covering condition is

$$| \overline{AN_9} | = | \overline{CN_9} | > | \overline{AS_1} | .$$

That is, $| c_2 - b_2 + 1 | > | a_3 + b_3 |$.

(5): The covering system of the face $EPAO$: $\Gamma_4(E)$, $\Gamma_1(A)$, $\Gamma_2(B)$. The covering condition is

$$| \overline{EM_9} | > | \overline{EN_1} | = | \overline{FN_1} | .$$

That is, $| a_1 + b_1 - 1 | > | a_2 + c_2 |$.

(6): The covering system of the face $EODQ$: $\Gamma_5(E)$, $\Gamma_2(A)$, $\Gamma_5(D)$. The covering conditions are

$$| \overline{ON_2} | = | \overline{R'N_4} | > | \overline{OS_9} | = | \overline{R'S_9} |,$$
$$| \overline{ON_4} | = | \overline{R'N_2} | > | \overline{OM_9} | .$$

That is,

$$| a_2 - b_2 - 1 | > | a_3 - c_3 |,$$
$$| a_2 + c_2 - 1 | > | a_1 + b_1 | .$$

For other faces of P_1, we have their corresponding covering systems:

- The covering system of $AODO'$ is : $\Gamma_2(F)$, $\Gamma_3(A)$, $\Gamma_4(D)$.
- The covering system of $DQCQ'$ is : $\Gamma_6(E)$, $\Gamma_1(D)$, $\Gamma_5(C)$.
- The covering system of $FP'AO'$ is : $\Gamma_1(F)$, $\Gamma_4(A)$, $\Gamma_6(B)$.
- The covering system of $FO'DQ'$ is : $\Gamma_3(F)$, $\Gamma_3(D)$, $\Gamma_6(C)$.
- The covering system of $FQ'CR'$ is : $\Gamma_1(C)$, $\Gamma_4(F)$, $\Gamma_2(D)$.
- The covering system of $FR'BP'$ is : $\Gamma_5(F)$, $\Gamma_2(C)$, $\Gamma_5(B)$.

Take $\mathbf{u}_1 = (0.38, 0.17, 0.62)$, $\mathbf{u}_2 = (0.62, 0.1, -0.38)$, $\mathbf{u}_3 = (-0.33, 0.67, 0.06)$, $\mathbf{u}_4 = -\mathbf{u}_3$, $\mathbf{u}_5 = -\mathbf{u}_2$, $\mathbf{u}_6 = -\mathbf{u}_1$, it is easy to verify $\mathbf{u}_i \in \partial(P_1)$ for $i = 1, ..., 6$ and conditions $(1) - (6)$ hold. Therefore, (3.1) holds. Then $b(P_1) \leq 6$. This completes the proof of the theorem. □

4. 3-dimensional cylinder

Denote by $H = K \times [-\lambda, \lambda] = \{(\mathbf{x}, y) : \mathbf{x} \in K,\ y \in [-\lambda, \lambda]\}$ a cylinder in $\mathbb{E}^3$, where K is a 2-dimensional non-parallelogram convex domain, $\lambda > 0$. By (1.3) and the fact that the blocking number is invariant under affine transformations, we have

$$\begin{aligned} b(H) &= b(H - H) \\ &= b((K - K) \times [-2\lambda, 2\lambda]) \\ &= b(C \times [-2\lambda, 2\lambda]) \\ &= b(C \times [-\lambda, \lambda]), \end{aligned}$$

determining $b(H)$ is equivalent to studying the blocking number of a 3-dimensional cylinder $H = C \times [-1, 1]$ whose base is a 2-dimensional symmetric convex domain C.

It is already known that $b(H) = 8$ (see [5]), if the base of the cylinder H is a parallelogram. The following theorem gives the exact blocking number of a 3-dimensional cylinder whose base is a 2-dimensional non-parallelogram convex domain. Before the proof of the Theorem 4.3, we need some lemmas.

Lemma 4.1 ([13]). *Let $H \subseteq \mathbb{E}^d$ be a cylinder H, whose base is a $d-1$-dimensional convex body K. Then we have*

$$2\gamma(K - K) \leq b(H) \leq 2b(K).$$

Especially, if K is symmetric, then $2\gamma(K) \leq b(H) \leq 2b(K)$ holds.

Lemma 4.2 ([1]). *Each 2-dimensional convex domain has an inscribed affine regular hexagon.*

Theorem 4.3. *For a cylinder H in $\mathbb{E}^3$ whose base is a 2-dimensional non-parallelogram convex domain, we have $b(H) = 6$.*

Proof. By the above it suffices to consider $H = C \times [-1, 1]$, where C is a 2-dimensional non-parallelogram and symmetric convex domain. By Lemma 4.1 and (1.1), we have $b(H) \geq 2\gamma(C) = 6$.

By Lemma 2.2 and Remark 2.3, its sufficient to find six points $\mathbf{u}_1$, $\mathbf{u}_2$, ..., $\mathbf{u}_6 \in \partial(H)$ such that

$$\partial(H) \subseteq \bigcup_{i=1}^{6} [\text{int}(H) + \mathbf{u}_i]. \tag{4.1}$$

By Lemma 4.2, C has an inscribed affine regular hexagon P. The vertices of P are denoted by $\mathbf{v}_1$, $\mathbf{v}_2$,..., $\mathbf{v}_6$ in an anti-clockwise order. See Figure 5. By the symmetry of C, we have $\mathbf{v}_1 = -\mathbf{v}_4$, $\mathbf{v}_2 = -\mathbf{v}_5$, $\mathbf{v}_3 = -\mathbf{v}_6$. Denote by C_1 and C_{-1} the two bases of H, respectively. It is obvious that $\partial(C)$ can not be covered by $\text{int}(H) + \mathbf{v}_1$, $\text{int}(H) + \mathbf{v}_3$, $\text{int}(H) + \mathbf{v}_5$. If we move the point $\mathbf{v}_2$ slightly along the boundary of C towards $\mathbf{v}_1$, then we get a point $\mathbf{v}_2'$. The symmetric point $\mathbf{v}_5$ of $\mathbf{v}_2$ also moves sightly towards $\mathbf{v}_4$ and we get $\mathbf{v}_5' = -\mathbf{v}_2'$. Now we can deduce

$$\partial(C) \subseteq \bigcup \{\text{int}(H) \pm \mathbf{v}_2', \text{int}(H) \pm \mathbf{v}_3\}.$$

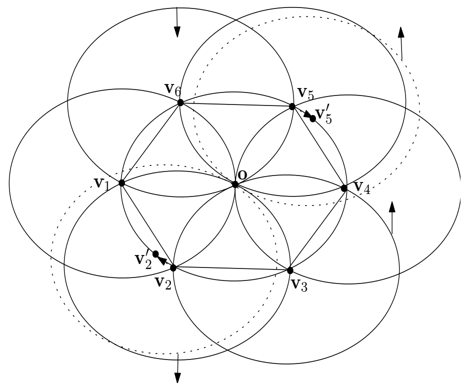

FIGURE 5. $\text{int}(C) + \mathbf{v}_i, i = 1, 2, ..., 6.$

Given a small positive number ϵ, let $\mathbf{u}_5 = \mathbf{v}_5' + (0, 0, \epsilon)$, $\mathbf{u}_6 = \mathbf{v}_6 - (0, 0, \epsilon)$, $\mathbf{u}_3 = \mathbf{v}_3 + (0, 0, \epsilon)$, and $\mathbf{u}_2 = \mathbf{v}_2' - (0, 0, \epsilon)$, then

$$\partial(C \times (1-\epsilon, -1+\epsilon)) \subseteq \bigcup \{\text{int}(H)+\mathbf{u}_5, \text{int}(H)+\mathbf{u}_6, \text{int}(H)+\mathbf{u}_3, \text{int}(H)+\mathbf{u}_2\}.$$

As for C_1 and C_{-1}, there always exists a positive number $\sigma < 1$, take $\mathbf{u}_1 = \sigma \mathbf{v}_1 + \mathbf{e}_1$, $\mathbf{u}_4 = \sigma \mathbf{v}_4 - \mathbf{e}_1$, where $\mathbf{e}_1 = (0, 0, 1)$. Then we have

$$\partial(C_1) \subseteq \bigcup \{\text{int}(H) + \mathbf{u}_3, \text{int}(H) + \mathbf{u}_5, \text{int}(H) + \mathbf{u}_1\}$$

and

$$\partial(C_{-1}) \subseteq \bigcup \{\text{int}(H) + \mathbf{u}_2, \text{int}(H) + \mathbf{u}_4, \text{int}(H) + \mathbf{u}_6\}.$$

Hence, (4.1) holds, that is to say, $b(H) \leq 6$. Therefore, $b(H) = 6$, this completes the proof of the theorem. □

5. ℓ_p UNIT BALL IN E^d

In this section, we estimate that the blocking number of $C_{3,p}$ for $1 \leq p < +\infty$ and $C_{d,p}$ for $\log_2 d < p < +\infty$.

Theorem 5.1. *$b(C_{3,p}) \leq 6$, for $1 \leq p < +\infty$.*

Proof. If $p = 1$ and $\log_2 3 < p < +\infty$, we already have $b(C_{3,p}) \leq 6$, see [13]. Hence we only need to consider the case when $1 < p \leq \log_2 3$.

Let $\mathbf{y}_1 = ((\frac{1}{2})^{\frac{1}{p}}, (\frac{1}{2})^{\frac{1}{p}}, 0)$, $\mathbf{y}_2 = (-(\frac{1}{2})^{\frac{1}{p}}, (\frac{1}{2})^{\frac{1}{p}}, 0)$, $\mathbf{y}_3 = ((\frac{1}{2})^{\frac{1}{p}}, -(\frac{1}{2})^{\frac{1}{p}}, 0)$, $\mathbf{y}_4 = (-(\frac{1}{2})^{\frac{1}{p}}, -(\frac{1}{2})^{\frac{1}{p}}, 0)$, $\mathbf{y}_5 = (0, 0, 1)$, $\mathbf{y}_6 = (0, 0, -1)$, we will show that

$$\partial(C_{3,p}) \subseteq \cup_{i=1}^{6} [\text{int}(C_{3,p}) + \mathbf{y}_i]. \tag{5.1}$$

Let

$$\Gamma = \{(x_1, x_2, x_3) : x_1^p + x_2^p + x_3^p = 1, x_i \geq 0, i = 1, 2, 3\}$$

be the surface of $C_{3,p}$ in the first octant. By the symmetry of $C_{3,p}$, it is sufficient to verify that

$$\Gamma \subseteq [\text{int}(C_{3,p}) + \mathbf{y}_1] \cup [\text{int}(C_{3,p}) + \mathbf{y}_5]. \tag{5.2}$$

Since

$$\left(1-\left(\frac{1}{2}\right)^{\frac{1}{p}}\right)^{p}+\frac{1}{2}<1,$$

we know $(1,0,0)$ and $(0,1,0)$ both belong to $\mathrm{int}(C_{3,p})+\mathbf{y}_1$. We also have

$$\Gamma_1=\{(x_1,x_2,x_3): x_1^p+x_2^p=1, x_3=0, x_i\geq 0, i=1,2\}\subseteq \mathrm{int}(C_{3,p})+\mathbf{y}_1,$$

since

$$\left(x_1-\left(\frac{1}{2}\right)^{\frac{1}{p}}\right)^{p}+\left(x_2-\left(\frac{1}{2}\right)^{\frac{1}{p}}\right)^{p}<x_1^p+x_2^p=1.$$

Let

$$\Gamma_2=\left\{(x_1,0,x_3): x_1^p+x_3^p=1, \left(\frac{1}{2}\right)^{\frac{1}{p}}<x_1\leq 1, 0\leq x_3<\left(\frac{1}{2}\right)^{\frac{1}{p}}\right\},$$

$$\Gamma_3=\left\{(0,x_2,x_3): x_2^p+x_3^p=1, \left(\frac{1}{2}\right)^{\frac{1}{p}}<x_2\leq 1, 0\leq x_3<\left(\frac{1}{2}\right)^{\frac{1}{p}}\right\},$$

It is easy to verify

$$\left(0,\left(\frac{1}{2}\right)^{\frac{1}{p}},\left(\frac{1}{2}\right)^{\frac{1}{p}}\right),\ \left(\left(\frac{1}{2}\right)^{\frac{1}{p}},0,\left(\frac{1}{2}\right)^{\frac{1}{p}}\right)\in\partial(C_{3,p})+\mathbf{y}_1,\ \mathrm{int}(C_{3,p})+\mathbf{y}_5,$$

$$\Gamma_2,\Gamma_3\subseteq \mathrm{int}(C_{3,p})+\mathbf{y}_1.$$

Let

$$\Gamma_2'=\left\{(x_1,0,x_3): x_1^p+x_3^p=1, \left(\frac{1}{2}\right)^{\frac{1}{p}}\leq x_3\leq 1, 0\leq x_1\leq\left(\frac{1}{2}\right)^{\frac{1}{p}}\right\},$$

$$\Gamma_3'=\left\{(0,x_2,x_3): x_2^p+x_3^p=1, \left(\frac{1}{2}\right)^{\frac{1}{p}}\leq x_3\leq 1, 0\leq x_2\leq\left(\frac{1}{2}\right)^{\frac{1}{p}}\right\},$$

It is easy to verify

$$\Gamma_2',\Gamma_3'\subseteq \mathrm{int}(C_{3,p})+\mathbf{y}_5.$$

Denote the intersection arc of $C_{3,p}+\mathbf{y}_5$ and Γ by

$$l=\left\{(x_1,x_2,x_3): x_3=\frac{1}{2}, x_1^p+x_2^p=1-\left(\frac{1}{2}\right)^{p}, x_i\geq 0, i=1,2\right\}.$$

Apart from l, the subset of Γ bounded by Γ_2', Γ_3' and l belong to $\mathrm{int}(C_{3,p})+\mathbf{y}_5$. Hence, to verify (5.2) is equivalent to confirm $l\subseteq \mathrm{int}(C_{3,p})+\mathbf{y}_1$. By a routine computation, we have $l\subseteq \mathrm{int}(C_{3,p})+\mathbf{y}_1$. Thus, (5.2) holds. Surfaces of $C_{3,p}$ in other quadrants can be covered similarly. Therefore, (5.1) holds and $b(C_{3,p})\leq 6$. This completes the proof of the theorem. □

Theorem 5.2. $b(C_{d,p})\leq 2d$, *for* $\log_2 d<p<+\infty$.

Proof. Suppose that $p < +\infty$. Take $\mathbf{y}_1 = (1, 0, \dots, 0)$, $\mathbf{y}_2 = (0, 1, \dots, 0)$, $\dots$, $\mathbf{y}_d = (0, 0, \dots, 1)$, then $\pm\mathbf{y}_i \in \partial(C_{d,p})$ for $1 \le i \le d$. It is easy to verify that

$$\left\{\mathbf{x} = (x_1, x_2, \dots, x_d) : \mathbf{x} \in \partial(C_{d,p}), x_i > \frac{1}{2}\right\} \subseteq \operatorname{int}(C_{d,p}) + \mathbf{y}_i.$$

If every boundary point z of $C_{d,p}$ has a coordinate z_i such that

$$|z_i| > \frac{1}{2},$$

then we have

$$\partial(C_{d,p}) \subseteq \cup_{i=1}^{d}(\operatorname{int}(C_{d,p})) \pm \mathbf{y}_i.$$

By Lemma 2.2 and Remark 2.3, $b(C_{d,p}) \le 2d$ holds.

Now we consider the subset

$$T = \left\{(t_1, t_2, \dots, t_d) : |t_i| \le \frac{1}{2}, i = 1, , d\right\}.$$

According to the above discussion, T must be a subset of $\operatorname{int}(C_{d,p})$. Then

$$(|t_1|^p + |t_2|^p + \dots + |t_d|^p)^{\frac{1}{p}} \le \left(\frac{d}{2^p}\right)^{\frac{1}{p}} \le \frac{d^{\frac{1}{p}}}{2} < 1,$$

and

$$\log_2 d < p.$$

This completes the proof of the theorem. □

Remark 5.3. By Theorem 5.2, we can easily deduce that $b(C_{3,p}) \le 6$ for $\log_2 3 < p < +\infty$ which has been already obtained by Yu [13].

Remark 5.4. By Theorem 5.2 and (1.2), we have $\gamma(C_{d,p}) \le b(C_{d,p}) \le 2d$, for $\log_2 d < p < +\infty$.

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