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CONGRUENCE PROPERTIES MODULO POWERS OF 3 FOR 6-COLORED GENERALIZED FROBENIUS PARTITIONS

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Abstract. In his 1984 AMS Memoir, Andrews introduced the family of functions $c\phi_k(n)$, which denotes the number of k-colored generalized Frobenius partitions of n. In this paper, we prove three congruences and three internal congruences modulo powers of 3 for $c\phi_6(n)$ by utilizing the generating function of $c\phi_6(3n+1)$ due to Hirschhorn. Finally, we conjecture two families of congruences and two families of internal congruences modulo arbitrary powers of 3 for $c\phi_6(n)$, which strengthen a conjecture due to Gu, Wang and Xia in 2016.

1. Introduction

In his 1984 Memoir of the American Mathematical Society, Andrews [1] introduced the notion of a generalized Frobenius partition of n, which is a two-rowed array of nonnegative integers of the form:

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix},$$

wherein each row, which is of the same length, is arranged in weakly decreasing order with $n = r + \sum_{i=1}^{r}(a_i + b_i)$. Furthermore, Andrews considered a variant of generalized Frobenius partition whose parts are taken from k copies of the nonnegative integers, which is called k-colored generalized Frobenius partitions. For any positive integer k, let $c\phi_k(n)$ denote the number of k-colored generalized Frobenius partitions of n. Andrews [1, Corollary 10.1] proved that for any $n \geq 0$,

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$$c\phi_2(5n+3) \equiv 0 \pmod{5}.$$

Since then, many scholars investigated extensively a number of congruence properties for $c\phi_k(n)$ with different moduli; see, for example, [2, 3, 5–15, 18–20, 20–35, 37–41].

In 2015, Baruah and Sarmah [3] derived an expression of the generating function of $c\phi_6(n)$ and established a 3-dissection formula of the generating function of $c\phi_6(n)$. As an immediate consequence, they [3, Corollary 4.1] proved that for any $n \geq 0$,

$$c\phi_6(3n+1) \equiv 0 \pmod{9},$$
$$c\phi_6(3n+2) \equiv 0 \pmod{9}.$$

Baruah and Sarmah [3] further conjectured that for any $n \geq 0$,

$$c\phi_6(3n+2) \equiv 0 \pmod{27}. \tag{1.1}$$

Soon after, Xia [38] proved (1.1) by utilizing the generating function of $c\phi_6(3n+2)$, derived by Baruah and Sarmah [3]. Moreover, Xia [38] also conjectured that for any $n \geq 0$,

$$c\phi_6(9n+7) \equiv 0 \pmod{27}, \tag{1.2}$$
$$c\phi_6(27n+16) \equiv 0 \pmod{243}. \tag{1.3}$$

Later, Hirscchorn [16] obtained another expression for 3-dissection formula of the generating function of $c\phi_6(n)$ and thus proved (1.1) and (1.2). Motivated by these work, Gu, Wang and Xia [15] proved eight congruences and four internal congruences modulo small powers of 3 for $c\phi_6(n)$. More precisely, they proved that

Theorem 1.1. [15, Theorem 1.1] *For any $n \geq 0$,*

$$c\phi_6(81n+61) \equiv 0 \pmod{3^4}, \tag{1.4}$$
$$c\phi_6(27n+16) \equiv 0 \pmod{3^5}, \tag{1.5}$$
$$c\phi_6(729n+547) \equiv 0 \pmod{3^5}, \tag{1.6}$$
$$c\phi_6(243n+142) \equiv 0 \pmod{3^6}, \tag{1.7}$$
$$c\phi_6(6561n+4921) \equiv 0 \pmod{3^6}, \tag{1.8}$$
$$c\phi_6(2187n+1276) \equiv 0 \pmod{3^7}, \tag{1.9}$$
$$c\phi_6(19683n+11482) \equiv 0 \pmod{3^7}, \tag{1.10}$$
$$c\phi_6(59049n+44287) \equiv 0 \pmod{3^7}, \tag{1.11}$$
$$c\phi_6(27n+7) \equiv 3c\phi_6(3n+1) \pmod{3^4}, \tag{1.12}$$
$$c\phi_6(81n+61) \equiv 3c\phi_6(9n+7) \pmod{3^5}, \tag{1.13}$$
$$c\phi_6(729n+547) \equiv 3c\phi_6(81n+61) \pmod{3^6}, \tag{1.14}$$
$$c\phi_6(6561n+4921) \equiv 3c\phi_6(729n+547) \pmod{3^7}. \tag{1.15}$$

Gu *et al.* [15] further posed the following conjecture:

Conjecture 1.2. *[15, Conjecture 1.2] For any $\alpha \geq 8$, there exist positive integers δ_k and k such that for any $n \geq 0$,*

$$c\phi_6\big(3^k n + \delta_k\big) \equiv 0 \pmod{3^\alpha}. \tag{1.16}$$

For (1.4)–(1.16), there are two natural questions. First, are there other congruences or internal congruences similar to (1.4)–(1.15) that do not appear in Theorem 1.1? Moreover, what are the exact values of k and δ_k in (1.16) when α equals 8? Further, is it possible to give a specific expression of arithmetic progression in (1.16)? Therefore, one purpose of this paper is to derive several congruences and internal congruences modulo 3^8 satisfied by $c\phi_6(n)$ similar to (1.4)–(1.15). Moreover, we also prove an internal congruence modulo 3^7 that is missed by Gu, Wang and Xia [15].

Theorem 1.3. *For any $n \geq 0$,*

$$c\phi_6(19683n + 11482) \equiv 0 \pmod{3^8}, \tag{1.17}$$

$$c\phi_6(177147n + 103336) \equiv 0 \pmod{3^8}, \tag{1.18}$$

$$c\phi_6(531441n + 398581) \equiv 0 \pmod{3^8}, \tag{1.19}$$

$$c\phi_6(243n + 142) \equiv 3c\phi_6(27n + 16) \pmod{3^7}, \tag{1.20}$$

$$c\phi_6(2187n + 1276) \equiv 3c\phi_6(243n + 142) \pmod{3^8}, \tag{1.21}$$

$$c\phi_6(59049n + 44287) \equiv 3c\phi_6(6561n + 4921) \pmod{3^8}. \tag{1.22}$$

We note that (1.17) is a stronger form of (1.10). The other purpose of this paper is to provide an explicit form for (1.16) based on Theorems 1.1 and 1.3.

Conjecture 1.4. *For any $n \geq 0$ and $\alpha \geq 1$,*

$$c\phi_6\left(3^{2\alpha}n + \frac{3^{2\alpha+1}+1}{4}\right) \equiv 0 \pmod{3^{\alpha+2}}, \tag{1.23}$$

$$c\phi_6\left(3^{2\alpha+1}n + \frac{7\times 3^{2\alpha}+1}{4}\right) \equiv 0 \pmod{3^{\alpha+4}}, \tag{1.24}$$

$$c\phi_6\left(3^{2\alpha+2}n + \frac{3^{2\alpha+3}+1}{4}\right) \equiv 3c\phi_6\left(3^{2\alpha}n + \frac{3^{2\alpha+1}+1}{4}\right) \pmod{3^{\alpha+4}}, \tag{1.25}$$

and

$$c\phi_6\left(3^{2\alpha+3}n + \frac{7\times 3^{2\alpha+2}+1}{4}\right) \equiv 3c\phi_6\left(3^{2\alpha+1}n + \frac{7\times 3^{2\alpha}+1}{4}\right) \pmod{3^{\alpha+6}}. \tag{1.26}$$

Remark: Two remarks on Conjecture 1.4 are in order. First, (1.2), (1.4), (1.6), (1.8), (1.11) and (1.19) are initial cases of (1.23); (1.3), (1.5), (1.7), (1.9) and (1.17) are initial cases of (1.24). However, (1.12) seems to be an isolated phenomenon. Second, (1.13)–(1.15) and (1.22) imply that (1.25) seems to hold. Moreover, (1.23) can be derived by (1.2), (1.25) and induction. Thus, a natural question is whether there exists a similar family of internal congruences modulo powers of 3 which implies (1.24). This question prompts us to discover congruences (1.20) and (1.21), which further led us to conjecture (1.26).

2. Preliminaries

To prove Theorem 1.3, we first collect some necessary notation and lemmas.

Throughout the rest of this paper, we always assume that q is a complex number such that $|q|<1$ and adopt the following customary notation:

$$(A;q)_\infty := \prod_{j=0}^{\infty}(1-Aq^j).$$

For notational convenience, we denote

$$E_k := (q^k;q^k)_\infty.$$

First, we need the following 3-dissections.

Lemma 2.1.

$$\frac{E_2^5}{E_1^2E_4^2} = \frac{E_{18}^5}{E_9^2E_{36}^2} + 2q\frac{E_6^2E_9E_{36}}{E_3E_{12}E_{18}}, \tag{2.1}$$

$$\frac{E_1E_4}{E_2} = \frac{E_3E_{12}E_{18}^5}{E_6^2E_9^2E_{36}^2} - q\frac{E_9E_{36}}{E_{18}}, \tag{2.2}$$

$$\frac{E_4^2}{E_2} = \frac{E_{12}E_{18}^2}{E_6E_{36}} + q^2\frac{E_{36}^2}{E_{18}}, \tag{2.3}$$

$$\frac{E_2^2}{E_4} = \frac{E_{18}^2}{E_{36}} - 2q^2\frac{E_6E_{36}^2}{E_{12}E_{18}}. \tag{2.4}$$

Proof. The identity (2.1) follows from Corollary (i) on page 49 of Berndt's book [4]. Moreover, from [4, p. 19, Corollary (ii)] we find that

$$\frac{E_2^2}{E_1} = \frac{E_6E_9^2}{E_3E_{18}} + q\frac{E_{18}^2}{E_9}. \tag{2.5}$$

Replacing q by $-q$ in (2.1) and (2.5) and utilizing the fact

$$(-q;-q)_\infty = \frac{E_2^3}{E_1E_4},$$

we obtain

$$\frac{E_1^2}{E_2} = \frac{E_9^2}{E_{18}} - 2q\frac{E_3E_{18}^2}{E_6E_9}, \tag{2.6}$$

and (2.2), respectively. Finally, (2.3) and (2.4) follow by replacing q by q^2 in (2.5) and (2.6), respectively. $\square$

Next, recall that $a(q)$ is one of Borweins' cubic theta functions, given by

$$a(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}.$$

Hirschhorn, Garvan and Borwein [17] established the following 3-dissections related to $a(q)$:

Lemma 2.2. [17, Eqs. (1.3) and (1.4)]

$$a(q) = a(q^3) + 6q\frac{E_9^3}{E_3}, \tag{2.7}$$

$$E_1^3 = a(q^3)E_3 - 3qE_9^3. \tag{2.8}$$

Hirschhorn *et al.* [17, Eq. (1.5)] also proved that

$$a(q) = 1 + 6\sum_{n=1}^{\infty}\left(\frac{q^{3n-2}}{1-q^{3n-2}} - \sum_{n=1}^{\infty}\frac{q^{3n-1}}{1-q^{3n-1}}\right),$$

from which one readily finds that

$$a(q) \equiv 1 \pmod 3 \qquad \text{and} \qquad a(q)^3 \equiv 1 \pmod 9. \tag{2.9}$$

Finally, with the help of the binomial theorem, one can easily establish the following congruence, which will be used frequently in the sequel.

Lemma 2.3. *For any $\alpha \geq 1$,*

$$E_1^{3^\alpha} \equiv E_3^{3^{\alpha-1}} \pmod{3^\alpha}. \tag{2.10}$$

3. Proof of Theorem 1.3

In this section, all the following congruences are modulo 3^8 unless otherwise specified.

Proof of Theorem 1.3.

Proof of (1.17). Hirscchorn [16] proved that (a misprint has been corrected)

$$\begin{aligned}\sum_{n=0}^{\infty} c\phi_6(3n+1)q^n = {} & 18a(q)^5\frac{E_2^5E_3^9}{E_1^{23}E_4^2} + 18a(q)^6\frac{E_3^9E_4E_6^2}{E_1^{23}E_2E_{12}} \\ & + 1701qa(q)^2\frac{E_2^5E_3^{18}}{E_1^{26}E_4^2} + 3402qa(q)^3\frac{E_3^{18}E_4E_6^2}{E_1^{26}E_2E_{12}} \\ & - 324qa(q)^5\frac{E_3^{12}E_{12}^2}{E_1^{24}E_6} + 13122q^2\frac{E_3^{27}E_4E_6^2}{E_1^{29}E_2E_{12}} \\ & - 17496q^2a(q)^2\frac{E_3^{21}E_{12}^2}{E_1^{27}E_6}.\end{aligned}$$

Utilizing (2.9), we obtain that

(3.1)
$$\begin{aligned}
&\sum_{n=0}^{\infty} c\phi_6(3n+1)q^n \\
&\equiv 18a(q)^5\frac{E_2^5E_3^9}{E_1^{23}E_4^2} + 18a(q)^6\frac{E_3^9E_4E_6^2}{E_1^{23}E_2E_{12}} + 1701qa(q)^2\frac{E_2^5E_3^{18}}{E_1^{26}E_4^2} \\
&\quad + 3402qa(q)^3\frac{E_3^{18}E_4E_6^2}{E_1^{26}E_2E_{12}} + 6237qa(q)^5\frac{E_3^{12}E_{12}^2}{E_1^{24}E_6} + 2187q^2a(q)^2\frac{E_3^{21}E_{12}^2}{E_1^{27}E_6} \\
&\equiv 18a(q)^5\frac{E_2^5}{E_1^2E_4^2}\frac{E_1^{708}}{E_3^{234}} + 18a(q)^6\frac{E_1E_4}{E_2}\frac{E_1^{705}E_6^2}{E_3^{234}E_{12}} + 1701qa(q)^2\frac{E_2^5}{E_1^2E_4^2}E_1^3E_3^9 \\
&\quad + 3402qa(q)^3\frac{E_1E_4}{E_2}\frac{E_3^9E_6^2}{E_{12}} + 6237qa(q)^5\frac{E_1^{57}E_{12}^2}{E_3^{15}E_6} + 2187q^2\frac{E_3^{12}E_{12}^2}{E_6},
\end{aligned}$$

where the last congruence follows from (2.9) and (2.10). According to (2.8), we find that

$$\begin{aligned}
E_1^{708} &\equiv a(q^3)^{236}E_3^{236} + 21qa(q^3)^{235}E_3^{235}E_9^3 + 252q^2a(q^3)^{234}E_3^{234}E_9^6 \\
&\quad + 81q^3a(q^3)^{233}E_3^{233}E_9^9 + 243q^4a(q^3)^{232}E_3^{232}E_9^{12} \pmod{729}, \qquad (3.2)\\
E_1^{705} &\equiv a(q^3)^{235}E_3^{235} + 24qa(q^3)^{234}E_3^{234}E_9^3 + 324q^2a(q^3)^{233}E_3^{233}E_9^6 \\
&\quad + 324q^3a(q^3)^{232}E_3^{232}E_9^9 + 486q^4a(q^3)^{231}E_3^{231}E_9^{12} \pmod{729}, \qquad (3.3)\\
E_1^{57} &\equiv a(q^3)^{19}E_3^{19} + 24qa(q^3)^{18}E_3^{18}E_9^3 \pmod{81}. \qquad (3.4)
\end{aligned}$$

Substituting (2.1), (2.2), (2.7), (2.8) and (3.2)–(3.4) into (3.1), collecting all the terms of the form q^{3n+2}, after simplification, we obtain that

$$\begin{aligned}
&\sum_{n=0}^{\infty} c\phi_6(9n+7)q^n \\
&\equiv 3^7\frac{E_1^{12}E_4^2}{E_2} + 3^7a(q)^{23}\frac{E_1^3E_3^3E_4^2}{E_2} + 2\times 3^7a(q)^2\frac{E_1^9E_3E_6^5}{E_{12}^2} \\
&\quad + 4\times 3^5a(q)^{239}\frac{E_3^4E_6^5}{E_{12}^2} + 28\times 3^3a(q)^{240}\frac{E_2^2E_3^4E_{12}}{E_4E_6} \\
&\quad + 7\times 3^6qa(q)^{237}\frac{E_2^2E_3^{13}E_{12}}{E_1^3E_4E_6}. \qquad (3.5)
\end{aligned}$$

In view of (2.9) and (2.10), we can rewrite (3.5) as

(3.6)
$$\begin{aligned}
\sum_{n=0}^{\infty} c\phi_6(9n+7)q^n &\equiv 2\times 3^7\frac{E_4^2}{E_2}E_3^4 + 2\times 3^7\frac{E_3^4E_6^5}{E_{12}^2} + 4\times 3^5a(q)^{239}\frac{E_3^4E_6^5}{E_{12}^2} \\
&\quad + 28\times 3^3a(q)^{240}\frac{E_2^2}{E_4}\frac{E_3^4E_{12}}{E_6} + 7\times 3^6q\frac{E_2^2}{E_4}\frac{E_1^6E_3^{10}E_{12}}{E_6}.
\end{aligned}$$

Moreover, with the help of (2.7), we find that

$$a(q)^{239} \equiv a(q^3)^{239} + 3qa(q^3)^{238}\frac{E_9^3}{E_3} + 9q^2a(q^3)^{237}\frac{E_9^6}{E_3^2} \pmod{27}, \tag{3.7}$$

$$\begin{aligned} a(q)^{240} \equiv{}& a(q^3)^{240} + 225qa(q^3)^{239}\frac{E_9^3}{E_3} \\ &+ 216q^2a(q^3)^{238}\frac{E_9^6}{E_3^2} + 27q^3a(q^3)^{237}\frac{E_9^9}{E_3^3} \pmod{243}. \end{aligned} \tag{3.8}$$

If we substitute (2.3), (2.4), (2.8), (3.7) and (3.8) into (3.6) and extract all the terms of the form q^{3n}, then replace q^3 by q, we deduce that

$$\begin{aligned} &\sum_{n=0}^{\infty} c\phi_6(27n+7)q^n \\ &\equiv 2\times 3^7\frac{E_1^4E_4E_6^2}{E_2E_{12}} + 2\times 3^7\frac{E_1^4E_2^5}{E_4^2} + 4\times 3^5a(q)^{239}\frac{E_1^4E_2^5}{E_4^2} \\ &\quad + 28\times 3^3a(q)^{240}\frac{E_1^4E_4E_6^2}{E_2E_{12}} + 3^6qa(q)^{237}\frac{E_1E_3^9E_4E_6^2}{E_2E_{12}} \\ &\quad + 4\times 3^6qa(q)^{239}\frac{E_1^{12}E_{12}^2}{E_6} + 4\times 3^5qa(q)^{239}\frac{E_1^3E_3^3E_{12}^2}{E_6}. \end{aligned} \tag{3.9}$$

Thanks to (2.9), we further get that

$$\begin{aligned} &\sum_{n=0}^{\infty} c\phi_6(27n+7)q^n \\ &\equiv 2\times 3^7\frac{E_1E_4}{E_2}\frac{E_3E_6^2}{E_{12}} + 2\times 3^7\frac{E_2^5}{E_1^2E_4^2}E_3^2 + 4\times 3^5a(q)^{239}\frac{E_2^5}{E_1^2E_4^2}E_1^6 \\ &\quad + 28\times 3^3a(q)^{240}\frac{E_1E_4}{E_2}\frac{E_1^3E_6^2}{E_{12}} + 3^6q\frac{E_1E_4}{E_2}\frac{E_3^9E_6^2}{E_{12}} \\ &\quad + 4\times 3^6qa(q)^{239}E_1^{12}\frac{E_{12}^2}{E_6} + 4\times 3^5qa(q)^{239}E_1^3\frac{E_3^3E_{12}^2}{E_6}. \end{aligned} \tag{3.10}$$

Plugging (2.1), (2.2), (2.8), (3.7) and (3.8) into (3.10) and taking all the terms in the form of q^{3n+2}, after simplification, we arrive at

$$\begin{aligned} &\sum_{n=0}^{\infty} c\phi_6(81n+61)q^n \\ &\equiv 3^6a(q)^{239}\frac{E_3^4E_6^5}{E_{12}^2} + 8\times 3^6a(q)^{237}\frac{E_1^9E_2^2E_3E_{12}}{E_4E_6} \\ &\quad + 43\times 3^4a(q)^{240}\frac{E_2^2E_3^4E_{12}}{E_4E_6} + 3^7qa(q)^{237}\frac{E_2^2E_3^{13}E_{12}}{E_1^3E_4E_6}. \end{aligned} \tag{3.11}$$

Now if we substitute (2.4) and (2.7)–(2.10) into (3.11) and collect all the terms of the form q^{3n}, then replace q^3 by q, we obtain that

$$\begin{aligned}
&\sum_{n=0}^{\infty} c\phi_6(243n+61)q^n \\
&\equiv 3^6 a(q)^{239}\frac{E_1^4E_2^5}{E_4^2} + 8\times 3^6 a(q)^3\frac{E_1^4E_4E_6^2}{E_2E_{12}} \\
&\quad + 43\times 3^4 a(q)^{240}\frac{E_1^4E_4E_6^2}{E_2E_{12}} + 3^7 q a(q)^{237}\frac{E_1E_3^9E_4E_6^2}{E_2E_{12}} \\
&\quad + 3^7 q\frac{E_1^{12}E_{12}^2}{E_6} + 3^6 q a(q)^{239}\frac{E_1^3E_3^3E_{12}^2}{E_6}.
\end{aligned} \tag{3.12}$$

In view of (2.9) and (2.10), we find that, modulo 3^8,

$$\begin{aligned}
&\sum_{n=0}^{\infty} c\phi_6(243n+61)q^n \\
&\equiv 3^6 a(q)^{239}\frac{E_2^5}{E_1^2E_4^2}E_1^6 + 8\times 3^6\frac{E_1E_4}{E_2}\frac{E_1^3E_6^2}{E_{12}} \\
&\quad + 43\times 3^4 a(q)^{240}\frac{E_1E_4}{E_2}\frac{E_1^3E_6^2}{E_{12}} + 3^7 q\frac{E_1E_4}{E_2}\frac{E_3^9E_6^2}{E_{12}} \\
&\quad + 3^7 q\frac{E_3^4E_{12}^2}{E_6} + 3^6 q a(q)^{239}\frac{E_1^3E_3^3E_{12}^2}{E_6}.
\end{aligned} \tag{3.13}$$

Plugging (2.1), (2.2), (2.8), (3.7) and (3.8) into (3.13) and picking all the terms of the form q^{3n+2}, after simplification, we deduce that

$$\begin{aligned}
&\sum_{n=0}^{\infty} c\phi_6(729n+547)q^n \\
&\equiv 3^7 a(q)^{239}\frac{E_3^4E_6^5}{E_{12}^2} + 2\times 3^7\frac{E_1^9E_2^2E_3E_{12}}{E_4E_6} \\
&\quad + 2\times 3^7\frac{E_2^2E_3^4E_{12}}{E_4E_6} + 13\times 3^5 a(q)^{240}\frac{E_2^2E_3^4E_{12}}{E_4E_6}.
\end{aligned} \tag{3.14}$$

With the help of (2.9) and (2.10), we further find that

$$\begin{aligned}
\sum_{n=0}^{\infty} c\phi_6(729n+547)q^n &\equiv 3^7\frac{E_3^4E_6^5}{E_{12}^2} + 4\times 3^7\frac{E_2^2}{E_4}\frac{E_3^4E_{12}}{E_6} \\
&\quad + 13\times 3^5 a(q)^{240}\frac{E_2^2}{E_4}\frac{E_3^4E_{12}}{E_6}.
\end{aligned} \tag{3.15}$$

Substituting (2.4) and (3.8) into (3.15), collecting all the terms of the form q^{3n}, then replacing q^3 by q, we deduce that

$$\sum_{n=0}^{\infty} c\phi_6(2187n+547)q^n \tag{3.16}$$
$$\equiv 3^7\frac{E_1^4E_2^5}{E_4^2}+3^7\frac{E_1^4E_4E_6^2}{E_2E_{12}}+3^7qa(q)^{239}\frac{E_1^3E_3^3E_{12}^2}{E_6}$$
$$+13\times 3^5a(q)^{240}\frac{E_1^4E_4E_6^2}{E_2E_{12}}$$
$$\equiv 2\times 3^7\frac{E_2^5}{E_1^2E_4^2}E_3^2+3^7q\frac{E_3^4E_{12}^2}{E_6}+13\times 3^5a(q)^{240}\frac{E_1E_4}{E_2}\frac{E_1^3E_6^2}{E_{12}},$$

where the last congruence follows from (2.9) and (2.10). Substituting (2.1), (2.2), (2.8) and (3.8) into (3.16), collecting all the terms of the form q^{3n+2}, after simplification, we find that

$$\sum_{n=0}^{\infty} c\phi_6(6561n+4921)q^n \equiv 3^6a(q)^{240}\frac{E_2^2E_3^4E_{12}}{E_4E_6}. \tag{3.17}$$

Thanks to (2.4) and (2.9), we further find that

$$\sum_{n=0}^{\infty} c\phi_6(6561n+4921)q^n \equiv 3^6\frac{E_2^2E_3^4E_{12}}{E_4E_6} \tag{3.18}$$

$$= 3^6\frac{E_3^4E_{12}}{E_6}\left(\frac{E_{18}^2}{E_{36}}-2q^2\frac{E_6E_{36}^2}{E_{12}E_{18}}\right). \tag{3.19}$$

The congruence (1.17) follows from (3.19) immediately.

Proof of (1.18). The congruence (3.19) implies that

$$\sum_{n=0}^{\infty} c\phi_6(19683n+4921)q^n \equiv 3^6\frac{E_1E_4}{E_2}\frac{E_1^3E_6^2}{E_{12}}. \tag{3.20}$$

Substituting (2.2) and (2.8) into (3.20) and picking all the terms of the form q^{3n+2}, after simplification, we deduce that

$$\sum_{n=0}^{\infty} c\phi_6(59049n+44287)q^n \equiv 3^7\frac{E_2^2E_3^4E_{12}}{E_4E_6}. \tag{3.21}$$

According to (2.4), we find that

$$\sum_{n=0}^{\infty} c\phi_6(59049n+44287)q^n \equiv 3^7\frac{E_3^4E_{12}}{E_6}\left(\frac{E_{18}^2}{E_{36}}-2q^2\frac{E_6E_{36}^2}{E_{12}E_{18}}\right). \tag{3.22}$$

The congruence (1.18) follows from (3.22) immediately.

Proof of (1.19). Taking out the term of the form q^{3n} in (3.22) gives that

$$\sum_{n=0}^{\infty} c\phi_6(177147n+44287)q^n \equiv 3^7\frac{E_1^4E_4E_6^2}{E_2E_{12}} \equiv 3^7\frac{E_1E_4}{E_2}\frac{E_3E_6^2}{E_{12}}. \tag{3.23}$$

Thanks to (2.2), we further see that

$$\sum_{n=0}^{\infty} c\phi_6(177147n+44287)q^n \equiv 3^7 \frac{E_3E_6^2}{E_{12}}\left(\frac{E_3E_{12}E_{18}^5}{E_6^2E_9^2E_{36}^2} - q\frac{E_9E_{36}}{E_{18}}\right). \tag{3.24}$$

Now (1.19) follows from (3.24) immediately.

Proof of (1.20). Plugging (2.9) into (3.11), we have

$$\begin{aligned}\sum_{n=0}^{\infty} c\phi_6(81n+61)q^n &\equiv 3^6 a(q)^{239}\frac{E_3^4E_6^5}{E_{12}^2} + 8\times 3^6\frac{E_2^2}{E_4}\frac{E_3^4E_{12}}{E_6} \\ &\quad + 43\times 3^4 a(q)^{240}\frac{E_2^2}{E_4}\frac{E_3^4E_{12}}{E_6} + 3^7 q\frac{E_2^2}{E_4}\frac{E_3^{12}E_{12}}{E_6}.\end{aligned} \tag{3.25}$$

It follows from (2.9), (2.10), (3.6) and (3.25) that

$$\begin{aligned}&\sum_{n=0}^{\infty}\big(c\phi_6(81n+61) - 3c\phi_6(9n+7)\big)q^n \\ &\equiv 3^6\frac{E_3^4E_6^5}{E_{12}^2} + 8\times 3^6\frac{E_2^2}{E_4}\frac{E_3^4E_{12}}{E_6} + 43\times 3^4 a(q)^{240}\frac{E_2^2}{E_4}\frac{E_3^4E_{12}}{E_6} \\ &\quad - 4\times 3^6\frac{E_3^4E_6^5}{E_{12}^2} - 28\times 3^4 a(q)^{240}\frac{E_2^2}{E_4}\frac{E_3^4E_{12}}{E_6} \\ &\equiv 8\times 3^6\frac{E_2^2}{E_4}\frac{E_3^4E_{12}}{E_6} + 5\times 3^5\frac{E_2^2}{E_4}\frac{E_3^4E_{12}}{E_6} \pmod{3^7}.\end{aligned}$$

According to (2.4), we obtain that

$$\begin{aligned}&\sum_{n=0}^{\infty}\big(c\phi_6(81n+61) - 3c\phi_6(9n+7)\big)q^n \\ &\equiv 8\times 3^6\frac{E_3^4E_{12}}{E_6}\left(\frac{E_{18}^2}{E_{36}} - 2q^2\frac{E_6E_{36}^2}{E_{12}E_{18}}\right) \\ &\quad + 5\times 3^5\frac{E_3^4E_{12}}{E_6}\left(\frac{E_{18}^2}{E_{36}} - 2q^2\frac{E_6E_{36}^2}{E_{12}E_{18}}\right) \pmod{3^7},\end{aligned}$$

from which we obtain (1.20).

Proof of (1.21). It follows from (2.9), (3.15), and (3.25) that

$$\begin{aligned}&\sum_{n=0}^{\infty}\big(c\phi_6(729n+547) - 3c\phi_6(81n+61)\big)q^n \\ &\equiv 3^7\frac{E_3^4E_6^5}{E_{12}^2} + 4\times 3^7\frac{E_2^2}{E_4}\frac{E_3^4E_{12}}{E_6} + 13\times 3^5 a(q)^{240}\frac{E_2^2}{E_4}\frac{E_3^4E_{12}}{E_6} \\ &\quad - 3^7\frac{E_3^4E_6^5}{E_{12}^2} - 8\times 3^7\frac{E_2^2}{E_4}\frac{E_3^4E_{12}}{E_6} - 43\times 3^5 a(q)^{240}\frac{E_2^2}{E_4}\frac{E_3^4E_{12}}{E_6} \\ &\equiv 2\times 3^7\frac{E_2^2}{E_4}\frac{E_3^4E_{12}}{E_6} + 8\times 3^6\frac{E_2^2}{E_4}\frac{E_3^4E_{12}}{E_6}.\end{aligned}$$

In view of (2.4), we find that

$$\begin{aligned}
&\sum_{n=0}^{\infty}\big(c\phi_6(729n+547)-3c\phi_6(81n+61)\big)q^n\\
&\equiv 2\times 3^7\frac{E_3^4E_{12}}{E_6}\left(\frac{E_{18}^2}{E_{36}}-2q^2\frac{E_6E_{36}^2}{E_{12}E_{18}}\right)\\
&\quad+8\times 3^6\frac{E_3^4E_{12}}{E_6}\left(\frac{E_{18}^2}{E_{36}}-2q^2\frac{E_6E_{36}^2}{E_{12}E_{18}}\right),
\end{aligned}$$

from which we obtain (1.21).

Proof of (1.22). The congruence (1.22) follows by combining (3.18) and (3.21) immediately.

This completes the proof of Theorem 1.3. □

4. Final remarks

We conclude this paper with three remarks.

First, with the help of (2.1), (2.2), (2.7) and the following 3-dissection due to Wang [36, Eq. (2.28)]:

$$\frac{1}{E_1^3}=\frac{E_9^3}{E_3^{12}}\big(a(q^3)^2E_3^2+3qa(q^3)E_3E_9^3+9q^2E_9^6\big),$$

after simplification, we find that there are 35 terms in the generating function of $c\phi_6(9n+7)$. Based on this generating function and following a similar technique of proving Theorem 1.3, one can also derive some congruences modulo 3^α for $c\phi_6(n)$ similar to (1.17)–(1.22), where $\alpha\geq 9$ is a fixed integer. However, the steps and process in proof will become more and more complicated as α increases.

Next, (3.11), (3.14), (3.18) and (3.22) seem to imply that for any $\alpha\geq 1$,

$$\sum_{n=0}^{\infty}c\phi_6\left(3^{2\alpha}n+\frac{3^{2\alpha+1}+1}{4}\right)q^n\equiv 3^{\alpha+2}\frac{E_2^2E_3^4E_{12}}{E_4E_6}\pmod{3^{\alpha+3}},\tag{4.1}$$

$$\sum_{n=0}^{\infty}c\phi_6\left(3^{2\alpha}n+\frac{3^{2\alpha+1}+1}{4}\right)q^n\equiv c_\alpha\cdot 3^{\alpha+2}\frac{E_2^2E_3^4E_{12}}{E_4E_6}\pmod{3^{\alpha+4}},\tag{4.2}$$

where c_α is an integer dependent on α. Obviously, the congruences (1.23) and (1.24) can be derived by combining (2.4), (4.1) and (4.2).

Finally, Andrews [1, Corollary 10.2] proved that for any prime k and any $n\geq 1$,

$$c\phi_k(n)\equiv\begin{cases}0 & \pmod{k^2}\qquad \text{if } k\nmid n,\\ c\phi_1(n/k) & \pmod{k^2}\qquad \text{if } k\mid n.\end{cases}\tag{4.3}$$

In 1987, Kolitsch [20] extended (4.3) to all positive values of k. More precisely, he proved that

$$\overline{c\phi}_k(n) := \sum_{d|\gcd(k,n)} \mu(d) c\phi_{k/d}\left(\frac{n}{d}\right) \equiv 0 \pmod{k^2}, \tag{4.4}$$

where $\mu(d)$ is the Möbius function. In particular, Kolitsch [20] also provided a combinatorial interpretation of the function $\overline{c\phi}_k(n)$, i.e., $\overline{c\phi}_k(n)$ denotes the number of the generalized Frobenius partitions of n with k colors whose order under cyclic permutation of the color is k. From (4.4) one sees that

$$\overline{c\phi}_6(n) = c\phi_6(n) - c\phi_3\left(\frac{n}{2}\right) - c\phi_2\left(\frac{n}{3}\right) + p\left(\frac{n}{6}\right). \tag{4.5}$$

Moreover, Wang [37, Eq. (1.44)] established the following family of congruences modulo powers of 3 for $c\phi_3(n)$:

$$c\phi_3\left(3^{2\alpha}n + \frac{7 \times 3^{2\alpha+1}+1}{8}\right) \equiv 0 \pmod{3^{4\alpha+5}}, \tag{4.6}$$

where $n \geq 0$ and $\alpha \geq 1$. The identity (4.5), together with (4.6), reveals that there is an inseparable relation on congruence properties modulo powers of 3 between $c\phi_6(n)$ and $\overline{c\phi}_6(n)$. Upon a little calculation, we find that if (1.23)–(1.26) are true, then the following are also valid:

i

$$\overline{c\phi}_6\left(3^{2\alpha}n + \frac{3^{2\alpha+1}+1}{4}\right) \equiv 0 \pmod{3^{\alpha+2}}, \tag{4.7}$$

$$\overline{c\phi}_6\left(3^{2\alpha+1}n + \frac{7 \times 3^{2\alpha}+1}{4}\right) \equiv 0 \pmod{3^{\alpha+4}}. \tag{4.8}$$

ii

$$\overline{c\phi}_6\left(3^{2\alpha+2}n + \frac{3^{2\alpha+3}+1}{4}\right) \equiv 3\overline{c\phi}_6\left(3^{2\alpha}n + \frac{3^{2\alpha+1}+1}{4}\right) \pmod{3^{\alpha+4}}, \tag{4.9}$$

$$\overline{c\phi}_6\left(3^{2\alpha+3}n + \frac{7 \times 3^{2\alpha}+1}{4}\right) \equiv 3\overline{c\phi}_6\left(3^{2\alpha+1}n + \frac{7 \times 3^{2\alpha}+1}{4}\right) \pmod{3^{\alpha+6}}. \tag{4.10}$$

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