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ENUMERATION OF PARALLELOGRAM POLYCUBES

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Abstract. In this paper, we give the Dirichlet generating function of parallelogram polycubes according to the volume and the width in terms of multivariate zeta functions. We also enumerate them according to the width, the length and the depth. All these results are generalized to polyhypercubes.

1. Introduction

In $\mathbb{Z}^2$, a *polyomino* is a finite connected union of cells without a cut point and defined up to translation [21]. This object, introduced in 1954 by Golomb [15], has been studied since and the main problem is finding the number of polyominoes for a given number n of cells. This problem is considered a hard problem in combinatorics. To date, no exact formula is known. However the number of polyominoes is known up to $n = 56$ (Jensen [20]). In the absence of formulas for the general case and to approximate their enumeration, polyominoes with special properties, such as convexity and direction, were defined and enumerated in the literature. Let us cite, for instance, the column-convex polyominoes [11], the convex polyominoes [9], the diagonally convex polyominoes [14] and the directed polyominoes [17]. Exact enumerations exist according to the number of cells for some of them and others were enumerated according to additional parameters such as the perimeter, the height and the width. One can find a survey in [18]. Several methods of enumeration were used as Temperley methodology [26], Bousquet Melou method [4] and ECO method [3]. One particular studied family is *parallelogram polyominoes*, the polyominoes of this class have columns without holes glued together with the bottoms and the tops of the columns forming two increasing sequences. Exact enumeration exists for them according to different parameters. Let us cite for instance the results of Delest and Viennot[12] according to the perimeter by using a bijection

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with Dyck paths, Delest, Dubernard and Dutour [10] according to the area, width, right and left corners, Bousquet-Melou [4] according to the area, width, height and length of the leftmost and rightmost column and more recently, Alofi and Dukes [2] established connections between EW-tableaux and parallelogram polyominoes.

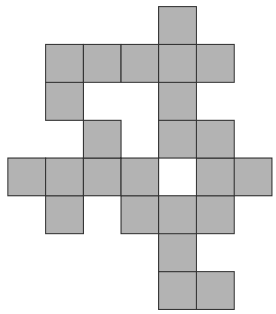

FIGURE 1. Example of a polyomino.

The extension of polyominoes in dimension 3 is called *polycubes*. In $\mathbb{Z}^3$, a unit cell is defined as a unit cube. A polycube is a finite face-connected union of elementary cells defined up to translation. As with the 2-dimensional case, the enumeration of polycubes with n cells is still an open problem. Many authors have enumerated the first values of polycubes. In 1971, Lunnon enumerated them up to $n = 7$ [22]. In 2008, Aleksandrowicz and Barequet gave the bound up to $n = 18$ [1] and the known upper bound is from Luther and Mertens up to $n = 19$ [23]. Unlike polyominoes, only a few classes of polycubes have been enumerated. Let us cite, for instance, the plane partitions [8], the directed plateau polycubes [6], and the partially directed snake polycubes [16]. However some tools were developed for the enumeration of polycubes, in particular the generic method [7], an extension of the Bousquet–Melou method [4] and the Dirichlet convolution for the enumeration of polycubes [5].

These methods only enumerate directed polycubes with convex restrictions. The problem for other classes is still open and no method for the 2-dimensional case has been adapted to the 3-dimensional case. One particular class that could not be enumerated is the class of parallelogram polycubes. In [6], the authors found a differential equation for the generating function according to the volume, the width, the area of the rightmost face, the height of the last plateau and the depth of the last plateau, but this equation could not be solved. Nevertheless, some asymptotic results, which are the only ones known, are given.

Our objective in this paper is to enumerate the family of parallelogram polycubes in two different ways by new approaches. The first one uses a Dirichlet generating function to enumerate them according to the width and the volume. It leads us to enumerate parallelogram polyominoes. Then we

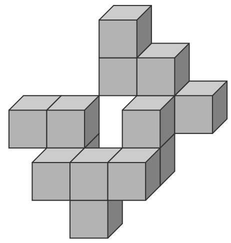

FIGURE 2. Example of a polycube.

generalize it to the polycube case. We also show the relation between these generating functions and the multivariate zeta functions [19]. The second enumeration is done according to the width, the height and the depth. In this case we project the polycubes and, using the known results for polyominoes, we deduce an explicit formula for polycubes. These two enumerations allow us to explore new parameters for polyominoes and polycubes different from the classic ones. This leads to the possibility of studying new enumerations. In Section 2, we give some definitions and notations. Then, in Section 3, we enumerate the parallelogram polycubes according to the width and the volume. In Section 4, we explore the parallelogram polycubes according to the width, the height and the depth. Finally, in the last Section, we generalize some results to any dimension $d \geq 4$.

2. PRELIMINARIES

Let $(0, \vec{i}, \vec{j})$ be an orthonormal coordinate. The area of a polyomino is the number of its cells, its width is the number of its columns and its height is the number of its rows.

A polyomino is said to be *column-convex* (resp. *row-convex*) if its intersection with any vertical (resp. horizontal) line is connected. If it is both column and row convex, it is called *convex* polyomino.

A North (resp. East) step is a movement of one unit in $\vec{i}$-direction (resp. $\vec{j}$-direction). From these two steps, a *directed* polyomino is defined as if from a distinguished cell called *root*, we can reach any other cell by a path that uses only North or East steps.

The *bottom* (resp. *top*) of a column is the height of the closet (resp. furthest) cell to the axis $(0, \vec{i})$.

A *parallelogram polyomino* is defined as a convex polyomino such that the bottoms and the tops of its columns form two increasing sequences, see Fig. 3.

Let $(0, \vec{i}, \vec{j}, \vec{k})$ be an orthonormal coordinate system. As for polyominoes, several parameters can be defined for a polycube. The *volume* is the number of its cubes. The *width* (resp. *height*, *depth*) of a polycube is the difference between its greatest and its smallest indices according to $\vec{i}$ (resp. $\vec{j}$, $\vec{k}$).

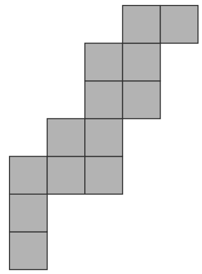

FIGURE 3. A parallelogram polyomino.

A polycube is said to be *directed* if each of its cells can be reached from a distinguished cell, called the *root*, by a path only made of East (one unit in the $\vec{i}$-direction), North (one unit in $\vec{j}$-direction) and Ahead (one unit in $\vec{k}$-direction) steps.

A *stratum* is a polycube of width 1 and a *plateau* is a rectangular stratum with no holes; it is the equivalent of a column for a polycube.

The front (resp. the back) of a plateau is the closest (resp. the furthest) side of the plane $(0, \vec{i}, \vec{j})$. The bottom (resp. the top) of a plateau is defined as the closest (resp. the furthest) side of the plane $(0, \vec{i}, \vec{k})$.

A *plateau polycube* is a polycube whose strata are plateaus glued together. A subclass of plateau polycubes are *parallelogram polycubes*. A parallelogram polycube is a plateau polycube such that the bottoms, the tops, the fronts and the backs of its plateaus form an increasing sequence, see Fig. 4.

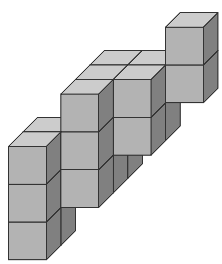

FIGURE 4. Example of a parallelogram polycube.

For more details about polyominoes and polycubes, one can see [6].

We define $S(X)$ of a finite set X as the set of all the ordered partitions of the set X. For example if $X = \{a, b, c\}$ then

$$\begin{aligned}S(X) = \{&(\{a\},\{b\},\{c\}),(\{a\},\{c\},\{b\}),(\{b\},\{a\},\{c\}),(\{b\},\{c\},\{a\}),\\&(\{c\},\{a\},\{b\}),(\{c\},\{b\},\{a\}),(\{a,b\},\{c\}),(\{a,c\},\{b\}),(\{b,c\},\{a\}),\\&(\{a\},\{b,c\}),(\{b\},\{a,c\}),(\{c\},\{a,b\}),(\{a,b,c\})\}.\end{aligned}$$

Given a sequence $\{a_{n_1,n_2,\ldots,n_k}\}_{n_1,n_2,\ldots,n_k\geq 1}$, the *ordinary generating function* OGF [27] is the formal power series of the form:

$$F(x_1,x_2,\ldots,x_k)=\sum_{n_1,n_2,\ldots,n_k\geq 1} a_{n_1,n_2,\ldots,n_k} x_1^{n_1}x_2^{n_2}\cdots x_k^{n_k}.$$

For this sequence, the *Dirichlet generating function* DGF [27] is the formal power series of the form:

$$V(x_1,x_2,\ldots,x_k)=\sum_{n_1,n_2,\ldots,n_k\geq 1} \frac{a_{n_1,n_2,\ldots,n_k}}{n_1^{x_1}n_2^{x_2}\cdots n_k^{x_k}}.$$

For $k\geq 1$, the *multivariate zeta functions* [19] are defined as:

$$\zeta_k(x_1,x_2,\ldots,x_k)=\sum_{n_1>n_2>\cdots>n_k\geq 1} \frac{1}{n_1^{x_1}n_2^{x_2}\cdots n_k^{x_k}}.$$

Note that the case $k=1$ is the classic Riemann zeta function [27], defined as

$$\zeta_1(x)=\sum_{n\geq 1}\frac{1}{n^x}.$$

3. Enumeration of parallelogram polycubes according to the width and the volume

In this section, we give an expression of the Dirichlet generating function of parallelogram polyominoes. Then we deduce a relation between this case and the case of polycubes.

We start by giving formulas for the number of parallelogram polyominoes according to the width and the area, and their extension for polycubes according to the width and the volume.

3.1. **Formulas for parallelogram polyominoes and polycubes.** Let $a_{m_1,m_2,\ldots,m_k}$ be the number of parallelogram polyominoes of width k and whose area (or height) of the ith column is equal m_i, where $1\leq i\leq k$. The following proposition is a consequence of the definition. However, as we use a similar reasoning in the 3-dimensional case, we give its proof.

Proposition 3.1. *For integer k and $m_1,m_2,\ldots,m_k\in\mathbb{N}$, we have:*

$$a_{m_1,m_2,\ldots,m_k}=\begin{cases}1, & \text{if } k=1,\\ \prod_{j=1}^{k-1}\min(m_j,m_{j+1}), & \text{otherwise.}\end{cases}$$

Proof. In the case $k=1$, the considered polyominoes (that are of width 1) are reduced to one column. Thus, for any value of m_1, there is only one possible polyomino. To determine $a_{m_1,\ldots,m_k}$ in the case $k\geq 2$, we have to build all possible corresponding polyominoes. We start by considering a column of height m_1 and we successively glue all the other columns one by one. When we add the ith column of height m_i onto the $(i+1)$th one of height m_{i+1} $(1\leq i\leq k-1)$, there are exactly $\min(m_i,m_{i+1})$ possibilities of gluing these two columns to obtain a parallelogram polyomino. An illustration of this building is given Fig. 5. As we do this for each pair of columns, we get the product. □

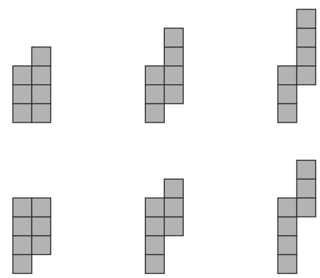

FIGURE 5. Example construction of different gluings to obtain a parallelogram polyomino.

Note that, summing over all ordered partitions of an integer n into k parts, we get $b_{k,n}$ the number of parallelogram polyominoes of width k and area n in the following corollary.

Corollary 3.2. *For integers $n \geq k \geq 1$,*

$$b_{k,n} = \sum_{\substack{m_1, m_2, \ldots, m_k \in \mathbb{N} \\ m_1 + m_2 + \cdots + m_k = n}} a_{m_1, m_2, \ldots, m_k}.$$

The first values of $b_{k,n}$ are given in the following table.

$n \backslash k$	1	2	3	4	5	6	7	8	9	10
1	1	0	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0	0
3	1	2	1	0	0	0	0	0	0	0
4	1	4	3	1	0	0	0	0	0	0
5	1	6	8	4	1	0	0	0	0	0
6	1	9	17	13	5	1	0	0	0	0
7	1	12	32	34	19	6	1	0	0	0
8	1	16	551	78	58	26	7	1	0	0
9	1	20	89	160	154	90	34	8	1	0
10	1	25	136	305	365	269	131	43	9	1

TABLE 1. The first values of $b_{k,n}$, the number of parallelogram polyominoes having k columns and area n.

Summing the values of each line we obtain the number of parallelogram polyominoes with n cells, which corresponds to sequence A006958 of the OEIS [24].

The Proposition 3.1 can be generalized to the 3-dimensional case in the following way.

Proposition 3.3. *Let $p_{n_1,n_2,\ldots,n_k}$ be the number of parallelogram polycubes of width k and whose ith plateau has a volume n_i with $i = 1,\ldots,k$.*

$$p_{n_1,n_2,\ldots,n_k} = \sum_{v_1|n_1,\ldots,v_k|n_k} \prod_{i=1}^{k-1} \min(v_i, v_{i+1}) \min\left(\frac{n_i}{v_i}, \frac{n_{i+1}}{v_{i+1}}\right).$$

Proof. First for a fixed volume of a plateau n_i, there are exactly $\tau(n_i)$ possible plateaus, where $\tau(n_i)$ is the number of divisors of n_i. Note that it corresponds to the sequence A000005 of the OEIS [24]. An example is shown in Fig. 6. Given a plateau of volume n_i and height v_i, we deduce that its depth is $\frac{n_i}{v_i}$. Recall that the width of a plateau is 1. Then we have to glue two plateaus in the same way as for polyominoes. For the 2-dimensional case, we glued according to the height but in the case of polycubes, we do it according to the height and the depth. Therefore for a plateau i of volume n_i and height v_i, and a plateau $i+1$ of volume n_{i+1} and height v_{i+1}, we have exactly $\min(v_i, v_{i+1}) \min(\frac{n_i}{v_i}, \frac{n_{i+1}}{v_{i+1}})$ possible gluings. An example is shown in Fig. 7. Moreover by summing for all possible heights of each plateau, we get the formula. □

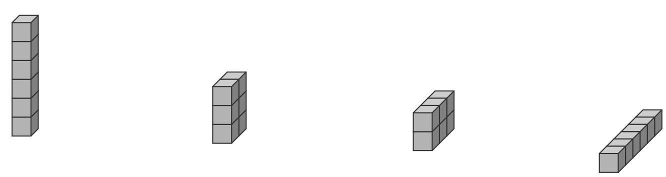

FIGURE 6. All plateaus of volume 6.

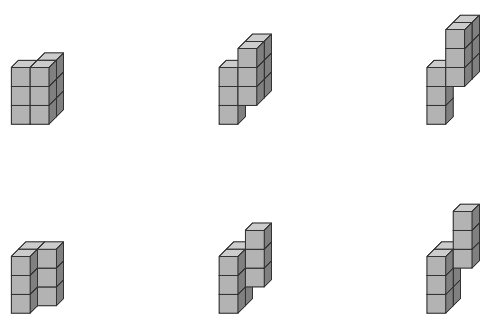

FIGURE 7. Example of gluing two plateaus.

Thus, by summing for all partitions of an integer n into k parts, we get $c_{k,n}$, the number of parallelogram polycubes of width k and volume n in the following corollary.

Corollary 3.4.

$$c_{k,n} = \sum_{\substack{n_1,n_2,\ldots,n_k \in \mathbb{N} \\ n_1+n_2+\cdots+n_k=n}} p_{n_1,n_2,\ldots,n_k}.$$

The first values of $c_{k,n}$, the number of parallelogram polycubes of width k and volume n, are given in the following table.

$n\backslash k$	1	2	3	4	5	6	7	8	9	10
1	1	0	0	0	0	0	0	0	0	0
2	2	1	0	0	0	0	0	0	0	0
3	2	4	1	0	0	0	0	0	0	0
4	3	10	6	1	0	0	0	0	0	0
5	2	18	22	8	1	0	0	0	0	0
6	4	32	59	38	10	1	0	0	0	0
7	2	44	132	132	58	12	1	0	0	0
8	4	70	264	374	245	82	14	1	0	0
9	3	84	469	916	836	406	110	16	1	0
10	4	126	808	2015	2438	1614	623	142	18	1

TABLE 2. The first values of $c_{k,n}$, the number of parallelogram polycubes of width k and area n.

Note that the values of the diagonals correspond to the results found experimentally in [6].

3.2. **Dirichlet generating function.** Let $V_k(x_1, x_2, \ldots, x_k)$ be the Dirichlet generating function of the parallelogram polyominoes of width k, where x_i codes the area of the ith column where $1 \le i \le k$, it is defined as follows

$$V_k(x_1, x_2, \ldots, x_k) := \sum_{n_1,n_2,\ldots,n_k \ge 1} \frac{\min(n_1, n_2) \cdots \min(n_{k-1}, n_k)}{n_1^{x_1} n_2^{x_2} \cdots n_k^{x_k}}$$

Before giving an algorithm to how express this generating function in terms of multivariate zeta functions, we give the formulas for $k = 1$, $k = 2$,$k = 3$ and $k = 4$.

- For $k = 1$,

$$V_1(x_1) = \sum_{n_1 \ge 1} \frac{1}{n_1^{x_1}} = \zeta_1(x_1).$$

- For $k = 2$,

$$V_2(x_1, x_2) = \sum_{n_1,n_2 \ge 1} \frac{\min(n_1, n_2)}{n_1^{x_1} n_2^{x_2}}$$
$$= \sum_{n_1>n_2\ge 1} \frac{1}{n_1^{x_1} n_2^{x_2-1}} + \sum_{n_2>n_1\ge 1} \frac{1}{n_1^{x_1-1} n_2^{x_2}} + \sum_{n_1=n_2\ge 1} \frac{1}{n_1^{x_1+x_2-1}}.$$

So

$$V_2(x_1, x_2) = \zeta_2(x_1, x_2 - 1) + \zeta_2(x_2, x_1 - 1) + \zeta_1(x_1 + x_2 - 1).$$

- For $k = 3$

$$
\begin{aligned}
V_3(x_1,x_2,x_3) &= \sum_{n_1,n_2,n_3\geq 1} \frac{\min(n_1,n_2)\min(n_2,n_3)}{n_1^{x_1}n_2^{x_2}n_3^{x_3}}\\
&= \sum_{n_1>n_2>n_3\geq 1} \frac{1}{n_1^{x_1}n_2^{x_2-1}n_3^{x_3-1}} + \sum_{n_1>n_3>n_2\geq 1} \frac{1}{n_1^{x_1}n_2^{x_2-2}n_3^{x_3}}\\
&+ \sum_{n_2>n_1>n_3\geq 1} \frac{1}{n_1^{x_1-1}n_2^{x_2}n_3^{x_3-1}} + \sum_{n_2>n_3>n_1\geq 1} \frac{1}{n_1^{x_1-1}n_2^{x_2}n_3^{x_3-1}}\\
&+ \sum_{n_3>n_1>n_2\geq 1} \frac{1}{n_1^{x_1-1}n_2^{x_2-1}n_3^{x_3}} + \sum_{n_3>n_2>n_1\geq 1} \frac{1}{n_1^{x_1-1}n_2^{x_2-1}n_3^{x_3}}\\
&+ \sum_{n_1=n_2>n_3\geq 1} \frac{1}{n_1^{x_1+x_2-1}n_3^{x_3-1}} + \sum_{n_1=n_3>n_2\geq 1} \frac{1}{n_1^{x_1+x_3}n_2^{x_2-2}}\\
&+ \sum_{n_2=n_3>n_1\geq 1} \frac{1}{n_1^{x_1-1}n_2^{x_2+x_3-1}} + \sum_{n_3>n_1=n_2\geq 1} \frac{1}{n_1^{x_1+x_2-2}n_3^{x_3}}\\
&+ \sum_{n_2>n_1=n_3\geq 1} \frac{1}{n_1^{x_1+x_3-2}n_2^{x_2}} + \sum_{n_1>n_2=n_3\geq 1} \frac{1}{n_1^{x_1}n_2^{x_2+x_3-2}}\\
&+ \sum_{n_1=n_2=n_3\geq 1} \frac{1}{n_1^{x_1+x_2+x_3-2}}.
\end{aligned}
$$

So

$$
\begin{aligned}
V_3(x_1,x_2,x_3) =&\zeta_3(x_1,x_2-1,x_3-1)+\zeta_3(x_1,x_3,x_2-2)\\
&+\zeta_3(x_2,x_1-1,x_3-1)+\zeta_3(x_2,x_3-1,x_1-1)\\
&+\zeta_3(x_3,x_1,x_2-2)+\zeta_3(x_3,x_2-1,x_1-1)\\
&+\zeta_2(x_1+x_2-1,x_3-1)+\zeta_2(x_1+x_3,x_2-2)\\
&+\zeta_2(x_2+x_3-1,x_1-1)+\zeta_2(x_3,x_1+x_2-2)\\
&+\zeta_2(x_2,x_1+x_3-2)+\zeta_2(x_1,x_2+x_3-2)\\
&+\zeta_1(x_1+x_2+x_3-2).
\end{aligned}
$$

Let X_k be the set of variables

$$X_k = \{x_1, x_2, \ldots, x_k\}.$$

For $k \geq 2$, we have the following theorem

Theorem 3.5. *For $k \geq 1$,*

$$V_k(x_1,x_2,\ldots,x_k) = \sum_{S=(S_1,S_2,\ldots,S_l)\in S(X)} \zeta_l(e_1,e_2,\ldots,e_l),$$

where, for $1 \leq i \leq l$, $e_i = \sum\limits_{x_j\in S_i} x_j - f_i$, $f_i = \sum_{j=1}^k f_{i,j}^+ + f_{i,j}^-$,

$$f_{i,j}^+ = \begin{cases} 1, & \textit{if } x_{j+1}\in S_t,\ 1\leq t\leq i-1\\ 0, & \textit{otherwise.}\end{cases}$$

and

$$f^-_{i,j} = \begin{cases} 1, & \text{if } x_{j-1} \in S_t,\ 1 \leq t \leq i \\ 0, & \text{otherwise.} \end{cases}$$

Proof. First, let us decompose the sum to make apparent all the n_j's possible orders, using the relations $>$ and $=$, $1 \leq j \leq k$. Note that the number of these decompositions is equal to the Fubini numbers which corresponds to the sequence A000670 of the OEIS [24].

Then to each decomposition we associate the partition $S = (S_1, S_2, \ldots, S_l) \in S(X)$ and to each x_j we associate a n_j, $1 \leq j \leq k$.

For $1 \leq j \leq k-1$ and $1 \leq i, t \leq l$, let $x_j \in S_i$ and $x_{j+1} \in S_t$ with $i > t$ (resp. $i < t$). This means that for the associated $n'_j s$, $n_j < n_{j+1}$ (resp. $n_j > n_{j+1}$), so $\min(n_j, n_{j+1}) = n_j$ (resp. $\min(n_j, n_{j+1}) = n_{j+1}$). It implies that in simplifying the fraction, we can decrease the power of n_j (resp. n_{j+1}). Thus, in the denominator of the fraction we get the factor $n_j^{x_j-1} n_{j+1}^{x_{j+1}}$ (resp. $n_j^{x_j} n_{j+1}^{x_{j+1}-1}$). So the variables in the zeta-functions are $e_i = x_j - 1$ and $e_t = x_{j+1}$ (resp. $e_i = x_j$ and $e_t = x_{j+1} - 1$).

If $x_j, x_{j+1} \in S_i$, then $n_j = n_{j+1}$. Using the same reasoning, we get in the zeta-function $e_i = x_j + x_{j+1} - 1$.

Applying the same reasoning for x_j and x_{j-1}, we add -1 to the variable e_i or e_t. The variable $-f_i$ is therefore defined to count all the -1's of the x_j's in S_i. Finally if S_i contains more than one element, e_i is the sum of its variables and $-f_i$. $\square$

Let $P_k(x_1, x_2, \ldots, x_k)$ be the Dirichlet generating function of parallelogram polycubes of width k and where x_i codes the volume of the ith plateau, with $1 \leq i \leq k$, defined by

$$P_k(x_1, x_2, \ldots, x_k) := \sum_{n_1, n_2, \ldots, n_k \geq 1} \frac{\sum_{v_1|n_1, \ldots, v_k|n_k} \prod_{j=1}^{k-1} \min(v_j, v_{j+1}) \min\left(\frac{n_j}{v_j}, \frac{n_{j+1}}{v_{j+1}}\right)}{n_1^{x_1} n_2^{x_2} \cdots n_k^{x_k}}.$$

This generating function can be expressed according to the generating function of parallelogram polyominoes $V_k(x_1, x_2, \ldots, x_k)$ in the following theorem.

Theorem 3.6. *For $k \geq 1$,*

$$P_k(x_1, x_2, \ldots, x_k) = \big(V_k(x_1, x_2, \ldots, x_k)\big)^2.$$

Proof. We know from Proposition 3.3 that the volume of each plateau is the product of its width and height. Therefore we can deduce the following decomposition.

$$\begin{aligned} P_k(x_1, x_2, \ldots, x_k) &= \sum_{n_1, n_2, \ldots, n_k \geq 1} \frac{\sum_{v_1|n_1, \ldots, v_k|n_k} \prod_{j=1}^{k-1} \min(v_j, v_{j+1}) \min\left(\frac{n_j}{v_j}, \frac{n_{j+1}}{v_{j+1}}\right)}{n_1^{x_1} n_2^{x_2} \cdots n_k^{x_k}} \\ &= \sum_{n_1, n_2, \ldots, n_k \geq 1} \frac{\prod_{j=1}^{k-1} \min(n_j, n_{j+1})}{n_1^{x_1} n_2^{x_2} \cdots n_k^{x_k}} \sum_{n_1, n_2, \ldots, n_k \geq 1} \frac{\prod_{j=1}^{k-1} \min(n_j, n_{j+1})}{n_1^{x_1} n_2^{x_2} \cdots n_k^{x_k}} \end{aligned}$$

Thus we get the result. $\square$

4. Enumeration according to the height, the width and the depth

As seen in Section 3, the enumeration of parallelogram polycubes is related to the case of polyominoes. In fact, by definition of parallelogram polycubes in [5], the projections on $(0, \vec{i}, \vec{j})$ and $(0, \vec{i}, \vec{k})$ of a parallelogram polycube gives two parallelogram polyominoes. Both have the same width as the polycube. Also, the height of the first polyomino is equal to the height of the polycube and the height of the second one is equal to the depth of the polycube. Moreover, each pair of parallelogram polyominoes having the same width corresponds to a unique parallelogram polycube. The example in Fig. 8 illustrates the projections of the previous polycube.

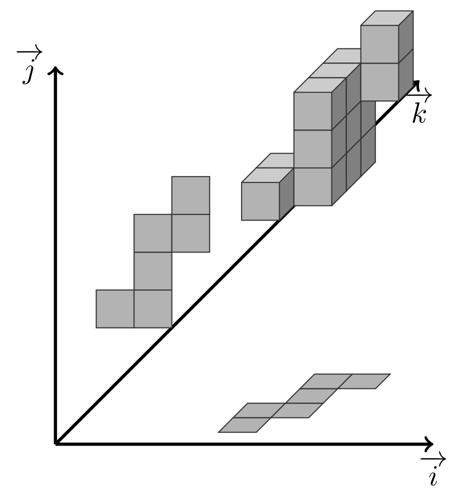

Figure 8. Parallelogram polycube and its projections.

In this section, we enumerate parallelogram polycubes according to the width, the height and the depth. We first start by giving in Lemma 4.1, the number of parallelogram polyominoes according to the width and height. Then, from this result, we deduce a formula for the polycubes.

Lemma 4.1. [13] *Let $k, n \in \mathbb{N}$ and $g_{k,n}$ denote the number of parallelogram polyominoes of width k and height n. Then,*

$$g_{k,n} = \frac{1}{n+k-1}\binom{n+k-1}{k}\binom{n+k-1}{k-1}.$$

From this lemma, the following theorem is deduced.

Theorem 4.2. *Let $s_{k,n,m}$ be the number of parallelogram polycubes, of width k, height n and depth m. Then for $k, n, m \geq 1$, we have,*

$$s_{k,n,m} = \frac{nm}{k^2(n+k-1)(m+k-1)}\binom{n+k-1}{k-1}^2\binom{m+k-1}{k-1}^2$$

Proof. Recall that for each pair of parallelogram polyominoes having the same width k, we build a unique polycube of width k. If the height of the first one is n, then the height of the polycube is also n and if the height of the second polyomino is m, it implies that the depth of the polycube m. From Lemma 4.1, the number of polyominoes of width k and height n is $\frac{1}{n+k-1}\binom{n+k-1}{k}\binom{n+k-1}{n}$, therefore by the rule of product, the formula is:

$$\frac{1}{(n+k-1)(m+k-1)}\binom{n+k-1}{k}\binom{n+k-1}{k-1}\binom{m+k-1}{k}\binom{m+k-1}{k-1}.$$

Using the binomial identity [25]

$$\binom{n+k-1}{k} = \binom{n+k}{k} - \binom{n+k-1}{k-1},$$

we obtain,

$$\begin{aligned}\binom{n+k-1}{k}\binom{n+k-1}{k-1} &= \left[\binom{n+k}{k} - \binom{n+k-1}{k-1}\right]\binom{n+k-1}{k-1}\\ &= \left[(\frac{n+k}{k}-1)\binom{n+k-1}{k-1}\right]\binom{n+k-1}{k-1}\\ &= \frac{n}{k}\binom{n+k-1}{k-1}^2,\end{aligned}$$

By replacing in the formula, we get the result. □

There are several appearances of specific instances of the numbers $s_{k,m,n}$ in the OEIS [24]. The following table gives these cases.

Sequence	Index OEIS
$s_{k,n,n}$	A174158
$s_{2,2,m}$	A045943
$s_{n,n,1}$	A000891
$a_l = \sum_{n+k=l} s_{k,n,n}$	A319743

TABLE 3. Sequences of the OEIS that appear in the enumeration of parallelogram polycubes.

5. Extension to any dimension

In this section, the results found for polycubes will be extended for any dimension $d \geq 4$. In $\mathbb{Z}^d$, a *polyhypercube* of dimension d, also called *d-polycube*, is a finite union of cells (unit hypercubes), connected by their hypercubes of dimension $d-1$, and defined up to translation.

Let $(0, \vec{i_1}, \vec{i_2}, \ldots, \vec{i_d})$ be an orthonormal coordinate system. The volume of a polyhypercube is the number of its hypercubes. The *width* is the difference between its greatest index and its smallest index according to $\vec{i_1}$. The *jth-height* is the difference between its greatest index and its smallest index

according to $\vec{i_j}$, with $2 \leq j \leq d$.
An elementary step is a positive move of one unit along the axis $\vec{i_j}$ with $1 \leq j \leq d$. A polyhypercube is directed if each cell can be reached from a distinguished cell, called the root, by a path made only by elementary steps.

The bottom (resp. top) according to $\vec{i_k}$, is the closest (resp. furthest) side of the hyperplane $(0, \vec{i_1}, \ldots, \vec{i_{k-1}}, \vec{i_{k+1}}, \ldots, \vec{i_d})$, with $k = 2, \ldots, d-1$. The bottom (resp. top) according to $\vec{i_d}$ is the closest (resp. furthest) side of the hyperplane $(0, \vec{i_1}, \ldots, \vec{i_{d-1}})$. A *stratum* is a polyhypercube of width one. A *plateau* is a hyperrectangular stratum without holes. A parallelogram polyhypercube is a polyhypercube whose bottoms and tops of its strata according to $\vec{i_k}$ form increasing sequences, $k = 2, \ldots, d$.

From these definitions, we deduce that the projection of a parallelogram polyhypercube on each plane gives a parallelogram polyomino. Therefore, the results of the two previous sections can be extended to any dimension $d \geq 4$.

Let $P_{d,k}(x_1, x_2, \ldots, x_k)$ be the generating function of parallelogram polyhypercube of dimension d and width k. Then, $P_{d,k}$ can be expressed according to V_k the generating function of parallelogram polyominoes of width k.

Theorem 5.1. *For $d \geq 4$ and $k \geq 1$,*

$$P_{d,k}(x_1, x_2, \ldots, x_k) = \big(V_k(x_1, x_2, \ldots, x_k)\big)^{d-1}.$$

Proof. The proof is the same as the case of dimension 3. It is deduced from the fact that the projection of a parallelogram polyhypercube on each plane gives a parallelogram polyomino. The volume of the ith hyperplateau is obtained by the product of the volumes of ith column of each polyomino obtained in the projection $1 \leq i \leq k$. □

Let $s_{k,n_1,\ldots,n_{d-1}}$ be the number of parallelogram polyhypercubes of dimension d, width k and the ith height is equal to n_i, with $i = 1, \ldots, d-1$. Using the same reasoning as for Theorem 4.2, we obtain

Theorem 5.2. *For $d \geq 4$, $k \geq 1$ and $n_i \geq 1$ with $i = 1 \ldots d-1$,*

$$s_{k,n_1,\ldots,n_{d-1}} = \frac{1}{k^{d-1}} \prod_{i=1}^{d-1} \frac{n_i}{n_i + k - 1} \binom{n_i + k - 1}{k - 1}^2.$$

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