Contributions to Discrete Mathematics

Volume 19, Number 3, Pages 241–257
ISSN 1715-0868

SPLIT LATTICE PATHS AND ROGERS–RAMANUJAN TYPE IDENTITIES

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Abstract. In this paper, an open problem posed by the second author [On q-series and split lattice paths, Graphs and Combinatorics, 2020] is addressed. Here, we provide combinatorial interpretations of four generalized basic series in terms of split lattice paths. Out of these series, two series have been studied by Adiga *et. al* [On Generalization of Some Combinatorial Identities, J. Ramanujan Soc. of Math. and Math. Sc., 2016] using split $(n+t)$-color partitions and R-weighted lattice paths but a direct one-to-one correspondence between these two classes was missing. We are successful in the quest of establishing bijections between the combinatorial graphical interpretations in terms of split lattice paths and combinatorial interpretations in terms of split $(n+t)$-color partitions using a purely algebraic approach. In this process, we encounter Rogers–Ramanujan type identities and we are able to provide their graphical interpretations using a constructive approach.

1. Introduction and Definitions

One of the most beautiful results in mathematics is the Rogesr–Ramanujan identities. After the discovery of such elegant sum–product identities, a quest was started to find more such types of identities in which several mathematicians have succeeded [8, 11, 12, 13]. As these identities have numerous applications in different fields [7], a new pursuit was started to explore such identities analytically, combinatorially, and graphically [2, 4, 9]. In 2014, Agarwal and Sood introduced a new combinatorial object, viz., split $(n+t)$-color partitions and used this new set of partitions to interpret unexplored basic series and basic series identities combinatorially [5, 14]. Recently one of the authors has introduced a new combinatorial object, viz., split lattice paths to study the graphical representations of Rogers–Ramanujan type identities [10]. In that paper, Gordon–McIntosh eighth order mock theta functions were explored graphically and an open problem was posed to explore Rogers–Ramanujan type identities graphically. To

Received by the editors May 23, 2022, and in revised form December 5, 2022.
2010 *Mathematics Subject Classification.* Primary 05A15, 05A17, 05A19, 11P81.
Key words and phrases. Basic series, split lattice paths, split $(n+t)$-color partitions.
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answer this posed problem, we provide graphical meanings to the following four generalized basic series constructively:
Let $m, l, p, t \in \mathbb{Z}^+$ and $k \in \{x : x = 2y; y \in \mathbb{Z}^+ \cup \{0\}\}$. For $| q |< 1$ and $1 \leq j \leq 4$, we define $g_j(q)$ by

$$g_1(q) = \sum_{\pi=0}^{\infty} \frac{q^{m\pi^2}(-q^l; q^{2l})_\pi}{(q^p; q^{2p})_\pi (q^k; q^k)_\pi}, \tag{1.1}$$

$$g_2(q) = \sum_{\pi=0}^{\infty} \frac{q^{m\pi^2+t\pi}(-q^l; q^{2l})_\pi}{(q^p; q^{2p})_{\pi+1} (q^k; q^k)_\pi}, \tag{1.2}$$

$$g_3(q) = \sum_{\pi=0}^{\infty} \frac{q^{m\pi^2+t\pi}(-q^l; q^{2l})_\pi}{(q^p; q^{2p})_\pi (q^k; q^k)_\pi}, \tag{1.3}$$

$$g_4(q) = \sum_{\pi=0}^{\infty} \frac{q^{m\pi^2-t\pi}(-q^l; q^{2l})_\pi}{(q^p; q^{2p})_\pi (q^k; q^k)_\pi} \text{ (provided } m > t\text{)}, \tag{1.4}$$

where the following standard notation is adopted:

$$(a; q)_\infty = (1 - a)(1 - aq) \cdots$$

$$(a; q)_n = \prod_{i=0}^{n-1} (1 - aq^i).$$

In this paper, we provide combinatorial interpretations of (1.1)–(1.4) using split lattice paths in Section 2. Recently, Adiga et. al [1] interpreted (1.1) and (1.2) in terms of split $(n + t)$-color partitions and R-weighted lattice paths but a direct one-to-one correspondence between the graphical aspect and the partition theoretic interpretation was missing. Hence in Section 3, the successful establishment of bijections between different classes of split $(n+t)$-color partitions and their graphical counterparts in terms of split lattice paths accomplishes the one-to-one correspondence which was expected in [1]. In Section 4, we provide some particular cases of these generalized basic series leading to entirely new 3-way combinatorial interpretations of some elegant Rogers–Ramanujan type identities found in [8]. Finally, we conclude by discussing the potential of this work in shedding light on the graphical aspects of some other Rogers–Ramanujan type identities in the future that have not been interpreted to date.

Before we state our main results we first recall some definitions:

Definition 1.1 ([3]). *A* partition with "$(n + t)$ copies of n", $t \geq 0$, *is a partition in which a part of size n, $n \geq 0$, can come in $(n + t)$ different colors denoted by subscripts: $n_1, n_2, \ldots, n_{n+t}$. Note that zeros are permitted if and only if t is greater than or equal to one. Also, zeros are not permitted to repeat in any partition.*

Remark: We note that if we take $t = 0$, then these are nothing but the n-color partitions. If the order of the parts is considered then these are n-color compositions.

Definition 1.3 ([3])**.** *The weighted difference of two parts g_k, h_l ($g \geq h$) is defined by $g - h - k - l$ and is denoted by $((g_k - h_l))$.*

Definition 1.4 ([5])**.** *Let a_p be a part in an $(n + t)$-color partition of a nonnegative integer ν. We split the color 'p' into two parts-'the green part' and 'the red part' denoted by 'g' and 'r' respectively, such that $1 \leq g \leq p$, $0 \leq r \leq p - 1$ and $p = g + r$. An $(n + t)$-color partition in which each part is split in this manner is called a split $(n + t)$-color partition.*

Example 1.5. *In 5_{2+1}, the green part is 2 and the red part is 1.*

Remark: We note that if $r = 0$, then it will not be written. Thus, for instance, we will write 5_3 for 5_{3+0}.

Definition 1.7 ([6])**.** *All "associated lattice paths" will be of finite length lying in the first quadrant. They start on the y-axis (origin included), end on the x-axis, and use three kinds of unitary steps:*

- *Northeast: from (x, y) to $(x + 1, y + 1)$.*
- *Southeast: from (x, y) to $(x + 1, y - 1)$, only allowed if $y > 0$.*
- *Horizontal: from (x, y) to $(x + 1, y)$, only allowed when the first step of a sequence of consecutive horizontal steps is preceded by a northeast step and the last is followed by a southeast step.*

The following terminology is used in describing associated lattice paths:

Truncated isosceles trapezoidal section (TITS)*: A section of the path which starts on the x-axis with northeast steps followed by horizontal steps and then followed by southeast steps ending on the x-axis forms a truncated isosceles trapezoidal section. Since the lower base lies on the x-axis and is not a part of the path, hence the term truncates.*

Slant section (SS)*: A section of the path consisting of only southeast steps which starts on the y-axis (origin not included) and ends on the x-axis.*

Height of a slant section*: It is 'd' if it starts from $(0, d)$. Clearly, an SS can only be at the beginning of the path. An associated lattice path can have at most one SS.*

Weight of a TITS*: Every TITS is represented by an ordered pair $\{u, v\}$, where u denotes its altitude and v the length of the upper base. The weight of a TITS with ordered pair $\{u, v\}$ is u.*

Weight of an associated lattice path*: It is the sum of weights of its TITSs. Note that the weight of the Slant Section is zero.*

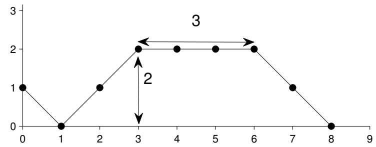


FIGURE 1. One SS of height 1 and one TITS with ordered pair $\{2, 3\}$

Example 1.8. *In Figure 1, the associated lattice path has one SS of height 1, one TITS with ordered pair* $\{2, 3\}$*, and its weight is 2.*

2. Split lattice paths and split $(n+t)$-color partitions

Now we recall the split lattice paths and describe the terminology used in these paths.

Definition 2.1 ([10])**.** *In* split lattice paths*, the length of the upper base 'v' of each TITS in an associated lattice path is split into two parts—the left part is called a 'ray' and the right part a 'segment' and their lengths are denoted by 'r' and 's' respectively, such that* $1 \le r \le v$, $0 \le s \le v-1$*, and* $v = r + s$*. An associated lattice path in which the lengths of the upper bases of all the TITSs are split into rays and segments is called a split lattice path. In a split lattice path, the ray is represented by a solid line and the segment is represented by a dotted line.*

Remark: For all types of calculations and logic, the length of the upper base 'v' is considered as a whole and not as its parts r and s, separately.

Definition 2.3 ([10])**.** *The following order is defined on the set of all TITSs of a split lattice path which firstly depends upon their weights and then on the length of their upper bases:*

If $u < w$ *then TITS with ordered pair* $\{u, v\}$ *will appear before the TITS with ordered pair* $\{w, x\}$ *and if* $u = w$ *then the TITS with ordered pair* $\{u, v\}$ *will appear before the TITS with ordered pair* $\{w, x\}$*, where* $v < x$*. Further if* $u = w$ *and* $v = x$*, then the order of these TITSs depends upon the length of the ray, that is, say* $v = r_1 + s_1$ *and* $x = r_2 + s_2$*, then TITS with ordered pair* $\{u, v\}$ *will appear before the TITS with ordered pair* $\{w, x\}$*, where* $r_1 < r_2$*. Thus, the TITSs satisfy the order:* $\{1, 1\} < \{1, 1+1\} < \{1, 2\} < \{2, 1\} < \{2, 1+1\} < \{2, 2\} < \{3, 1\} < \{3, 1+1\} < \{3, 2\} < \{3, 1+2\} < \{3, 2+1\} < \{3, 3\} < \cdots$.

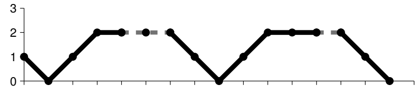


FIGURE 2. One SS of height 1 and two TITSs with same ordered pair $\{2, 3\}$.

Example 2.4. *In Figure 2, the split lattice path has one SS of height 1 and two TITSs with ordered pair* $\{2,3\}$*. Here, the weights and the lengths of the upper bases (as a whole) of both the TITSs are equal. Thus given Definition 2.3, TITS with a ray of length 1 will appear before the TITS with a ray of length 2 in the corresponding split lattice path. Here, the weight of the split lattice path is 4.*

3. Split Lattice Paths and Generalized Series

Adiga et. al [1] proved that the q-series (1.1) and (1.2) have their combinatorial counterparts in terms of split $(n+t)$-color partitions. A slight modification in the combinatorial interpretation of (1.1) leads us to the combinatorial interpretations of basic series (1.3) and (1.4) in terms of split $(n+t)$-color partitions. These results are enumerated by $P^{(m,l)}_{1_{(p,k)}}(\nu)$ and $P^{(m,t,l)}_{j_{(p,k)}}(\nu)$, $2 \leq j \leq 4$ as defined in Theorems 3.1–3.4. It is worth noting that the direct bijections between combinatorial identities (1.1) and (1.2) were not established in terms of split $(n+t)$-color partitions and R-weighted lattice paths in [1]. Our objective in this section is to extend the results of these four generalized basic series in terms of split lattice paths and provide a one-to-one correspondence between different classes of split lattice paths and split $(n+t)$-color partitions algebraically.

Theorem 3.1. *Let* $\mathcal{P}^{(m,l)}_{1_{(p,k)}}(\pi,\nu)$ *denote the set of split n-color partitions of* ν *with exactly* π *parts which satisfy the following conditions:*

1. *parts and subscripts have the same parity,*
2. *if* u_i *is the smallest or the only part in the partition, then* $u \equiv i \pmod{k}$*,*
3. *the red part of the subscript is 0 or* l*,*
4. *the green part is congruent to* $m \pmod{p}$*,*
5. *the weighted difference between any two consecutive parts is nonnegative and is congruent to* $0 \pmod{k}$*.*

Let $\mathcal{Q}^{(m,l)}_{1_{(p,k)}}(\pi,\nu)$ *denote the set of split lattice paths of weight* ν *with exactly* π *TITSs and no SS, satisfying the following conditions:*

1. *for any TITS with ordered pair* $\{u,v\}$*,* v *does not exceed* u *and* $u \equiv v \pmod{2}$*,*
2. *the path begins with a TITS with ordered pair* $\{u,v\}$*, where* $u \equiv v \pmod{k}$*,*
3. *the length of the segment is 0 or* l*,*
4. *the length of a ray is congruent to* $m \pmod{p}$*,*
5. *for any two TITSs with respective ordered pairs* $\{u_1,v_1\}$ *and* $\{u_2,v_2\}$ $(u_1 \leq u_2)$*,* $u_2 - v_2 \equiv u_1 + v_1 \pmod{k}$ *holds.*

Let $\mathcal{P}_{1_{(p,k)}}^{(m,l)}(\nu) = \bigcup_{\pi=0}^{\infty} \mathcal{P}_{1_{(p,k)}}^{(m,l)}(\pi,\nu)$ *and* $\mathcal{Q}_{1_{(p,k)}}^{(m,l)}(\nu) = \bigcup_{\pi=0}^{\infty} \mathcal{Q}_{1_{(p,k)}}^{(m,l)}(\pi,\nu)$*, then* $\sum_{\nu=0}^{\infty} P_{1_{(p,k)}}^{(m,l)}(\nu)q^{\nu} = \sum_{\nu=0}^{\infty} Q_{1_{(p,k)}}^{(m,l)}(\nu)q^{\nu} = g_1(q)$*, where* $P_{1_{(p,k)}}^{(m,l)}(\nu) = |\mathcal{P}_{1_{(p,k)}}^{(m,l)}(\nu)|$ *and* $Q_{1_{(p,k)}}^{(m,l)}(\nu) = |\mathcal{Q}_{1_{(p,k)}}^{(m,l)}(\nu)|$.

Example. $\mathcal{P}_{1_{(1,4)}}^{(2,2)}(12) = 10$*, the relevant partitions are:* $12_{12}, 12_{10+2}, 12_8, 12_{6+2}, 12_4, 12_{2+2}, 10_6 + 2_2, 10_{4+2} + 2_2, 10_2 + 2_2, 9_3 + 3_3$.

Theorem 3.2. *Let* $\mathcal{P}_{2_{(p,k)}}^{(m,t,l)}(\pi,\nu)$ *denote the set of split* $(n+t)$*-color partitions of* ν *with exactly* π *parts which satisfy the following conditions:*

1. *the parts and their subscripts have the same parity if* t *is even, otherwise the parity is opposite,*
2. *the red part of the subscripts is 0 or* l*,*
3. *the green part is congruent to* $m \pmod p$ *and it is greater than or equal to* m*,*
4. *the smallest part is of the form* u_{u+t} *and* $u \equiv 0 \pmod p$*,*
5. *the red part of the subscript of the smallest part is 0,*
6. *the weighted difference between any two consecutive parts is nonnegative and is congruent to* $0 \pmod k$.

Let $\mathcal{Q}_{2_{(p,k)}}^{(m,t,l)}(\pi,\nu)$ *denote the set of split lattice paths of weight* ν *with exactly* π *TITSs satisfying the following conditions:*

1. *for any TITS with ordered pair* $\{u,v\}$*,* v *does not exceed* $u+t$ *and* $u \equiv v \pmod 2$ *if* t *is even, otherwise* $u \not\equiv v \pmod 2$*,*
2. *the length of the segment is 0 or* l*,*
3. *the length of ray is congruent to* $m \pmod p$*,*
4. *there is an SS of height* t *or a TITS with ordered pair* $\{u, u+t\}$ *and* $u \equiv 0 \pmod p$*,*
5. *the length of the segment of the first TITS is 0,*
6. *for any two TITSs with respective ordered pairs* $\{u_1, v_1\}$ *and* $\{u_2, v_2\}$ $(u_1 \le u_2)$*,* $u_2 - v_2 \equiv u_1 + v_1 \pmod k$ *holds.*

Let $\mathcal{P}_{2_{(p,k)}}^{(m,t,l)}(\nu) = \bigcup_{\pi=0}^{\infty} \mathcal{P}_{2_{(p,k)}}^{(m,t,l)}(\pi,\nu)$ *and* $\mathcal{Q}_{2_{(p,k)}}^{(m,t,l)}(\nu) = \bigcup_{\pi=0}^{\infty} \mathcal{Q}_{2_{(p,k)}}^{(m,t,l)}(\pi,\nu)$*, then* $\sum_{\nu=0}^{\infty} P_{2_{(p,k)}}^{(m,t,l)}(\nu)q^{\nu} = \sum_{\nu=0}^{\infty} Q_{2_{(p,k)}}^{(m,t,l)}(\nu)q^{\nu} = g_2(q)$*, where* $P_{2_{(p,k)}}^{(m,t,l)}(\nu) = |\mathcal{P}_{2_{(p,k)}}^{(m,t,l)}(\nu)|$ *and* $Q_{2_{(p,k)}}^{(m,t,l)}(\nu) = |\mathcal{Q}_{2_{(p,k)}}^{(m,t,l)}(\nu)|$.

Example. $\mathcal{P}_{2_{(2,4)}}^{(2,1,1)}(13) = 7$*, the relevant partitions are:* $13_{12} + 0_1, 13_8 + 0_1, 13_4 + 0_1, 10_{4+1} + 3_2 + 0_1, 9_2 + 4_{2+1} + 0_1, 11_6 + 2_3, 11_2 + 2_3$.

Theorem 3.3. *Let* $\mathcal{P}_{3_{(p,k)}}^{(m,t,l)}(\pi,\nu)$ *denote the set of split* n*-color partitions of* ν *with exactly* π *parts which satisfy the following conditions:*

1. *if* u_i *is the smallest or the only part in the partition, then* $u \equiv i + t \pmod k$ *and* $u \ge i + t$*,*
2. *the parts and subscripts have the same parity if* t *is even, otherwise the parity is opposite.*

(3) the red part of the subscript is 0 or l,
(4) the green part is greater than or equal to m and it is congruent to $m \pmod p$,
(5) the weighted difference between any two consecutive parts is nonnegative and is congruent to $0 \pmod k$,
(6) all parts are greater than or equal to $m+t$.

Let $\mathcal{Q}_{3_{(p,k)}}^{(m,t,l)}(\pi,\nu)$ denote the set of split lattice paths of weight ν with exactly π TITSs and no SS, satisfying the following conditions:

(1) the path begins with a TITS with ordered pair $\{u,v\}$ where $u \equiv v+t \pmod k$,
(2) for any TITS with ordered pair $\{u,v\}$, v does not exceed u and $u \equiv v (mod\ 2)$ if t is even, otherwise $u \not\equiv v \pmod 2$,
(3) the length of the segment is 0 or l,
(4) the length of ray is congruent to $m \pmod p$,
(5) for any two TITSs with respective ordered pairs $\{u_1,v_1\}$ and $\{u_2,v_2\}$ $(u_1 \le u_2)$, $u_2 - v_2 \equiv u_1 + v_1 \pmod k$ holds,
(6) the altitudes of all TITSs are greater than or equal to $m+t$.

Let $\mathcal{P}_{3_{(p,k)}}^{(m,t,l)}(\nu) = \bigcup_{\pi=0}^{\infty} \mathcal{P}_{3_{(p,k)}}^{(m,t,l)}(\pi,\nu)$ and $\mathcal{Q}_{3_{(p,k)}}^{(m,t,l)}(\nu) = \bigcup_{\pi=0}^{\infty} \mathcal{Q}_{3_{(p,k)}}^{(m,t,l)}(\pi,\nu)$, then $\sum_{\nu=0}^{\infty} P_{3_{(p,k)}}^{(m,t,l)}(\nu)q^\nu = \sum_{\nu=0}^{\infty} Q_{3_{(p,k)}}^{(m,l,t)}(\nu)q^\nu = g_3(q)$, where $P_{3_{(p,k)}}^{(m,t,l)}(\nu) = |\mathcal{P}_{3_{(p,k)}}^{(m,t,l)}(\nu)|$ and $Q_{3_{(p,k)}}^{(m,t,l)}(\nu) = |\mathcal{Q}_{3_{(p,k)}}^{(m,t,l)}(\nu)|$.

Example. $\mathcal{P}_{3_{(1,4)}}^{(1,3,1)}(11) = 6$, *the relevant partitions are:* 11_8, 11_{7+1}, 11_4, 11_{3+1}, $7_2 + 4_1$, $7_{1+1} + 4_1$.

Theorem 3.4. *Let $\mathcal{P}_{4_{(p,k)}}^{(m,t,l)}(\pi,\nu)$ denote the set of split n-color partitions of ν with exactly π parts which satisfy the following conditions:*

(1) if u_i is the smallest or the only part in the partition, then $u \equiv i \pmod k$,
(2) the parts and their subscripts have same parity,
(3) the red part of the subscript is 0 or l,
(4) the green part is greater than or equal to $m-t$ and it is congruent to $(m-t) \pmod p$,
(5) the weighted difference between any two consecutive parts is greater than or equal to $2t$ and is congruent to $2t \pmod k$.

Let $\mathcal{Q}_{4_{(p,k)}}^{(m,t,l)}(\pi,\nu)$ denote the set of split lattice paths of weight ν with exactly π TITSs and no SS, satisfying the following conditions:

(1) the path begins with a TITS with ordered pair $\{u,v\}$ where $u \equiv v \pmod k$,
(2) for any TITS with ordered pair $\{u,v\}$, v does not exceed u and $u \equiv \pmod 2$,
(3) the length of the segment is 0 or l,
(4) the length of ray is $\ge m-t$ and it is congruent to $(m-t) \pmod p$,

(5) *for any two TITSs with respective ordered pairs* $\{u_1, v_1\}$ *and* $\{u_2, v_2\}$ $(u_1 \le u_2)$, $u_2 - v_2 \equiv u_1 + v_1 + 2t \pmod{k}$ *holds.*

Let $\mathcal{P}^{(m,t,l)}_{4_{(p,k)}}(\nu) = \bigcup_{\pi=0}^{\infty} \mathcal{P}^{(m,t,l)}_{4_{(p,k)}}(\pi, \nu)$ *and* $\mathcal{Q}^{(m,t,l)}_{4_{(p,k)}}(\nu) = \bigcup_{\pi=0}^{\infty} \mathcal{Q}^{(m,t,l)}_{4_{(p,k)}}(\pi, \nu)$, *then* $\sum_{\nu=0}^{\infty} P^{(m,t,l)}_{4_{(p,k)}}(\nu) q^\nu = \sum_{\nu=0}^{\infty} Q^{(m,t,l)}_{4_{(p,k)}}(\nu) q^\nu = g_4(q)$, *where* $P^{(m,t,l)}_{4_{(p,k)}}(\nu) = |\mathcal{P}^{(m,t,l)}_{4_{(p,k)}}(\nu)|$ *and* $Q^{(m,t,l)}_{4_{(p,k)}}(\nu) = |\mathcal{Q}^{(m,t,l)}_{4_{(p,k)}}(\nu)|$.

Example. $\mathcal{P}^{(5,3,1)}_{4_{(2,6)}}(20) = 8$, *the relevant partitions are:* $20_{20}, 20_{14}, 20_8, 20_2, 18_8 + 2_2, 18_2 + 2_2, 17_{4+1} + 3_{2+1}, 16_2 + 4_4$.

Notation. In Theorems 3.1–3.4, the notation $|S|$ denotes the cardinality of the set S.

3.1. **Proof of Theorem 3.1.** We will show that the infinite summation series $g_1(q)$ generates the split lattice paths enumerated by $Q^{(m,l)}_{1_{(p,k)}}(\pi, \nu)$ constructively. For this, we define a set of triplets (ς, ξ, σ) denoted by $\mathcal{R}^{(m,l)}_{1_{(p,k)}}(\pi, \nu)$ where $\varsigma = (\varsigma_1, \varsigma_2, \dots, \varsigma_\pi), \xi = (\xi_1, \xi_2, \dots, \xi_\pi)$, and $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_\pi)$ are π tuples of integers and satisfy:

(i) $\varsigma_i \in \mathbb{Z}_{\ge 0} = \{0, 1, 2, 3, \dots\}$, for $1 \le i \le \pi$,
(ii) $\xi_i \in \mathbb{Z}_{\ge 0} = \{0, 1, 2, 3, \dots\}$, for $1 \le i \le \pi$,
(iii) $\sigma_i = 0$ or 1, for $1 \le i \le \pi$,
(iv) if the weight of the triplet (ς, ξ, σ) is defined by

$$w(\varsigma, \xi, \sigma) = m\pi^2 + \sum_{i=1}^{\pi} i\varsigma_i k + \sum_{i=1}^{\pi} (2i-1)\xi_i p + \sum_{i=1}^{\pi} (2i-1)\sigma_i l,$$

then (ς, ξ, σ) should satisfy $w(\varsigma, \xi, \sigma) = \nu$.

We can easily establish that the generating function for $\mathcal{R}^{(m,l)}_{1_{(p,k)}}(\pi, \nu)$ is

$$\sum_{\nu=0}^{\infty} R^{(m,l)}_{1_{(p,k)}}(\pi, \nu) q^\nu = \sum_{\pi=0}^{\infty} \frac{q^{m\pi^2}(-q^l; q^{2l})_\pi}{(q^p; q^{2p})_\pi (q^k; q^k)_\pi},$$

where $R^{(m,l)}_{1_{(p,k)}}(\pi, \nu) = |\mathcal{R}^{(m,l)}_{1_{(p,k)}}(\pi, \nu)|$. In the second and third steps, we will prove that there is a bijection between the set of split lattice paths enumerated by $\mathcal{Q}^{(m,l)}_{1_{(p,k)}}(\pi, \nu)$ and set of triplets enumerated by $\mathcal{R}^{(m,l)}_{1_{(p,k)}}(\pi, \nu)$.

Proof. In the first step, we shall prove that

$$\sum_{\nu=0}^{\infty} Q^{(m,l)}_{1_{(p,k)}}(\nu) q^\nu = \sum_{\pi=0}^{\infty} \frac{q^{m\pi^2}(-q^l; q^{2l})_\pi}{(q^p; q^{2p})_\pi (q^k; q^k)_\pi}. \tag{3.1}$$

In $\frac{q^{m\pi^2}(-q^l; q^{2l})_\pi}{(q^p; q^{2p})_\pi (q^k; q^k)_\pi}$, the factor $q^{m\pi^2}$ generates a split lattice path with π TITSs such that ith TITS has ordered pair $\{(2i-1)m, m\}$. For $\pi = 3$ and $m = 1$, the path begins as:

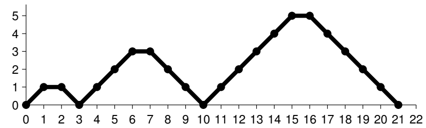


FIGURE 3. Split lattice path for $\pi = 3$ and $m = 1$

In Figure 3, we consider two successive TITSs, say, ith and $(i+1)$th with corresponding ordered pairs $\{(2i-1)m, m\}$ and $\{(2i+1)m, m\}$, respectively. The factor $1/(q^k; q^k)_\pi$ generates π nonnegative multiples of k say $\varsigma_1 \geq \varsigma_2 \geq \cdots \geq \varsigma_\pi \geq 0$, which is encoded by increasing the altitude of ith TITS by $\varsigma_{\pi-i+1}$, $1 \leq i \leq \pi$. Thus the ordered pair associated with ith TITS becomes $\{(2i-1)m + \varsigma_{\pi-i+1}, m\}$. Consider the ith and $(i+1)$st TITSs as shown in Figure 4.

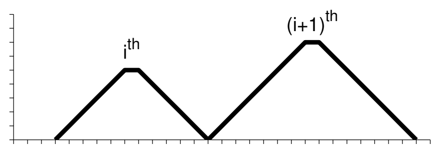


FIGURE 4. ith and $(i+1)$th TITSs

The factor $1/(q^p; q^{2p})_\pi$ generates π nonnegative multiples of $(2i-1)\,p$ say $\xi_1 \times p \geq \xi_2 \times 3p \geq \cdots \geq \xi_\pi \times (2\pi-1)p \geq 0$, which is encoded by increasing the altitude of the ith TITS by $2p(\xi_\pi + \xi_{\pi-1} + \cdots + \xi_{\pi-i+2}) + \xi_{\pi-i+1}p$ and length of ray by $\xi_{\pi-i+1}p$. Thus, the ordered pair associated with ith TITS becomes $\{(2i-1)m + \varsigma_{\pi-i+1} + 2p(\xi_\pi + \xi_{\pi-1} + \cdots + \xi_{\pi-i+2}) + \xi_{\pi-i+1}p, m + \xi_{\pi-i+1}p\}$. Now Figure 4 changes to Figure 5.

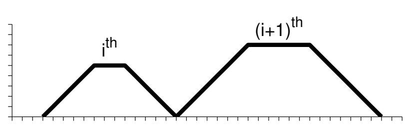


FIGURE 5. ith and $(i+1)$th TITSs

The factor $(-q^l; q^{2l})_\pi$ generates distinct nonnegative multiples of $(2i-1)\,l$ say $\sigma_1 \times l \geq \sigma_2 \times 3l \geq \cdots \geq \sigma_\pi \times (2\pi-1)l \geq 0$, which is encoded by increasing the altitude of the ith TITS by $2l(\sigma_\pi + \sigma_{\pi-1} + \cdots + \sigma_{\pi-i+2}) + \sigma_{\pi-i+1}l$ and by putting a segment of length $\sigma_{\pi-i+1}l$ adjacent to the ray. Figure 5 now changes to Figure 6.

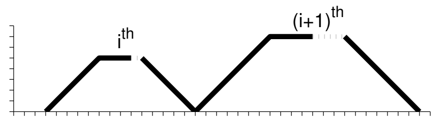


FIGURE 6. ith and $(i+1)$th TITSs

Every split lattice path enumerated by $Q^{(m,l)}_{1_{(p,k)}}(\pi,\nu)$ is uniquely generated in this manner.

3.2. **Bijections between certain restricted classes of split** $(n+t)$**-color partitions and split lattice paths.** We see that there exists a well-defined map between the triplets $(\varsigma,\xi,\sigma)=\Big((\varsigma_i)^{\pi}_{i=1},(\xi_i)^{\pi}_{i=1},(\sigma_i)^{\pi}_{i=1}\Big)$of the set $\mathcal{R}^{(m,l)}_{1_{(p,k)}}(\pi,\nu)$ and the π TITSs represented by the ordered pairs $\{u_i,v_i\}^{\pi}_{i=1}=\{u_i,r_i+s_i\}^{\pi}_{i=1}$ of the set $\mathcal{Q}^{(m,l)}_{1_{(p,k)}}(\pi,\nu)$, say,

$$\varphi_1:\mathcal{Q}^{(m,l)}_{1_{(p,k)}}(\pi,\nu)\to\mathcal{R}^{(m,l)}_{1_{(p,k)}}(\pi,\nu)$$

defined by

$$\varphi_1\Big((\varsigma_i)^{\pi}_{i=1},(\xi_i)^{\pi}_{i=1},(\sigma_i)^{\pi}_{i=1}\Big)=\{u_{\pi-i+1},r_{\pi-i+1}+s_{\pi-i+1}\}^{\pi}_{i=1}\,,$$

such that $\forall\ 1\le i\le\pi$

$$\varphi_1:\begin{cases}\xi_i p=r_{\pi-i+1}-m\\ \sigma_i l=s_{\pi-i+1}\end{cases}\tag{3.2}$$

and $u_{\pi-i+1}=(2(\pi-i+1)-1)m+\varsigma_i+2p(\xi_\pi+\xi_{\pi-1}+\cdots+\xi_{i+1})+\xi_i p\ +2l(\sigma_\pi+\sigma_{\pi-1}+\cdots+\sigma_{i+1})+\sigma_i l$.

In the second step we now show that $\varphi_1(\varsigma,\xi,\sigma)$ is actually an element of $\mathcal{Q}^{(m,l)}_{1_{(p,k)}}(\pi,\nu)$. Let us denote the ith and $(i+1)$th TITSs by $\{u_i,v_i\}$ and $\{u_{i+1},v_{i+1}\}$ respectively. Then

$$\begin{aligned}u_i&=(2i-1)m+\varsigma_{\pi-i+1}+2p(\xi_\pi+\xi_{\pi-1}+\cdots+\xi_{\pi-i+2})+\xi_{\pi-i+1}p+\\&\quad+2l(\sigma_\pi+\sigma_{\pi-1}+\cdots+\sigma_{\pi-i+2})+\sigma_{\pi-i+1}l\\ v_i&=m+\xi_{\pi-i+1}p+\sigma_{\pi-i+1}l\\ u_{i+1}&=(2i+1)m+\varsigma_{\pi-i+1}+2p(\xi_\pi+\xi_{\pi-1}+\cdots+\xi_{\pi-i+1})+\xi_{\pi-i}p+\\&\quad 2l(\sigma_\pi+\sigma_{\pi-1}+\cdots+\sigma_{\pi-i+1})+\sigma_{\pi-i}l\\ v_{i+1}&=m+\xi_{\pi-i}p+\sigma_{\pi-i}l.\end{aligned}$$

Clearly, $v_i\le u_i$ and the parity of u_i and v_i both depends upon $m+\xi_{\pi-i+1}p+\sigma_{\pi-i+1}l$. If $m+\xi_{\pi-i+1}p+\sigma_{\pi-i+1}l$ is even then both u_i and v_i are even and vice-versa. This confirms that for any TITS with ordered pair $\{u,v\}$, v does not exceed u and $u\equiv v \pmod 2$. If u_i denotes the first TITS in the split lattice path then it will correspond to the smallest part or the singleton part in the corresponding split n-color partition and $u-i=\varsigma_\pi\equiv 0 \pmod k$.

Since $r_i = m+\xi_{\pi-i+1}p$, therefore the green part of the split n-color partition is congruent to $m \pmod p$. Also, we know that from condition (iii) on the triplets of the set $\mathcal{R}_{1_{(p,k)}}^{(m,l)}(\pi,\nu)$, $\sigma_i = 0$ or 1 for all $1 \le i \le \pi$. Thus the length of the segment denoted by $s_i = \sigma_{\pi-i+1}l$ is either 0 or l in the split lattice path enumerated by $\mathcal{Q}_{1_{(p,k)}}^{(m,l)}(\pi,\nu)$. Now, we consider $u_{i+1} - v_{i+1} = u_i + v_i + \varsigma_{\pi-i} - \varsigma_{\pi-i+1}$ which implies that weighted difference between any two consecutive parts is congruent to $0 \pmod k$.

In the third step, we show that φ_1 is injective as well as surjective.

(1) φ_1 is injective:

Suppose $\Big((\varsigma_i)_{i=1}^{\pi}, (\xi_i)_{i=1}^{\pi}, (\sigma_i)_{i=1}^{\pi}\Big)$ and $\Big((\alpha_i)_{i=1}^{\pi}, (\beta_i)_{i=1}^{\pi}, (\gamma_i)_{i=1}^{\pi}\Big)$ are any two triplets in the set $\mathcal{R}_{1_{(p,k)}}^{(m,l)}(\pi,\nu)$ such that for all $1 \le i \le \pi$:

$$\varphi_1\Big((\varsigma_i)_{i=1}^{\pi}, (\xi_i)_{i=1}^{\pi}, (\sigma_i)_{i=1}^{\pi}\Big) = \varphi_1\Big((\alpha_i)_{i=1}^{\pi}, (\beta_i)_{i=1}^{\pi}, (\gamma_i)_{i=1}^{\pi}\Big)$$

implies

$$\{u_{\pi-i+1}, r_{\pi-i+1} + s_{\pi-i+1}\} = \{u'_{\pi-i+1}, r'_{\pi-i+1} + s'_{\pi-i+1}\} \tag{3.3}$$

We know from Definition 2.3 that two TITSs are equal if their weights as well as the lengths of their rays and segments are equal. Thus from (3.3), we have

$$\begin{array}{lllll} & r_{\pi-i+1} = r'_{\pi-i+1} & \text{and} & s_{\pi-i+1} = s'_{\pi-i+1} & \\ \Rightarrow & m+\xi_i p = m+\beta_i p & \text{and} & \sigma_i l = \gamma_i l & \text{using (3.2)} \\ \Rightarrow & \xi_i = \beta_i & \text{and} & \sigma_i = \gamma_i & \text{for } 1 \le i \le \pi \end{array}$$

$$\Rightarrow \varsigma_i = \alpha_i \text{ for } 1 \le i \le \pi$$

$$\Rightarrow \Big((\varsigma_i)_{i=1}^{\pi}, (\xi_i)_{i=1}^{\pi}, (\sigma_i)_{i=1}^{\pi}\Big) = \Big((\alpha_i)_{i=1}^{\pi}, (\beta_i)_{i=1}^{\pi}, (\gamma_i)_{i=1}^{\pi}\Big),$$

hence φ_1 is injective.

(2) φ_1 is surjective:

Consider a TITS with ordered pair $\{u_i, v_i\}$ for $1 \le i \le \pi$ of the split lattice path enumerated by $Q_{1_{(p,k)}}^{(m,l)}(\pi,\nu)$, where $v_i = r_i + s_i$ for $1 \le i \le \pi$. Now by the conditions of Theorem 3.1 on the split lattice paths in the set $\mathcal{Q}_{1_{(p,k)}}^{(m,l)}(\pi,\nu)$, we know that every $r_i \ge m$ and $s_i = 0$ or l for $1 \le i \le \pi$. So, by definition of the map φ_1 we have $\xi_i p = r_{\pi-i+1} - m \ge 0$ and $\sigma_i l = s_{\pi-i+1} = 0$ or l for $1 \le i \le \pi$. Thus $\Big((\varsigma_i)_{i=1}^{\pi}, (\xi_i)_{i=1}^{\pi}, (\sigma_i)_{i=1}^{\pi}\Big)$ is a triplet in the set $\mathcal{R}_{1_{(p,k)}}^{(m,l)}(\pi,\nu)$. Hence φ_1 is surjective and the inverse mapping is:

$$(\varphi_1)^{-1} : \begin{cases} r_i = \xi_{\pi-i+1}p + m, & \\ s_i = \sigma_{\pi-i+1}l, & 1 \le i \le \pi \end{cases}$$

and $u_i = (2i-1)m + \varsigma_{\pi-i+1} + 2p(\xi_\pi + \xi_{\pi-1} + \cdots + \xi_{\pi-i+2}) + \xi_{\pi-i+1}p + 2l(\sigma_\pi + \sigma_{\pi-1} + \cdots + \sigma_{\pi-i+2}) + \sigma_{\pi-i+1}l$.

This completes the bijection. □

3.3. **Outline of the proofs of theorems 3.2–3.4.**

Proof of Theorem 3.2. The proof is treated in the same manner as Theorem 3.1. Here, the only difference is that there are two extra factors namely $q^{t\pi}$ and $(1-q^{(2\pi+1)p})^{-1}$. The factor $q^{t\pi}$ puts t southeast steps: $(0,t),(1,t-1),\dots,(t-1,1),(t,0)$ thus there are now $\pi+1$ TITSs and the path begins with an SS of height t or a TITS with ordered pair $\{u,u+t\}$. Therefore, we define the set of triplets (ς,ξ,σ) denoted by $\mathcal{R}^{(m,t,l)}_{2_{(p,k)}}(\pi,\nu)$ where $\varsigma=(\varsigma_1,\varsigma_2,\dots,\varsigma_\pi)$ and $\sigma=(\sigma_1,\sigma_2,\dots,\sigma_\pi)$ are π-tuples of integers and $\xi=(\xi_1,\xi_2,\dots,\xi_{\pi+1})$ are $(\pi+1)$-tuples of integers and satisfy:

(i) $\varsigma_i\in\mathbb{Z}_{\geq 0}=\{0,1,2,3,\dots\}$, for $1\leq i\leq\pi$,
(ii) $\xi_i\in\mathbb{Z}_{\geq 0}=\{0,1,2,3,\dots\}$, for $1\leq i\leq\pi+1$,
(iii) $\sigma_i=0$ or 1, for $1\leq i\leq\pi$,
(iv) if the weight of the triplet (ς,ξ,σ) is defined by

$$w(\varsigma,\xi,\sigma)=m\pi^2+t\pi+\sum_{i=1}^{\pi}i\varsigma_i k+\sum_{i=1}^{\pi+1}(2i-1)p\xi_i+\sum_{i=1}^{\pi}(2i-1)l\sigma_i,$$

then (ς,ξ,σ) should satisfy $w(\varsigma,\xi,\sigma)=\nu$.

Clearly, the generating function for $\mathcal{R}^{(m,t,l)}_{2_{(p,k)}}(\pi,\nu)$ will be

$$\sum_{\nu=0}^{\infty}\mathcal{R}^{(m,t,l)}_{2_{(p,k)}}(\pi,\nu)q^\nu=\sum_{\pi=0}^{\infty}\frac{q^{m\pi^2+t\pi}(-q^l;q^{2l})_\pi}{(q^p;q^{2p})_{\pi+1}(q^k;q^k)_\pi}.$$

We will consider two cases to establish bijection between the triplets of the set $\mathcal{R}^{(m,t,l)}_{2_{(p,k)}}(\pi,\nu)$ and the set of split lattice paths $\mathcal{Q}^{(m,t,l)}_{2_{(p,k)}}(\pi,\nu)$.

Case I When $\xi_{\pi+1}=0$, then the factor $\frac{q^{t\pi}}{(1-q^{(2\pi+1)p})}$ makes an SS of height t at the beginning of the path which clearly corresponds to the part 0_t in the split $(n+t)$-color partition.

Case II When $\xi_{\pi+1}\neq 0$, then the factor $\frac{q^{t\pi}}{(1-q^{(2\pi+1)p})}$ makes the split lattice path to begin with a TITS with ordered pair $\{\xi_{\pi+1},t+\xi_{\pi+1}\}$ which corresponds to the part u_{u+t} in the split $(n+t)$-color partition.

In both cases, we define the map

$$\varphi_2:\mathcal{R}^{(m,t,l)}_{2_{(p,k)}}(\pi,\nu)\to\mathcal{Q}^{(m,t,l)}_{2_{(p,k)}}(\pi,\nu)\text{ by :}$$

$$\varphi_2:\begin{cases}\xi_i p=r_{\pi-i+2}-m\\ \sigma_i l=s_{\pi-i+2}.\end{cases}\tag{3.4}$$

Proof of Theorem 3.3. The proof is treated in the same manner as the proof of Theorem 3.1 except for the fact that here the extra factor $q^{t\pi}$ ensures that the first TITS in the split lattice path corresponds to the smallest part in the split n-color partition of the form $u-i \equiv t \pmod{k}$. This restricts the part size and changes the parity of the parts and the subscripts depending upon the value of t.

Proof of Theorem 3.4. The proof is treated in the same manner as the proof of Theorem 3.1 except for the fact that the extra factor $q^{-t\pi}$ decreases the length of altitudes and upper bases of all the TITSs by t due to which the weighted difference between any two consecutive split n-color parts is congruent to $2t \pmod{k}$.

4. Particular cases leading to Rogers–Ramanujan Type Identities

For some particular values of m, t, l, p, and k, identities (1.1)–(1.4) lead us to the following Rogers–Ramanujan type identities found in [8].

$$\begin{aligned}\sum_{\pi=0}^{\infty} \frac{q^{\pi^2}(-q;q^2)_\pi}{(q;q^2)_\pi (q^4;q^4)_\pi} \\ = \prod_{n=1}^{\infty} \frac{(1+q^{2n-1})(1+q^{10n-2})(1+q^{10n-5})(1+q^{10n-8})}{(1-q^{2n})(1-q^{10n})^{-1}(1-q^{10n-3})^{-1}(1-q^{10n-7})^{-1}},\end{aligned} \tag{4.1}$$

$$\begin{aligned}\sum_{\pi=0}^{\infty} \frac{q^{\pi^2+2\pi}(-q;q^2)_\pi}{(q^2;q^4)_\pi (q^4;q^4)_\pi} \\ = \prod_{n=1}^{\infty} \frac{(1+q^{2n-1})(1-q^{28n-16})(1-q^{28n-12})}{(1-q^{2n})(1-q^{14n})^{-1}(1-q^{14n-1})^{-1}(1-q^{14n-13})^{-1}},\end{aligned} \tag{4.2}$$

$$\begin{aligned}\sum_{\pi=0}^{\infty} \frac{q^{3\pi^2-2\pi}(-q;q^2)_\pi}{(q^2;q^4)_\pi (q^4;q^4)_\pi} \\ = \prod_{n=1}^{\infty} \frac{(1+q^{2n-1})(1-q^{10n})(1-q^{10n-3})(1-q^{10n-7})}{(1-q^{2n})(1-q^{20n-16})^{-1}(1-q^{20n-4})^{-1}},\end{aligned} \tag{4.3}$$

$$\begin{aligned}\sum_{\pi=0}^{\infty} \frac{q^{\pi^2}(-q^3;q^6)_\pi}{(q^2;q^4)_\pi (q^4;q^4)_\pi} \\ = \prod_{n=1}^{\infty} \frac{(1+q^{2n-1})(1-q^{12n})(1-q^{12n-2})(1-q^{12n-10})}{(1-q^{2n})(1-q^{24n-16})^{-1}(1-q^{24n-8})^{-1}},\end{aligned} \tag{4.4}$$

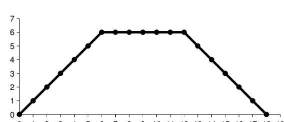
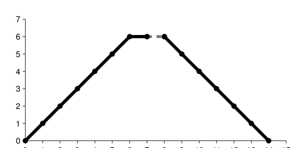
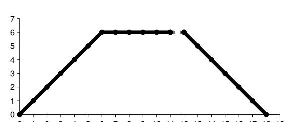
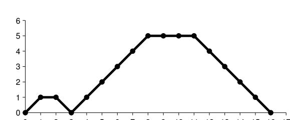
$$\begin{aligned}(4.5)\qquad &\sum_{\pi=0}^{\infty}\frac{q^{\pi^2+2\pi}(-q;q^2)_\pi}{(q;q^2)_{\pi+1}(q^4;q^4)_\pi}\\ &\qquad=\prod_{n=1}^{\infty}\frac{(1+q^{2n-1})(1-q^{20n})(1-q^{20n-5})(1-q^{20n-15})}{(1-q^{2n})}.\end{aligned}$$

Now, Theorems 3.1–3.4 enable us to provide the following combinatorial interpretation of the identities (4.1)–(4.5) in terms of split $(n+t)$-color partitions, split lattice paths, and ordinary partitions, respectively:

Theorem 4.1. *Let* $\mathcal{X}_1(\nu)=\sum_{l=0}^{\nu}\mathcal{S}_1(\nu-l)\mathcal{T}_1(\nu)$*, where* $\mathcal{S}_1(\nu)$ *denotes the number of partitions of* ν *into parts congruent to* $\pm2,\pm4,\pm8 \pmod{20}$ *and* $\mathcal{T}_1(\nu)$ *denotes the number of partitions of* ν *into distinct parts congruent to* $\pm1,\pm2,5 \pmod{10}$ *where the parts congruent to* $5 \pmod{10}$ *are counted twice. Then* $\mathcal{P}^{(1,1)}_{1_{(1,4)}}(\nu)=\mathcal{Q}^{(1,1)}_{1_{(1,4)}}(\nu)=\mathcal{X}_1(\nu)$, $\forall\ \nu\geq0$*, where* $\mathcal{P}^{(1,1)}_{1_{(1,4)}}(\nu)$ *and* $\mathcal{Q}^{(1,1)}_{1_{(1,4)}}(\nu)$ *are defined as in Theorem 3.1.*

Example. $\mathcal{P}^{(1,1)}_{1_{(1,4)}}(6)=6$*, since the relevant partitions are:* 6_6, 6_{5+1}, 6_2, 6_{1+1}, 5_3+1_1, $5_{2+1}+1_1$. *Also,* $\mathcal{X}_1(6)=\sum_{a=0}^{6}\mathcal{S}_1(6-a)\mathcal{T}_1(6)=\mathcal{S}_1(6)\mathcal{T}_1(0)+\mathcal{S}(5)\mathcal{T}_1(1)+\cdots+\mathcal{S}_1(0)\mathcal{T}_1(6)=2(1)+0(1)+2(1)+0(2)+1(0)+0(2)+1(2)=6.$

Table 1 represents the relevant partitions enumerated by $\mathcal{P}^{(1,1)}_{1_{(1,4)}}$ and the corresponding split lattice paths enumerated by $\mathcal{Q}^{(1,1)}_{1_{(1,4)}}$ for $\nu=6$.

split $n-$color partitions	split lattice paths	split $n-$color partitions	split lattice paths
6_6		6_{1+1}	
6_{5+1}		5_3+1_1	

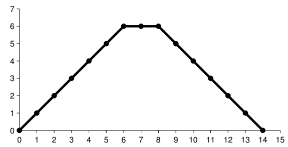
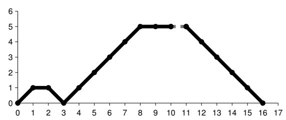
6_2		$5_{2+1}+1_1$	

TABLE 1. Split n-color partitions enumerated by $\mathcal{P}^{(1,1)}_{1_{(1,4)}}$ and split lattice paths enumerated by $\mathcal{Q}^{(1,1)}_{1_{(1,4)}}$ for $\nu = 6$

Theorem 4.2. *Let* $\mathcal{X}_2(\nu) = \sum_{l=0}^{\nu} \mathcal{S}_2(\nu - l)\mathcal{T}_2(\nu)$, *where* $\mathcal{S}_2(\nu)$ *denotes the number of partitions of* ν *into parts congruent to* $0, \pm 4, \pm 6, \pm 8, \pm 10, 14 \pmod{28}$ *and* $\mathcal{T}_2(\nu)$ *denotes the number of partitions of* ν *into distinct parts congruent to* $\pm 3, \pm 5, 7 \pmod{14}$. *Then* $\mathcal{P}^{(1,2,1)}_{3_{(2,4)}}(\nu) = \mathcal{Q}^{(1,2,1)}_{3_{(2,4)}}(\nu) = \mathcal{X}_2(\nu)$ *for all* $\nu \geq 0$, *where* $\mathcal{P}^{(1,2,1)}_{3_{(2,4)}}(\nu)$ *and* $\mathcal{Q}^{(1,2,1)}_{3_{(2,4)}}(\nu)$ *are defined as in Theorem 3.3.*

Example. $\mathcal{P}^{(1,2,1)}_{3_{(2,4)}}(6) = 1$, *since the relevant partition is:* 6_{3+1}. *Also,* $\mathcal{X}_2(6) = \sum_{a=0}^{6} \mathcal{S}_2(6-a)\mathcal{T}_2(a) = \mathcal{S}_2(6)\mathcal{T}_2(0) + \mathcal{S}_2(5)\mathcal{T}_2(1) + \cdots + \mathcal{S}_2(0)\mathcal{T}_2(6) = 1(1) + 0(0) + 1(0) + 0(1) + 0(0) + 0(1) + 1(0) = 1.$

Theorem 4.3. *Let* $\mathcal{X}_3(\nu) = \sum_{l=0}^{\nu} \mathcal{S}_3(\nu - l)\mathcal{T}_3(\nu)$, *where* $\mathcal{S}_3(\nu)$ *denotes the number of partitions of* ν *into parts congruent to* $\pm 2, \pm 8 \pmod{20}$ *and* $\mathcal{T}_3(\nu)$ *denotes the number of partitions of* ν *into distinct parts congruent to* $\pm 1, 5 \pmod{10}$. *Then* $\mathcal{P}^{(3,2,1)}_{4_{(2,4)}}(\nu) = \mathcal{Q}^{(3,2,1)}_{4_{(2,4)}}(\nu) = \mathcal{X}_3(\nu)$ *for all* $\nu \geq 0$, *where* $\mathcal{P}^{(3,2,1)}_{4_{(2,4)}}(\nu)$ *and* $\mathcal{Q}^{(3,2,1)}_{4_{(2,4)}}(\nu)$ *are defined as in Theorem 3.4.*

Example. $\mathcal{P}^{(3,2,1)}_{3_{(2,4)}}(6) = 2$, *since the relevant partitions are:* 6_{5+1}, 6_{1+1}. *Also,* $\mathcal{X}_3(6) = \sum_{a=0}^{6} \mathcal{S}_3(6-a)\mathcal{T}_3(a) = \mathcal{S}_3(6)\mathcal{T}_3(0) + \mathcal{S}_3(5)\mathcal{T}_3(1) + \cdots + \mathcal{S}_3(0)\mathcal{T}_3(6) = 1(1) + 0(1) + 1(0) + 0(0) + 1(0) + 0(1) + 1(1) = 2.$

Theorem 4.4. *Let* $\mathcal{X}_4(\nu)$ *denote the number of partitions of* ν *such that odd parts are distinct and even parts are congruent to* $\pm 4, \pm 6 \pmod{24}$. *Then* $\mathcal{P}^{(1,3)}_{1_{(2,4)}}(\nu) = \mathcal{Q}^{(1,3)}_{1_{(2,4)}}(\nu) = \mathcal{X}_4(\nu)$ *for all* $\nu \geq 0$, *where* $\mathcal{P}^{(1,3)}_{1_{(2,4)}}(\nu)$ *and* $\mathcal{Q}^{(1,3)}_{1_{(2,4)}}(\nu)$ *are defined as in Theorem 3.1.*

Example. $\mathcal{P}^{(1,3)}_{1_{(2,4)}}(6) = 2$, *since the relevant partitions are:* 6_{3+3}, $5_3 + 1_1$. *Also,* $\mathcal{X}_4(6) = 2$, *the relevant ordinary partitions are:* 6, $5+1$.

Theorem 4.5. *Let* $\mathcal{X}_5(\nu)$ *denote the number of partitions of* ν *into parts congruent to* $\pm 1, \pm 3, \pm 4, \pm 7, \pm 8, \pm 9 \pmod{20}$.

Then $\mathcal{P}_{2_{(1,4)}}^{(1,2,1)}(\nu) = \mathcal{Q}_{2_{(1,4)}}^{(1,2,1)}(\nu) = \mathcal{X}_5(\nu)$ *for all* $\nu \geq 0$*, where* $\mathcal{P}_{2_{(1,4)}}^{(1,2,1)}(\nu)$ *and* $\mathcal{Q}_{2_{(1,4)}}^{(1,2,1)}(\nu)$ *are defined as in Theorem 3.2.*

Example. $\mathcal{P}_{2_{(1,4)}}^{(1,2,1)}(6) = 4$*, since the relevant partitions are:* 6_8*,* $6_4 + 0_2$*,* $6_{3+1} + 0_2$*,* $5_1 + 1_3$*. Also,* $\mathcal{X}_5(6) = 4$*, the relevant ordinary partitions are:* $3+3$*,* $3+1^3$*,* $4+1^2$*,* 1^6*.*

5. Conclusion

In this paper, the graphical aspect of four generalized basic series (1.1)–(1.4) have been discussed in terms of split lattice paths. Further, bijections have been established between two different infinite classes of combinatorial identities. The generalized combinatorial identities enable us to provide a new insight to explore Rogers–Ramanujan type identities graphically. Since in Theorems 3.1–3.4, we have considered 'k' as a nonnegative even integer for all combinatorial interpretations, now the questions may arise:

(1) If 'k' is taken to be an odd integer, can we have such types of combinatorial interpretations?
(2) If so, will these interpretations lead us to the combinatorial interpretations of some Rogers–Ramanujan type identities?

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