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A BIJECTION BETWEEN THE SETS OF (a, b, b^2)-GENERALIZED MOTZKIN PATHS AVOIDING uvv-PATTERNS AND uvu-PATTERNS

YIDONG SUN, CHENG SUN, AND XIULI HAO

Abstract. A generalized Motzkin path, called G-Motzkin path for short, of length n is a lattice path from $(0,0)$ to $(n,0)$ in the first quadrant of the XY-plane that consists of up steps $\mathrm{u}=(1,1)$, horizontal steps $\mathrm{h}=(1,0)$, vertical steps $\mathrm{v}=(0,-1)$ and down steps $\mathrm{d}=(1,-1)$. An (a,b,c)-G-Motzkin path is a weighted G-Motzkin path such that the u-steps, h-steps, v-steps and d-steps are weighted respectively by $1, a, b$ and c. Let τ be a word on $\{\mathrm{u},\mathrm{h},\mathrm{v},\mathrm{d}\}$, denote by $\mathcal{G}_n^{\tau}(a,b,c)$ the set of τ-avoiding (a,b,c)-G-Motzkin paths of length n for a pattern τ. In this paper, we consider the uvv-avoiding (a,b,c)-G-Motzkin paths and provide a direct bijection σ between $\mathcal{G}_n^{\mathrm{uvv}}(a,b,b^2)$ and $\mathcal{G}_n^{\mathrm{uvu}}(a,b,b^2)$. Finally, the set of fixed points of σ is also described and counted.

1. Introduction

Lattice paths, as an important class of research in Combinatorics, have produced many interesting results in recent years, with common lattice paths such as Dyck [8, 16, 14], Motzkin [4, 1, 13, 18], Schröder [7] and Delannoy [2, 3, 19] lattice paths. A *generalized Motzkin path*, called *G-Motzkin path* for short, of length n is a lattice path from $(0,0)$ to $(n,0)$ in the first quadrant of the XY-plane that consists of up steps $\mathrm{u}=(1,1)$, horizontal steps $\mathrm{h}=(1,0)$, vertical steps $\mathrm{v}=(0,-1)$ and down steps $\mathrm{d}=(1,-1)$. Other related lattice paths with various steps, including vertical steps permitted, have been considered by [9, 10, 11, 20, 21, 24]. See Figure 1 for a G-Motzkin path of length 23.

An (a,b,c)-*G-Motzkin path* is a weighted G-Motzkin path P such that the u-steps, h-steps, v-steps and d-steps of P are weighted respectively by $1, a, b$ and c. The *weight* of P, denoted by $w(\mathrm{P})$, is the product of the weights of

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Corresponding Author: Yidong Sun, sydmath@dlmu.edu.cn.

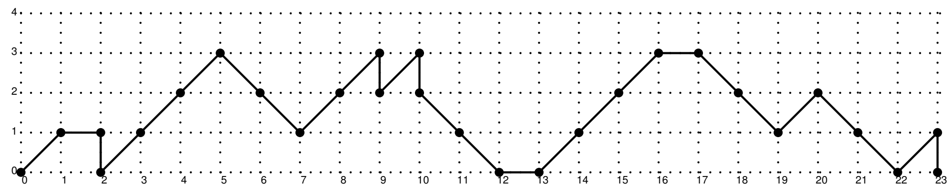

FIGURE 1. A G-Motzkin path of length 23.

each step of P. For example, $w(\text{uhuduuvvdhuudv}) = a^2b^3c^3$. The *weight* of a subset $\mathcal{A}$ of the set of weighted G-Motzkin paths, denoted by $w(\mathcal{A})$, is the sum of the weights of all paths in $\mathcal{A}$. Denote by $G_n(a,b,c)$ the weight of the set $\mathcal{G}_n(a,b,c)$ of all (a,b,c)-G-Motzkin paths of length n. Let τ be a word on $\{\text{u},\text{h},\text{v},\text{d}\}$, a G-Motzkin path P is called τ-avoiding, if the pattern τ is not a subpath of P. Denote by $G_n^\tau(a,b,c)$ the weight of the set $\mathcal{G}_n^\tau(a,b,c)$ of all τ-avoiding (a,b,c)-G-Motzkin paths of length n, that is the weight of the subset of all (a,b,c)-G-Motzkin paths of length n avoiding the pattern τ. Figure 1 is an example of a G-Motzkin path of length 23 avoiding the pattern uvv, but not avoiding the pattern uvu.

Recently, Sun et al. [20, 21] have derived the generating functions of $G_n(a,b,c)$ and $G_n^{\text{uvu}}(a,b,c)$ as follows

$$\begin{aligned} G(a,b,c;x) &= \sum_{n=0}^\infty G_n(a,b,c)x^n \\ &= \frac{1-ax-\sqrt{(1-ax)^2-4x(b+cx)}}{2x(b+cx)} = \frac{1}{1-ax}C\Big(\frac{x(b+cx)}{(1-ax)^2}\Big), \end{aligned} \tag{1.1}$$

$$\begin{aligned} G^{\text{uvu}}(a,b,c;x) &= \sum_{n=0}^\infty G_n^{\text{uvu}}(a,b,c)x^n \\ &= \frac{(1-ax)(1+bx)-\sqrt{(1-ax)^2(1+bx)^2-4x(1+bx)(b+cx)}}{2x(b+cx)} \\ &= \frac{1}{1-ax}C\Big(\frac{x(b+cx)}{(1-ax)^2(1+bx)}\Big), \end{aligned} \tag{1.2}$$

where

$$C(x) = \sum_{n=0}^\infty C_n x^n = \frac{1-\sqrt{1-4x}}{2x} \tag{1.3}$$

is the generating function for the well-known Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$, counting the number of Dyck paths of length $2n$ [17, 16].

A *Dyck path* of length $2n$ is a G-Motzkin path of length $2n$ with no h-steps or v-steps. A *Motzkin path* of length n is a G-Motzkin path of length n with no v-steps. An (a,b)-*Dyck path* is a weighted Dyck path with u-steps weighted by 1, d-steps in ud-peaks weighted by a and other d-steps weighted by b. An (a,b)-*Motzkin path* of length n is an $(a,0,b)$-G-Motzkin path of length n. A *Schröder path* of length $2n$ is a path from $(0,0)$ to $(2n,0)$ in the first quadrant of the XY-plane that consists of up steps $\mathrm{u}=(1,1)$, horizontal steps $\mathrm{H}=(2,0)$ and down steps $\mathrm{d}=(1,-1)$. An (a,b)-*Schröder path* is a weighted Schröder path such that the u-steps, H-steps and d-steps are weighted respectively by $1,a$ and b. 0 Let $\mathcal{C}_n(a,b),\mathcal{M}_n(a,b)$ and $\mathcal{S}_n(a,b)$ be respectively the sets of (a,b)-Dyck paths of length $2n$, (a,b)-Motzkin paths of length n and (a,b)-Schröder paths of length $2n$. Let $C_n(a,b),M_n(a,b)$ and $S_n(a,b)$ be their weights with $C_0(a,b)=M_0(a,b)=S_0(a,b)=1$ respectively. It is not difficult to deduce that [6]

$$\begin{aligned}
C_n(a,b) &= \sum_{k=1}^{n}\frac{1}{n}\binom{n}{k-1}\binom{n}{k}a^kb^{n-k},\\
M_n(a,b) &= \sum_{k=0}^{n}\binom{n}{2k}C_ka^{n-2k}b^k,\\
S_n(a,b) &= \sum_{k=0}^{n}\binom{n+k}{2k}C_ka^{n-k}b^k,
\end{aligned}$$

and their generating functions are

$$\begin{aligned}
C(a,b;x) &= \sum_{n=0}^{\infty}C_n(a,b)x^n = \frac{1-(a-b)x-\sqrt{(1-(a-b)x)^2-4bx}}{2bx},\\
M(a,b;x) &= \sum_{n=0}^{\infty}M_n(a,b)x^n = \frac{1-ax-\sqrt{(1-ax)^2-4bx^2}}{2bx^2},\\
S(a,b;x) &= \sum_{n=0}^{\infty}S_n(a,b)x^n = \frac{1-ax-\sqrt{(1-ax)^2-4bx}}{2bx}.
\end{aligned}$$

There are close relation formulas between $C_n(a,b),M_n(a,b)$ and $S_n(a,b)$. More precisely, Chen and Pan [6] derived the following equivalent relations

$$S_n(a,b)=C_n(a+b,b)=(a+b)M_{n-1}(a+2b,(a+b)b)$$

for $n\geq 1$ and provided some combinatorial proofs. Sun et al. [20] obtained that

$$G_n^{\mathrm{uvu}}(a,b,b^2)=S_n(a,b)$$

for $n\geq 0$ and presented bijections between the sets $\mathcal{G}_n^{\mathrm{uvu}}(a,b,b^2)$ and $\mathcal{S}_n(a,b)$ as well as the set $\mathcal{C}_n(a+b,b)$. For example, we give a one-to-one weight-preserving correspondence between $\mathcal{G}_2^{\mathrm{uvu}}(a,b,b^2)$, $\mathcal{S}_2(a,b)$ and $\mathcal{G}_2^{\mathrm{uvv}}(a,b,b^2)$ in Table 1.

In the literature, there are many papers dealing with (a,b)-Motzkin paths or (a,b)-Motzkin numbers. For example, Chen and Wang [5] explored the connection between noncrossing linked partitions and $(3,2)$-Motzkin paths, established a one-to-one correspondence between the set of noncrossing linked partitions of $\{1,\ldots,n+1\}$ and the set of large $(3,2)$-Motzkin

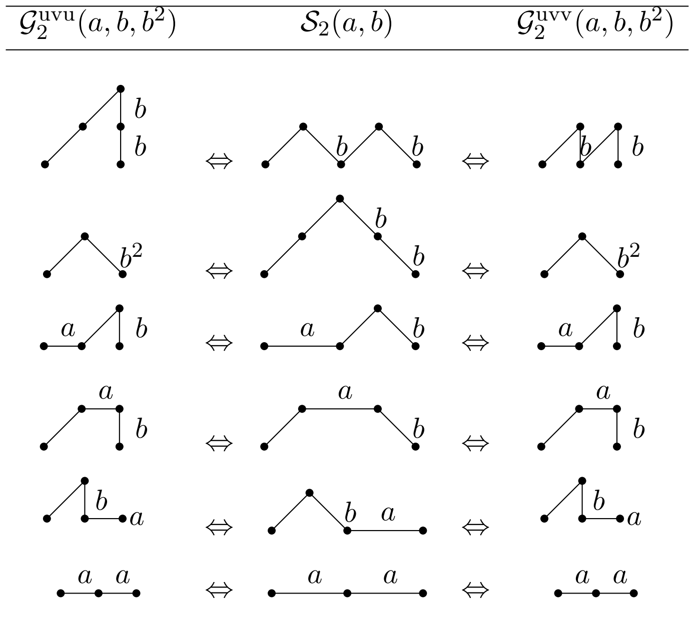

TABLE 1. The relation of $\mathcal{G}_2^{\mathrm{uvu}}(a,b,b^2)$, $\mathcal{S}_2(a,b)$ and $\mathcal{G}_2^{\mathrm{uvv}}(a,b,b^2)$.

paths of length n, which leads to a simple explanation of the well-known relation between the large and the little Schröder numbers. Yan [23] found a bijective proof between the set of restricted $(3,2)$-Motzkin paths of length n and the set of the Schröder paths of length $2n$. Recently, Sun [22] has given some identities related to the (a,b)-Motzkin numbers.

In the present paper we concentrate on the uvv-avoiding G-Motzkin paths, that is, the G-Motzkin paths with no uvv patterns. Precisely, the next section considers the enumerations of the set of uvv-avoiding (a,b,c)-G-Motzkin paths and the set of uvv-avoiding (a,b,c)-G-Motzkin paths with no h-steps on the x-axis, and find that $G_n^{\mathrm{uvv}}(a,b,b^2) = G_n^{\mathrm{uvu}}(a,b,b^2)$. The third section provides a direct bijection σ between the set $\mathcal{G}_n^{\mathrm{uvv}}(a,b,b^2)$ of uvv-avoiding (a,b,b^2)-G-Motzkin paths and the set $\mathcal{G}_n^{\mathrm{uvu}}(a,b,b^2)$ of uvu-avoiding (a,b,b^2)-G-Motzkin paths. Finally, the set of fixed points of σ is also described and counted.

2. UVV-AVOIDING (a,b,c)-G-MOTZKIN PATHS

In this section, we first consider the uvv-avoiding (a,b,c)-G-Motzkin paths which involve some classical structures as special cases, and count the set of uvv-avoiding (a,b,c)-G-Motzkin paths with no h-steps on the x-axis.

Let $G^{\mathrm{uvv}}(a,b,c;x) = \sum_{n=0}^{\infty} G_n^{\mathrm{uvv}}(a,b,c)x^n$ be the generating function for the uvv-avoiding (a,b,c)-G-Motzkin paths. According to the method of the first return decomposition [8], any uvv-avoiding (a,b,c)-G-Motzkin path P can be decomposed as one of the following four forms:

$$\mathrm{P}=\varepsilon,\ \mathrm{P}=\mathrm{h}_a\mathrm{Q}_1,\ \mathrm{P}=\mathrm{u}\mathrm{Q}_2\mathrm{d}_c\mathrm{Q}_1 \text{ or } \mathrm{P}=\mathrm{u}\mathrm{Q}_3\mathrm{v}_b\mathrm{Q}_1,$$

where x_t denotes the x-steps with weight t for $\mathrm{x}\in\{\mathrm{h},\mathrm{v},\mathrm{d}\}$, Q_1 and Q_2 are (possibly empty) uvv-avoiding (a,b,c)-G-Motzkin paths, and Q_3 is any uvv-avoiding (a,b,c)-G-Motzkin paths with no uv-step at the end of Q_3. Then we get the relation

$$\begin{aligned}G^{\mathrm{uvv}}(a,b,c;x) &= 1+axG^{\mathrm{uvv}}(a,b,c;x)+cx^2G^{\mathrm{uvv}}(a,b,c;x)^2\\ &\quad + bx\big(G^{\mathrm{uvv}}(a,b,c;x)-bxG^{\mathrm{uvv}}(a,b,c;x)\big)G^{\mathrm{uvv}}(a,b,c;x)\\ &= 1+axG^{\mathrm{uvv}}(a,b,c;x)+(b+(c-b^2)x)xG^{\mathrm{uvv}}(a,b,c;x)^2. \end{aligned}\tag{2.1}$$

Solving this, we have

$$\begin{aligned}G^{\mathrm{uvv}}(a,b,c;x) &= \frac{1-ax-\sqrt{(1-ax)^2-4x(b+(c-b^2)x)}}{2x(b+(c-b^2)x)}\\ &= \frac{1}{1-ax}C\Big(\frac{x(b+(c-b^2)x)}{(1-ax)^2}\Big).\end{aligned}\tag{2.2}$$

When $a=b=c=1$, $G^{\mathrm{uvv}}(1,1,1;x)=\frac{1-x-\sqrt{(1-x)^2-4x}}{2x}$, which is just the generating function of the large Schröder numbers [15].

By (1.3), taking the coefficient of x^n in $G^{\mathrm{uvv}}(a,b,c;x)$, we derive the following result

Proposition 2.1. *For any integer $n\geq 0$, we have*

$$G_n^{\mathrm{uvv}}(a,b,c)=\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{k}{j}\binom{n+k-j}{2k}C_k a^{n-k-j}b^{k-j}(c-b^2)^j.$$

Setting $T=xG(a,b,c;x)$, (2.1) produces

$$T=x\frac{1+aT+(c-b^2)T^2}{1-bT},\tag{2.3}$$

using the Lagrange inversion formula [12], taking the coefficient of x^{n+1} in T in three different ways, we derive the following result

Proposition 2.2. *For any integer $n \geq 0$, we have*

$$
\begin{aligned}
G_n^{\mathrm{uvv}}(a,b,c) &= \frac{1}{n+1}\sum_{k=0}^{[\frac{n}{2}]}\sum_{j=0}^{n-2k}\binom{n+1}{k}\binom{n+1-k}{j} \\
&\qquad\qquad \cdot\binom{2n-2k-j}{n-2k-j}a^j b^{n-2k-j}(c-b^2)^k \\
&= \frac{1}{n+1}\sum_{k=0}^{n}\sum_{j=0}^{[\frac{n-k}{2}]}\binom{n+1}{k}\binom{n+1-k}{j} \\
&\qquad\qquad \cdot\binom{2n-k-2j}{n-k-2j}a^k b^{n-k-2j}(c-b^2)^j \\
&= \frac{1}{n+1}\sum_{k=0}^{n}\sum_{j=0}^{n-k}\binom{n+1}{k}\binom{k}{j} \\
&\qquad\qquad \cdot\binom{2n-k-j}{n-k-j}a^{k-j} b^{n-k-j}(c-b^2)^j.
\end{aligned}
$$

Exactly, by (1.1), (1.2) and (2.2), it can be deduced that

$$G^{\mathrm{uvv}}(a,b,b^2+c;x) = G(a,b,c;x), \quad G^{\mathrm{uvv}}(a,b,b^2;x) = G^{\mathrm{uvu}}(a,b,b^2;x).$$

That is $G_n^{\mathrm{uvv}}(a,b,b^2+c) = G_n(a,b,c)$ and $G_n^{\mathrm{uvv}}(a,b,b^2) = G_n^{\mathrm{uvu}}(a,b,b^2)$. The first identity has a direct combinatorial interpretation if one notices that each d_{b^2+c}-step of $\mathrm{P} \in \mathcal{G}_n^{\mathrm{uvv}}(a,b,b^2+c)$ can be regarded equivalently as the corresponding d_c-step and $\mathrm{uv}_b\mathrm{v}_b$-step of $\mathrm{P}' \in \mathcal{G}_n(a,b,c)$. The combinatorial interpretation of the second identity will be given in the next section.

When (a,b,c) is specialized, $G^{\mathrm{uvv}}(a,b,c;x)$ and $G_n^{\mathrm{uvv}}(a,b,c)$ can reduce to some well-known generating functions and classical combinatorial sequences involving the Catalan numbers C_n, Motzkin numbers M_n, the large Schröder numbers S_n, $(a+b,b)$-Catalan number $C_n(a+b,b)$, (a,b)-Motzkin number $M_n(a,b)$ and (a,b)-Schröder number $S_n(a,b)$ respectively. See Table 2 for example.

Denote by $\bar{G}_n^{\mathrm{uvv}}(a,b,c)$ the weight of the set $\bar{\mathcal{G}}_n^{\mathrm{uvv}}(a,b,c)$ of all uvv-avoiding (a,b,c)-G-Motzkin paths of length n such that the paths have no h-steps on the x-axis. Set $\bar{\mathcal{G}}^{\mathrm{uvv}}(a,b,c) = \bigcup_{n\geq 0}\bar{\mathcal{G}}_n^{\mathrm{uvv}}(a,b,c)$.

Let $\bar{G}^{\mathrm{uvv}}(a,b,c;x) = \sum_{n=0}^{\infty}\bar{G}_n^{\mathrm{uvv}}(a,b,c)x^n$ be the generating function for the uvv-avoiding (a,b,c)-G-Motzkin paths in $\bar{\mathcal{G}}^{\mathrm{uvv}}(a,b,c)$. According to the method of the first return decomposition, any paths $\mathrm{P} \in \bar{\mathcal{G}}^{\mathrm{uvv}}(a,b,c)$ can be decomposed as one of the following three forms:

$$\mathrm{P} = \varepsilon, \ \mathrm{P} = \mathrm{uQ_2d}_c\mathrm{Q}_1 \text{ or } \mathrm{P} = \mathrm{uQ_3v}_b\mathrm{Q}_1,$$

(a,b,c)	$G^{\mathrm{uvv}}(a,b,c;x)$	$G_n^{\mathrm{uvv}}(a,b,c)$	*Senquences*
$(0,1,1)$	$C(x)=\frac{1-\sqrt{1-4x}}{2x}$	C_n	[15, A000108]
$(1,0,1)$	$M(x)=\frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}$	M_n	[15, A001006]
$(1,1,1)$	$S(x)=\frac{1-x-\sqrt{1-6x+x^2}}{2x}$	S_n	[15, A006318]
$(1,0,2)$	$\frac{1-x-\sqrt{1-2x-7x^2}}{4x}$	a_n	[15, A025235]
$(-3,4,16)$	$\frac{1+3x-\sqrt{1-10x+9x^2}}{8x}$	a_n	[15, A059231]
$(a,0,b)$	$\frac{1-ax-\sqrt{(1-ax)^2-4bx^2}}{2bx^2}$	$M_n(a,b)$	
(a,b,b^2)	$\frac{1-ax-\sqrt{(1-ax)^2-4bx}}{2bx}$	$C_n(a+b,b)$ *or* $S_n(a,b)$	

TABLE 2. The specializations of $G^{\mathrm{uvv}}(a,b,c;x)$ and $G_n^{\mathrm{uvv}}(a,b,c)$.

where $\mathrm{Q}_1 \in \bar{\mathcal{G}}^{\mathrm{uvv}}(a,b,c)$, $\mathrm{Q}_2 \in \mathcal{G}^{\mathrm{uvv}}(a,b,c)$ and $\mathrm{Q}_3 \in \mathcal{G}^{\mathrm{uvv}}(a,b,c)$ has no uv-step at the end of Q_3. Then we get the relation

$$\begin{aligned}\bar{G}^{\mathrm{uvv}}(a,b,c;x) = {} & 1 + cx^2 G^{\mathrm{uvv}}(a,b,c;x)\bar{G}^{\mathrm{uvv}}(a,b,c;x) \\ & + bx\big(G^{\mathrm{uvv}}(a,b,c;x) - bxG^{\mathrm{uvv}}(a,b,c;x)\big)\bar{G}^{\mathrm{uvv}}(a,b,c;x),\end{aligned}$$

which, by (2.1) and (2.3), leads to

$$\begin{aligned} x\bar{G}^{\mathrm{uvv}}(a,b,c;x) &= \frac{x}{1-(b+(c-b^2)x)xG^{\mathrm{uvv}}(a,b,c;x)} \\ &= \frac{xG^{\mathrm{uvv}}(a,b,c;x)}{1+axG^{\mathrm{uvv}}(a,b,c;x)} \\ &= \frac{T}{1+aT}. \end{aligned}$$

By the Lagrange inversion formula, taking the coefficient of x^{n+1} in $x\bar{G}^{\mathrm{uvv}}(a,b,c;x)$ in three different ways, we derive the following result

Proposition 2.3. *For any integer $n \geq 0$, we have*

$$\begin{aligned}
\bar{G}_n^{\mathrm{uvv}}(a,b,c) &= \sum_{i=0}^{n+1}(-1)^i \frac{i+1}{n+1} \sum_{k=0}^{[\frac{n}{2}]} \sum_{j=0}^{n-2k} \binom{n+1}{k}\binom{n+1-k}{j} \\
&\qquad \cdot \binom{2n-i-2k-j}{n-i-2k-j} a^{i+j} b^{n-i-2k-j} (c-b^2)^k \\
&= \sum_{i=0}^{n+1}(-1)^i \frac{i+1}{n+1} \sum_{k=0}^{n} \sum_{j=0}^{[\frac{n-k}{2}]} \binom{n+1}{k}\binom{n+1-k}{j} \\
&\qquad \cdot \binom{2n-i-k-2j}{n-i-k-2j} a^{i+k} b^{n-i-k-2j} (c-b^2)^j \\
&= \sum_{i=0}^{n+1}(-1)^i \frac{i+1}{n+1} \sum_{k=0}^{n} \sum_{j=0}^{n-k} \binom{n+1}{k}\binom{k}{j} \\
&\qquad \cdot \binom{2n-i-k-j}{n-i-k-j} a^{i+k-j} b^{n-i-k-j} (c-b^2)^j.
\end{aligned}$$

3. A BIJECTION BETWEEN THE SETS $\mathcal{G}_n^{\mathrm{uvv}}(a,b,b^2)$ AND $\mathcal{G}_n^{\mathrm{uvu}}(a,b,b^2)$

In this section, we give a direct bijection between the set $\mathcal{G}_n^{\mathrm{uvv}}(a,b,b^2)$ of uvv-avoiding (a,b,b^2)-G-Motzkin paths and the set $\mathcal{G}_n^{\mathrm{uvu}}(a,b,b^2)$ of uvu-avoiding (a,b,b^2)-G-Motzkin paths.

Theorem 3.1. *There exists a bijection σ between $\mathcal{G}_n^{\mathrm{uvv}}(a,b,b^2)$ and $\mathcal{G}_n^{\mathrm{uvu}}(a,b,b^2)$ for any integer $n \geq 0$.*

Proof. Given any $\mathrm{Q} \in \mathcal{G}_n^{\mathrm{uvv}}(a,b,b^2)$ for $n \geq 0$, when $n = 0, 1$ and 2, we define

$$\sigma(\varepsilon) = \varepsilon,\ \sigma(\mathrm{h}_a) = \mathrm{h}_a,\ \sigma(\mathrm{uv}_b) = \mathrm{uv}_b.$$

For $n \geq 2$, Q is uvv-avoiding, there are six cases to be considered to define $\sigma(\mathrm{Q})$ recursively.

CASE 1: $\mathrm{Q} = \mathrm{h}_a\mathrm{Q}'$ with $\mathrm{Q}' \in \mathcal{G}_{n-1}^{\mathrm{uvv}}(a,b,b^2)$. We define

$$\sigma(\mathrm{Q}) = \mathrm{h}_a\sigma(\mathrm{Q}').$$

CASE 2: $\mathrm{Q} = \mathrm{uv}_b\mathrm{h}_a\mathrm{Q}'$ with $\mathrm{Q}' \in \mathcal{G}_{n-2}^{\mathrm{uvv}}(a,b,b^2)$. We define

$$\sigma(\mathrm{Q}) = \mathrm{uv}_b\mathrm{h}_a\sigma(\mathrm{Q}').$$

CASE 3: $\mathrm{Q} = \mathrm{uv}_b\mathrm{Q}''\mathrm{Q}'$ such that $\mathrm{Q}'' \in \mathcal{G}_k^{\mathrm{uvv}}(a,b,b^2)$ is primitive and $\mathrm{Q}' \in \mathcal{G}_{n-k-1}^{\mathrm{uvv}}(a,b,b^2)$ for certain $1 \leq k \leq n-1$. We define

$$\sigma(\mathrm{Q}) = \mathrm{u}\sigma(\mathrm{Q}'')\mathrm{v}_b\sigma(\mathrm{Q}').$$

In this case, one can notice that there exist uvu's in Q, but not in $\sigma(\mathrm{Q})$.

Case 4: $\mathrm{Q} = \mathrm{u}^i\mathrm{ud}_{b^2}\mathrm{v}_b^i\mathrm{Q}'$ with $\mathrm{Q}' \in \mathcal{G}_{n-2-i}^{\mathrm{uvv}}(a,b,b^2)$ for $0 \le i \le n-2$. We define

$$\sigma(\mathrm{Q}) = \begin{cases} \mathrm{u}^j\mathrm{uv}_b\mathrm{d}_{b^2}^j\sigma(\mathrm{Q}'), & \text{if } i = 2j-1 \ge 1, \\ \mathrm{u}^{j+1}\mathrm{d}_{b^2}^{j+1}\sigma(\mathrm{Q}'), & \text{if } i = 2j \ge 0. \end{cases}$$

Case 5: $\mathrm{Q} = \mathrm{u}^i\mathrm{uQ}''\mathrm{d}_{b^2}\mathrm{v}_b^i\mathrm{Q}'$ such that $\mathrm{Q}'' \in \mathcal{G}_k^{\mathrm{uvv}}(a,b,b^2)$ is nonempty and $\mathrm{Q}' \in \mathcal{G}_{n-k-i}^{\mathrm{uvv}}(a,b,b^2)$ for certain $1 \le k \le n-i$ and $0 \le i \le n-2$. We define

$$\sigma(\mathrm{Q}) = \begin{cases} \mathrm{u}^j\sigma(\mathrm{Q}''\mathrm{uv}_b)\mathrm{d}_{b^2}^j\sigma(\mathrm{Q}'), & \text{if } i = 2j-1 \ge 1, \\ \mathrm{u}^{j+1}\sigma(\mathrm{Q}''\mathrm{uv}_b)\mathrm{v}_b\mathrm{d}_{b^2}^j\sigma(\mathrm{Q}'), & \text{if } i = 2j \ge 0. \end{cases}$$

Case 6: $\mathrm{Q} = \mathrm{u}^i\mathrm{Q}''\mathrm{v}_b^i\mathrm{Q}'$ such that $\mathrm{Q}'' \in \mathcal{G}_k^{\mathrm{uvv}}(a,b,b^2)$ is not primitive and $\mathrm{Q}' \in \mathcal{G}_{n-k-i}^{\mathrm{uvv}}(a,b,b^2)$ for certain $1 \le k \le n-i$ and $1 \le i \le n-1$, where Q'' does not end with uv_b since Q is uvv-avoiding. We define

$$\sigma(\mathrm{Q}) = \begin{cases} \mathrm{u}^j\sigma(\mathrm{Q}'')\mathrm{v}_b\mathrm{d}_{b^2}^{j-1}\sigma(\mathrm{Q}'), & \text{if } i = 2j-1 \ge 1, \\ \mathrm{u}^j\sigma(\mathrm{Q}'')\mathrm{d}_{b^2}^j\sigma(\mathrm{Q}'), & \text{if } i = 2j \ge 2. \end{cases}$$

From the definition of σ, one can deduce by induction that $\sigma(\mathrm{Q})$ is uvu-avoiding and the following assertions hold:

- In Case 3, $\sigma(\mathrm{Q}'')$ must be primitive and not be $\mathrm{uuv}_b\mathrm{v}_b$ since Q'' is primitive;
- In Case 5, $\sigma(\mathrm{Q}''\mathrm{uv}_b)$ has the form $\mathrm{P}_1\mathrm{uuv}_b\mathrm{v}_b$ or $\mathrm{P}_2\mathrm{uv}_b$ since Q'' is nonempty, where both $\mathrm{P}_1 \in \mathcal{G}_{r-1}^{\mathrm{uvu}}(a,b,b^2)$ and $\mathrm{P}_2 \in \mathcal{G}_r^{\mathrm{uvu}}(a,b,b^2)$ must not end with uv_b for certain $r \ge 1$;
- In Case 6, $\sigma(\mathrm{Q}'')$ is not primitive and does not end with uv_b or $\mathrm{uuv}_b\mathrm{v}_b$ since Q'' is not primitive and does not end with uv_b.

Conversely, the inverse procedure can be handled as follows. Given any $\mathrm{P} \in \mathcal{G}_n^{\mathrm{uvu}}(a,b,b^2)$ for $n \ge 0$, when $n = 0, 1$, we define

$$\sigma^{-1}(\varepsilon) = \varepsilon,\ \sigma^{-1}(\mathrm{h}_a) = \mathrm{h}_a,\ \sigma^{-1}(\mathrm{uv}_b) = \mathrm{uv}_b.$$

For $n \ge 2$, there are five cases to be considered to define $\sigma^{-1}(\mathrm{P})$ recursively.

Case I: $\mathrm{P} = \mathrm{h}_a\mathrm{P}'$ with $\mathrm{P}' \in \mathcal{G}_{n-1}^{\mathrm{uvu}}(a,b,b^2)$. We define

$$\sigma^{-1}(\mathrm{P}) = \mathrm{h}_a\sigma^{-1}(\mathrm{P}').$$

Case II: $\mathrm{P} = \mathrm{uv}_b\mathrm{h}_a\mathrm{P}'$ with $\mathrm{P}' \in \mathcal{G}_{n-2}^{\mathrm{uvu}}(a,b,b^2)$. We define

$$\sigma^{-1}(\mathrm{P}) = \mathrm{uv}_b\mathrm{h}_a\sigma^{-1}(\mathrm{P}').$$

CASE III: $\mathrm{P} = \mathrm{uP''v}_b\mathrm{P'}$ such that $\mathrm{P''} \in \mathcal{G}_k^{\mathrm{uvu}}(a, b, b^2)$ and $\mathrm{P'} \in \mathcal{G}_{n-1-k}^{\mathrm{uvu}}(a, b, b^2)$ for certain $1 \le k \le n-1$. We define

$$\sigma^{-1}(\mathrm{P}) = \begin{cases} \mathrm{uv}_b\sigma^{-1}(\mathrm{P''})\sigma^{-1}(\mathrm{P'}), & \text{if } \mathrm{P''}(\neq \mathrm{u}^2\mathrm{v}_b^2) \text{ is primitive,} \\ \mathrm{u}\sigma^{-1}(\mathrm{P''})\mathrm{v}_b\sigma^{-1}(\mathrm{P'}), & \text{if } \mathrm{P''}(\neq \varepsilon) \text{ is not primitive} \\ & \quad \text{and does not end with} \\ & \quad \mathrm{u}^2\mathrm{v}_b^2 \text{ or } \mathrm{uv}_b, \\ \mathrm{u}\sigma^{-1}(\mathrm{P}_1\mathrm{uv}_b)\mathrm{d}_{b^2}\sigma^{-1}(\mathrm{P'}), & \text{if } \mathrm{P''} = \mathrm{P}_1\mathrm{uuv}_b\mathrm{v}_b, \\ \mathrm{u}\sigma^{-1}(\mathrm{P}_2)\mathrm{d}_{b^2}\sigma^{-1}(\mathrm{P'}), & \text{if } \mathrm{P''} = \mathrm{P}_2\mathrm{uv}_b, \end{cases}$$

where both $\mathrm{P}_1 \in \mathcal{G}_{r-1}^{\mathrm{uvu}}(a, b, b^2)$ and $\mathrm{P}_2 \in \mathcal{G}_r^{\mathrm{uvu}}(a, b, b^2)$ must not end with uv_b for certain $r \ge 1$, since P is uvu-avoiding.
CASE IV: $\mathrm{P} = \mathrm{u}^j\mathrm{P''d}_{b^2}^j\mathrm{P'}$ such that $\mathrm{P''} \in \mathcal{G}_k^{\mathrm{uvu}}(a, b, b^2)$ and $\mathrm{P'} \in \mathcal{G}_{n-2j-k}^{\mathrm{uvu}}(a, b, b^2)$ for certain $0 \le k \le n-2j$ and the maximum $j \ge 1$. We define

$$\sigma^{-1}(\mathrm{P}) = \begin{cases} \mathrm{u}^{2j-1}\mathrm{d}_{b^2}\mathrm{v}_b^{2j-2}\sigma^{-1}(\mathrm{P'}), & \text{if } \mathrm{P''} = \varepsilon, \\ \mathrm{u}^{2j}\sigma^{-1}(\mathrm{P}_1\mathrm{uv}_b)\mathrm{d}_{b^2}\mathrm{v}_b^{2j-1}\sigma^{-1}(\mathrm{P'}), & \text{if } \mathrm{P''} = \mathrm{P}_1\mathrm{uuv}_b\mathrm{v}_b, \\ \mathrm{u}^{2j}\sigma^{-1}(\mathrm{P}_2)\mathrm{d}_{b^2}\mathrm{v}_b^{2j-1}\sigma^{-1}(\mathrm{P'}), & \text{if } \mathrm{P''} = \mathrm{P}_2\mathrm{uv}_b, \\ \mathrm{u}^{2j}\sigma^{-1}(\mathrm{P''})\mathrm{v}_b^{2j}\sigma^{-1}(\mathrm{P'}), & \text{if } \mathrm{P''}(\neq \varepsilon) \text{ is not primi-} \\ & \quad \text{tive and does not end} \\ & \quad \text{with } \mathrm{u}^2\mathrm{v}_b^2 \text{ or } \mathrm{uv}_b, \end{cases}$$

where both $\mathrm{P}_1 \in \mathcal{G}_{r-1}^{\mathrm{uvu}}(a, b, b^2)$ and $\mathrm{P}_2 \in \mathcal{G}_r^{\mathrm{uvu}}(a, b, b^2)$ must not end with uv_b for certain $r \ge 1$, since P is uvu-avoiding.
CASE V: $\mathrm{P} = \mathrm{u}^j\mathrm{uP''v}_b\mathrm{d}_{b^2}^j\mathrm{P'}$ such that $\mathrm{P''} \in \mathcal{G}_k^{\mathrm{uvu}}(a, b, b^2)$ and $\mathrm{P'} \in \mathcal{G}_{n-2j-1-k}^{\mathrm{uvu}}(a, b, b^2)$ for certain $0 \le k \le n-2j-1$ and the maximum $j \ge 1$. We define

$$\sigma^{-1}(\mathrm{P}) = \begin{cases} \mathrm{u}^{2j}\mathrm{d}_{b^2}\mathrm{v}_b^{2j-1}\sigma^{-1}(\mathrm{P'}), & \text{if } \mathrm{P''} = \varepsilon, \\ \mathrm{u}^{2j+1}\sigma^{-1}(\mathrm{P}_1\mathrm{uv}_b)\mathrm{d}_{b^2}\mathrm{v}_b^{2j}\sigma^{-1}(\mathrm{P'}), & \text{if } \mathrm{P''} = \mathrm{P}_1\mathrm{uuv}_b\mathrm{v}_b, \\ \mathrm{u}^{2j+1}\sigma^{-1}(\mathrm{P}_2)\mathrm{d}_{b^2}\mathrm{v}_b^{2j}\sigma^{-1}(\mathrm{P'}), & \text{if } \mathrm{P''} = \mathrm{P}_2\mathrm{uv}_b, \\ \mathrm{u}^{2j+1}\sigma^{-1}(\mathrm{P''})\mathrm{v}_b^{2j+1}\sigma^{-1}(\mathrm{P'}), & \text{if } \mathrm{P''}(\neq \varepsilon) \text{ is not primi-} \\ & \quad \text{tive and does not end} \\ & \quad \text{with } \mathrm{u}^2\mathrm{v}_b^2 \text{ or } \mathrm{uv}_b, \end{cases}$$

where both $\mathrm{P}_1 \in \mathcal{G}_{r-1}^{\mathrm{uvu}}(a, b, b^2)$ and $\mathrm{P}_2 \in \mathcal{G}_r^{\mathrm{uvu}}(a, b, b^2)$ must not end with uv_b for certain $r \ge 1$, since P is uvu-avoiding.

It is not difficult to verify that $\sigma^{-1}\sigma = \sigma\sigma^{-1} = 1$, both σ and σ^{-1} are two weight-keeping mappings and $\sigma^{-1}(\mathrm{P})$ is uvv-avoiding by induction on the length of P. Hence, σ is a desired bijection between $\mathcal{G}_n^{\mathrm{uvv}}(a, b, b^2)$ and $\mathcal{G}_n^{\mathrm{uvu}}(a, b, b^2)$. This completes the proof of Theorem 3.1. □

In order to give a more intuitive view on the bijection σ, a pictorial description of σ is presented for

$$\mathrm{Q} = \mathrm{u}^3\mathrm{d}_{b^2}\mathrm{v}_b^2\mathrm{u}^2\mathrm{d}_{b^2}\mathrm{v}_b\mathrm{u}^5\mathrm{v}_b\mathrm{d}_{b^2}\mathrm{v}_b^3\mathrm{h}_a\mathrm{u}^2\mathrm{h}_a\mathrm{d}_{b^2}\mathrm{v}_b\mathrm{u}^3\mathrm{v}_b\mathrm{uv}_b\mathrm{h}_a\mathrm{v}_b^2 \in \mathcal{G}_{23}^{\mathrm{uvv}}(a, b, b^2),$$

and

$$\sigma(\mathrm{Q}) = \mathrm{u}^2\mathrm{d}_{b^2}^2\mathrm{u}^2\mathrm{v}_b\mathrm{d}_{b^2}\mathrm{u}^4\mathrm{v}_b\mathrm{d}_{b^2}^2\mathrm{h}_a\mathrm{uh}_a\mathrm{uv}_b\mathrm{d}_{b^2}\mathrm{u}^3\mathrm{v}_b^2\mathrm{h}_a\mathrm{d}_{b^2} \in \mathcal{G}_{23}^{\mathrm{uvu}}(a, b, b^2).$$

See Figure 2 for detailed illustrations.

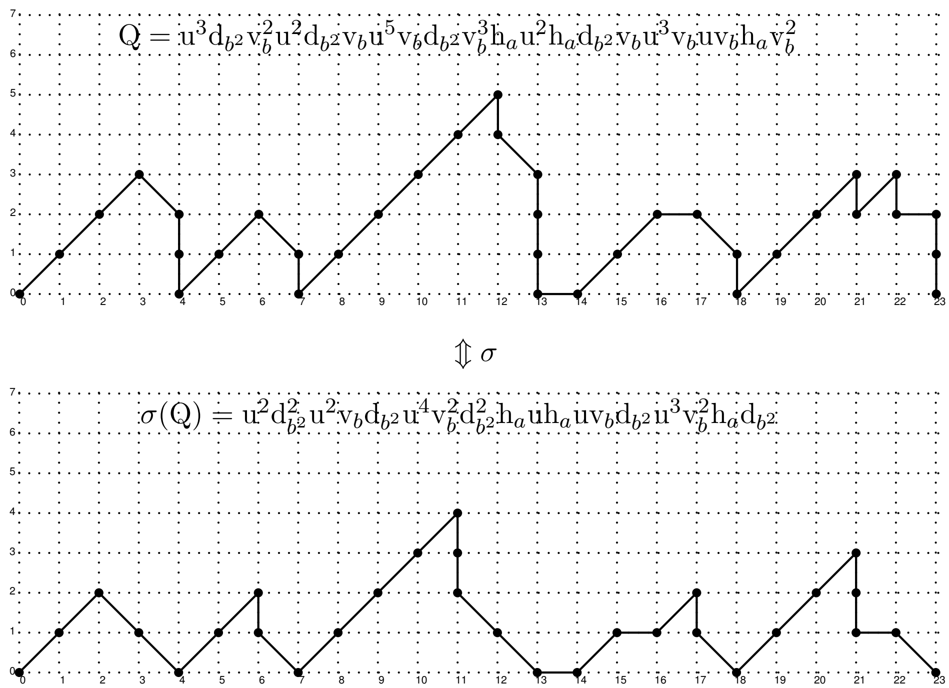

FIGURE 2. An example of the bijection σ described in the proof of Theorem 3.1.

4. Counting the set of fixed points of the bijection σ

In this section, we will count the set of fixed points of the bijection σ presented in Section 3.

Let $\mathcal{F}_n = \{\mathrm{Q} \in \mathcal{G}_n^{\mathrm{uvv}}(a, b, b^2) | \sigma(\mathrm{Q}) = \mathrm{Q}\}$ and $\mathcal{F} = \bigcup_{n\geq 0} \mathcal{F}_n$, set $F_n = |\mathcal{F}_n|$. It is easy to verify the few initial values for F_n, see Table 4.1. It should be noted that the sequence F_n currently does not match any of the sequences in OEIS [15].

n	0	1	2	3	4	5	6	7	8	9	10
F_n	1	2	5	13	39	125	421	1478	5329	19658	73783

Table 4.1. The first values of F_n.

According to the definition of σ, any $\mathrm{Q} \in \mathcal{F}_n$ must belong to the set, $\mathcal{G}_n^{\{\mathrm{uvv},\mathrm{uvu}\}}(a, b, b^2)$, of (a, b, b^2)-G-Motzkin paths avoiding both the uvv and uvu patterns, since Q is uvv-avoiding and $\sigma(\mathrm{Q})$ is uvu-avoiding. But there exists $\mathrm{P} \in \mathcal{G}_n^{\{\mathrm{uvv},\mathrm{uvu}\}}(a, b, b^2)$ such that $\sigma(\mathrm{P}) \neq \mathrm{P}$. For example, $\sigma(\mathrm{uudv}) = \mathrm{uuvd} \neq \mathrm{uudv}$ for $n = 3$. From the proof of Theorem 3.1, one can deduce that $\mathrm{Q} \notin \mathcal{F}_n(\sigma)$ if Q is being in the following situations, 1) in the whole Case 3; 2) in the Case 4 when $i \geq 1$; 3) in the whole Case 5; and 4) in the Case 6 when $i \geq 2$. Equivalently, one can derive that

- In Case 1, $\mathrm{Q} = \mathrm{h}_a\mathrm{Q}' \in \mathcal{F}_n$ if and only if $\mathrm{Q}' \in \mathcal{F}_{n-1}$;
- In Case 2, $\mathrm{Q} = \mathrm{uv}_b\mathrm{h}_a\mathrm{Q}' \in \mathcal{F}_n$ if and only if $\mathrm{Q}' \in \mathcal{F}_{n-2}$;
- In Case 4 when $i = 0$, $\mathrm{Q} = \mathrm{ud}_{b^2}\mathrm{Q}' \in \mathcal{F}_n$ if and only if $\mathrm{Q}' \in \mathcal{F}_{n-2}$;
- In Case 6 when $i = 1$, $\mathrm{Q} = \mathrm{uQ}''\mathrm{v}_b\mathrm{Q}' \in \mathcal{F}_n$ if and only if $\mathrm{Q}'' \in \mathcal{F}_k$ and $\mathrm{Q}' \in \mathcal{F}_{n-k-1}$ for certain $1 \leq k \leq n-1$ such that $\mathrm{Q}''(\neq \varepsilon)$ is not primitive and does not end with uv_b.

Let $\mathcal{A}_n$ be the subset of $\mathrm{Q} \in \mathcal{F}_n$ such that Q is not primitive and does not end with uv_b, $\mathcal{B}_n$ be the subset of $\mathrm{Q} \in \mathcal{F}_n$ such that Q ends with uv_b, and $\mathcal{C}_n$ be the subset of $\mathrm{Q} \in \mathcal{F}_n$ such that Q is primitive and does not end with uv_b. Set $a_n = |\mathcal{A}_n|, b_n = |\mathcal{B}_n|, c_n = |\mathcal{C}_n|$. Firstly, $\mathcal{F}_n$ is the disjoint union of $\mathcal{A}_n, \mathcal{B}_n$ and $\mathcal{C}_n$, i.e., $F_n = a_n + b_n + c_n$ for $n \geq 0$; Secondly, $\mathcal{C}_0 = \mathcal{C}_1 = \emptyset$, $\mathcal{C}_2 = \{\mathrm{uh}_a\mathrm{v}_b, \mathrm{ud}_{b^2}\}$ and $\mathcal{C}_n = \mathrm{u}\mathcal{A}_{n-1}\mathrm{v}_b$ for $n \geq 3$, i.e., $c_n = a_{n-1}$ for $n \geq 3$ with $c_0 = c_1 = 0$ and $c_2 = c_3 = 2$; Thirdly, $\mathcal{B}_n$ is the disjoint union of $\mathcal{A}_{n-1}\mathrm{uv}_b$ and $\mathcal{C}_{n-1}\mathrm{uv}_b$ for $n \geq 1$, i.e., $b_n = a_{n-1} + c_{n-1}$ for $n \geq 1$ with $b_0 = 0$ and $b_1 = b_2 = 1$. These together generate the following Lemma.

Lemma 4.1. *For any integer $n \geq 4$, we have*

$$F_n = a_n + 2a_{n-1} + a_{n-2} \tag{4.1}$$

with $a_0 = a_1 = 1, a_2 = 2, a_3 = 7$ and $a_4 = 23$.

On the other hand, the family $\mathcal{F}$ can be partitioned into the form:

$$\mathcal{F} = \varepsilon + \mathrm{h}_a\mathcal{F} + \mathrm{uv}_b\mathrm{h}_a\mathcal{F} + \mathrm{ud}_{b^2}\mathcal{F} + \mathrm{u}\mathcal{A}'\mathrm{v}_b\mathcal{F},$$

where $\mathcal{A}' = \mathcal{A} - \varepsilon$ and $\mathcal{A} = \bigcup_{n\geq 0} \mathcal{A}_n$. This leads to the following recurrence for F_n.

Lemma 4.2. *For any integer $n \geq 1$, we have*

$$F_{n+1} = F_n + 2F_{n-1} + \sum_{k=1}^{n} a_k F_{n-k} \tag{4.2}$$

with $F_0 = 1, F_1 = 2$.

Let $F(x)=\sum_{n\geq 0}F_nx^n$ and $A(x)=\sum_{n\geq 0}a_nx^n$. By (4.1), we have

$$\begin{aligned} F(x) &= 1+2x+5x^2+13x^3+\sum_{n\geq 4}(a_n+2a_{n-1}+a_{n-2})x^n \\ &= 1+2x+5x^2+13x^3+\big(A(x)-1-x-2x^2-7x^3\big) \\ &\quad +2x\big(A(x)-1-x-2x^2\big)+x^2\big(A(x)-1-x\big) \\ &= (1+x)^2A(x)-x+x^3. \end{aligned} \tag{4.3}$$

By (4.2), we obtain

$$\begin{aligned} F(x) &= 1+2x+x(F(x)-1)+2x^2F(x)+x(A(x)-1)F(x) \\ &= 1+x+2x^2F(x)+xA(x)F(x). \end{aligned} \tag{4.4}$$

Eliminating $A(x)$ in (4.3) and (4.4) produces

$$xF(x)^2-(1+x)(1+x-3x^2-x^3)F(x)+(1+x)^3=0.$$

Solving this, we have

$$\begin{aligned} F(x) &= \frac{(1+x)(1+x-3x^2-x^3)}{2x} \\ &\quad -\frac{(1+x)\sqrt{(1+x-3x^2-x^3)^2-4x(1+x)}}{2x} \\ &= \frac{(1+x)^2}{1+x-3x^2-x^3}C\Big(\frac{x(1+x)}{(1+x-3x^2-x^3)^2}\Big). \end{aligned} \tag{4.5}$$

By (4.5), taking the coefficient of x^n in $F(x)$, we get the explicit formula for the number F_n of the fixed points of the bijection σ, namely,

Theorem 4.3. *For any integer $n\geq 0$, we have*

$$F_n=\sum_{k=0}^{n}\sum_{j=0}^{[\frac{n-k}{2}]}\sum_{i=0}^{j}(-1)^{n-k-i}\binom{2k+j}{j}\binom{j}{i}\binom{n-j-i-2}{n-k-2j-i}3^{j-i}C_k,$$

where C_k is the k-th Catalan number.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Department of Mathematics, Dalian Maritime University, 116026 Dalian, PR China
E-mail address: sydmath@dlmu.edu.cn

Department of Mathematics, Dalian Maritime University, 116026 Dalian, PR China

Center for Combinatorics, LPMC, Nankai University, 300071 Tianjin, PR China
E-mail address: scmath@mail.nankai.edu.cn

Department of Mathematics, Dalian Maritime University, 116026 Dalian, PR China
E-mail address: hxlmath@dlmu.edu.cn