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THE FLAG f- AND h- VECTORS OF GENERALIZED SQUARE POSETS

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Abstract. Tiling a quadrant of the plane with squares gives rise to the Square poset. Adjustments in the tiling tactic generate a series of posets that we refer to as the generalized Square posets. We study the flag f- and h-vectors of this class of generalized Square posets.

1. Introduction

The theory of posets plays an important unifying role in enumerative combinatorics. As the classical case, the Square poset can show its such feature in different contexts. For instance, enumerations on the Square poset have close relations to partition numbers, Catalan numbers, binomial coefficients, etc.

Meanwhile the Square poset possesses many interesting properties that have attracted much attention. One particular instance worth noting is that the number of n-element order ideals in the Square poset is the number of partitions of n. Propp [5] extended this property to a class of generalized Square posets (Hexagonal, Rhomb, Tilt, etc.; we will introduce them later in this section). Therein many meaningful identities were achieved, or reproved combinatorially. In this paper, we aim to study the flag f- and h-vectors of the aforementioned class of generalized Square posets.

Whilst we are introducing the definitions of flag f- and h-vectors, the reader is assumed to be familiar with all the basic concepts of modern enumerative combinatorics (as can be found in e.g. in [10, Ch3]). Let P be a finite poset of rank n, with rank function $\rho : P \to [0, n]$. Here $[0, n]$ is the set $\{0, 1, 2, \ldots, n\}$. If $S \subseteq [0, n]$ then define the subposet

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$P_S = \{t \in P : \rho(t) \in S\}$, called the *S-rank-selected* subposet of P. Now define $\tilde{f}_P(S)$ (or simply $\tilde{f}(S)$) to be the number of maximal chains of P_S. For instance, $\tilde{f}(i)$ (short for $\tilde{f}(\{i\})$) is exactly the number of elements of P of rank i. The function $\tilde{f} : 2^{[0,n]} \to Z$ is called the *flag f-vector* of P. Also define $\tilde{h}_P(S) = \tilde{h}(S)$ by

$$\tilde{h}(S) = \sum_{T \subseteq S} (-1)^{\sharp(S-T)} \tilde{f}(T).$$

The function $\tilde{h}$ is called the *flag h-vector* of P. The flag f- and h-vectors are natural extension of f- and h-vectors which are classic face enumerating vectors in complex. They occur naturally in diverse areas of mathematics and have been widely studied. For example, $\tilde{h}_P(S) \geq 0$ when P is a Cohen-Macaulay poset. The behavior of the flag h-vector has a close relationship with the combinatorial properties of the poset. Many classical results have been shown in the thesis of Richard Stanley [7]. For instance, one of the so many interesting results is when the poset P is a distributive lattice, its flag h-vector has a combinatorial interpretation. [1, 3, 4, 6, 8] are also a few recommended references of the many in the literature. These two functions of the Square poset can produce a number of beautiful identities and counting results, and can be computed explicitly. In light of this, we set out to study these two functions of some generalised Square posets.

Recall that the *Square* poset is determined by the set $\mathbb{N} \times \mathbb{N}$ (here $\mathbb{N} = \{0, 1, 2, \ldots\}$) and the relation $(a', b') \leq (a, b)$ if and only if $a' \leq a$ and $b' \leq b$. This poset can be obtained by tiling a quadrant of the plane with squares, shown as Figure 1 (see [5], [9], [10, Chap3] for more information); here Figure 1 requires us (so do its variants in the following) to imagine and keep in mind its infinite picture as it is. Based on Figure 1, by adjusting the tiling tactic accordingly, we obtain a series of posets that we refer to as the generalized Square posets: the Hexagonal poset, Rhomb poset, Tilt poset, Tilt $^{(1)}$ poset, and the Punc poset (see Figure 2,3,6,7,8 respectively).

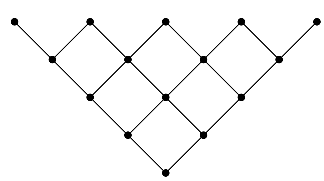

FIGURE 1. The Square poset

Let $S = \{i_1, i_2, \ldots, i_t\}$ be a set of integers with $i_1 \geq 0$. We emphasize this notation holds throughout the whole paper. It is known that the flag f- and h-vectors of the Square poset are given as follows.

$$\begin{aligned}\tilde{f}(S) &= (i_1 + 1)(i_2 - i_1 + 1) \cdots (i_t - i_{t-1} + 1),\\ \tilde{h}(S) &= i_1(i_2 - i_1 - 1) \cdots (i_t - i_{t-1} - 1).\end{aligned}$$

As a simple application, it is straightforward to see that the number of saturated chains from $(0,0)$ (the bottom element) to level n is 2^n.

Now, by replacing the squares in Figure 1 with hexagons and adding a new minimal element, we obtain the poset *Hexagonal* as Figure 2. In mathematical words, its elements are $\{(a,b)|a,b \in \mathbb{N}, a \geq b\}$ and we say $(a',b') \leq (a,b)$ if

- $a' \leq a$ and $b' \leq b$ when $a'+b'$ is odd,
- $a' < a$ and $b' \leq b$ when $a'+b'$ is even.

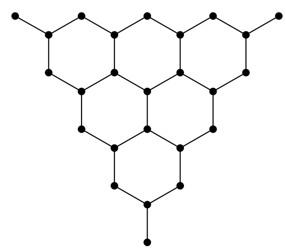

FIGURE 2. The Hexagonal poset

Let $S = \{i_1, \ldots, i_t\}$ and $n_j = \lfloor i_j/2 \rfloor$. Then the flag f- and h-vectors of the poset Hexagonal have been given as follows [2].

$$\begin{aligned} \tilde{f}(S) &= (n_1+1)(n_2-n_1+1)(n_3-n_2+1)\cdots(n_t-n_{t-1}+1), \\ \tilde{h}(S) &= n_1(n_2-n_1-1)(n_3-n_2-1)\cdots(n_t-n_{t-1}-1). \end{aligned}$$

By the formula for the flag f-vector, it is easy to see that the number of saturated chains from $(0,0)$ (the bottom element) to level n is $2^{\lfloor \frac{n}{2} \rfloor}$.

The following part of this paper is organized as follows. Section 2 investigates the Rhomb poset. The Tilt poset, Tilt $^{(1)}$ poset, and the Punc poset are studied in Section 3 as a package of the sub-posets of Square.

2. THE RHOMB POSET

The *Rhomb* poset is pictorially defined by Figure 3, i.e., decompose each hexagon in Hexagonal poset into three rhombi. Mathematically, we can define that the elements of the Rhomb poset are the triples $(a,b,c) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ which satisfy

- $c = a+b+1$ if $a+b+c$ is odd,
- $c = a+b$ or $c = a+b+2$, if $a+b+c$ is even.

We declare that $(a',b',c') \leq (a,b,c)$ if $a' \leq a$, $b' \leq b$ and $c' \leq c$. The Rhomb poset has the unique minimun $(0,0,0)$, and obviously it is ranked by the sum of entries of the triple. See Figure 4 for illustration of the elements of rank 0 to 4.

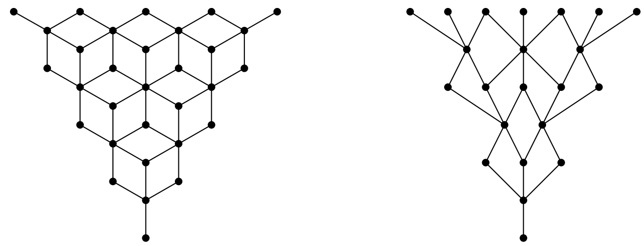

FIGURE 3. The Rhomb poset and its Hasse diagram

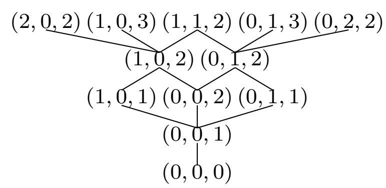

FIGURE 4. Rank 0 to 4 of the Rhomb poset

Theorem 2.1. *Let $S = \{i_1, i_2, \ldots, i_t\}$ and $S_m = \{i_1, i_2, \ldots, i_{t-m}\}$ be the sets of rank numbers of the Rhomb poset. Define*

$$o(i_k) = \begin{cases} 1, & i_k \text{ odd} \\ 0, & i_k \text{ even} \end{cases},$$

where $k = 1, \ldots, t$. Then

$$\tilde{f}(i_j) = \frac{i_j + 1}{2^{o(i_j)}},\ \tilde{f}(i_j, i_k) = \frac{\tilde{f}(i_j)(i_k - i_j + 2) + o(i_j) - 1}{2^{o(i_k)}}$$

and

$$\tilde{f}(S) = \frac{\tilde{f}(S_1)(i_t - i_{t-1} + 2) + (o(i_{t-1}) - 1)\tilde{f}(S_2)}{2^{o(i_t)}}.$$

Proof. Note that $\tilde{f}(i_j)$ is the number of elements of rank i_j. By definition of the Rhomb poset, the element (a, b, c) in rank i_j satisfies that $a + b + c = i_j$ and $c = a + b + 1$ if i_j is odd. That is $a + b = (i_j - 1)/2$. Then there are $(i_j - 1)/2 + 1$ ways to choose $a \in \mathbb{N}$ and $b \in \mathbb{N}$, i.e., $\tilde{f}(i_j) = (i_j + 1)/2$ if i_j is odd. Similarly, $\tilde{f}(i_j) = i_j/2 + 1 + (i_j - 2)/2 + 1 = i_j + 1$ if i_j is even. That is $\tilde{f}(i_j) = \frac{i_j+1}{2^{o(i_j)}}$. Recall that $\tilde{f}(i_j, i_k)$ is the number of chains between rank i_j and i_k. If i_j is odd, then there are $(i_k - i_j + 2)/2^{o(i_k)}$ elements of rank i_k, and all of them are comparable with exactly one element of rank i_j. Hence the number of chains between rank i_j and i_k is $\tilde{f}(i_j)(i_k - i_j + 2)/2^{o(i_k)}$. If i_j is even, then there are $(i_j + 2)/2$ elements of rank i_j generating $(i_k - i_j + 1)/2^{o(i_k)}$ chains, and there are $i_j/2$ elements of rank i_j generating $(i_k - i_j + 3)/2^{o(i_k)}$

chains. Hence the number of chains between rank i_j and i_k is

$$\frac{i_j+2}{2}\frac{i_k-i_j+1}{2^{o(i_k)}}+\frac{i_j}{2}\frac{i_k-i_j+3}{2^{o(i_k)}}=\frac{\tilde{f}(i_j)(i_k-i_j+2)-1}{2^{o(i_k)}}.$$

Combine these two cases in view of the parity of i_j, we get that

$$\tilde{f}(i_j,i_k)=\frac{\tilde{f}(i_j)(i_k-i_j+2)+o(i_j)-1}{2^{o(i_k)}}.$$

Now the formula of $\tilde{f}(S)$ is true for $t=1$ and $t=2$. We then turn to prove the case $t>2$, also having to consider the parity of i_{t-1}. If i_{t-1} is odd, then there are $(i_t-i_{t-1}+2)/2^{o(i_t)}$ elements of rank i_t, and all of them are comparable with exactly one element of rank i_{t-1}. So the number of chains between rank i_1 and i_t is

$$\frac{\tilde{f}(S_1)(i_t-i_{t-1}+2)}{2^{o(i_t)}}.$$

If i_{t-1} is even, then the number of the chains from each element of rank i_{t-1} to i_t produces an interlacing sequence of $(i_t-i_{t-1}+1)/2^{o(i_t)}$ and $(i_t-i_{t-1}+3)/2^{o(i_t)}$, starting with the former element. Therefore the number of the chains from each element of rank i_{t-2} to rank i_t via rank i_{t-1} is the sum of an interlacing sequence of $(i_t-i_{t-1}+1)/2^{o(i_t)}$ and $(i_t-i_{t-1}+3)/2^{o(i_t)}$. These sequences vary in length but are all arrayed in the same manner that starts and terminates with the former of the two elements. Hence the number of chains from each element of rank i_{t-2} to rank i_t is

$$\frac{a(i_t-i_{t-1}+2)}{2^{o(i_t)}}-\frac{1}{2^{o(i_t)}},$$

where a is exactly the length of the corresponding interlacing sequence. Taking $(i_t-i_{t-1}+2)/2^{o(i_t)}$ as the number of chains generated by each element of rank i_{t-1} to rank i_t, we get

$$\frac{\tilde{f}(S_1)(i_t-i_{t-1}+2)}{2^{o(i_t)}}$$

chains from i_1 to i_t. However, the number of the total chains is technically overcounted by $\tilde{f}(S_2)/2^{o(i_t)}$ that should be subtracted. Hence,

$$\tilde{f}(S)=\frac{\tilde{f}(S_1)(i_t-i_{t-1}+2)}{2^{o(i_t)}}-\frac{\tilde{f}(S_2)}{2^{o(i_t)}}.$$

This completes the proof. □

An example shall make the proof clearer for the case when i_{t-1} is even, say $i_{t-1}=4$ as shown in Figure 5. The number of the chains from each element of rank 4 to those of rank 5 is the sequence $1,2,1,2,1$. Therefore the number of the chains from element p_1 or p_3 in rank 2 to rank 5 is $1+2+1=\frac{3(5-4+2)}{2}-\frac{1}{2}=4$, and the number of the chains from p_2 in rank

2 to rank 5 is $1+2+1+2+1 = \frac{5(5-4+2)}{2} - \frac{1}{2} = 7$. Hence the sum of chains from rank 1 to rank 5 is

$$\frac{\tilde{f}(\{1,2,4\})(5-4+2)}{2} - \frac{\tilde{f}(\{1,2\})}{2} = 15.$$

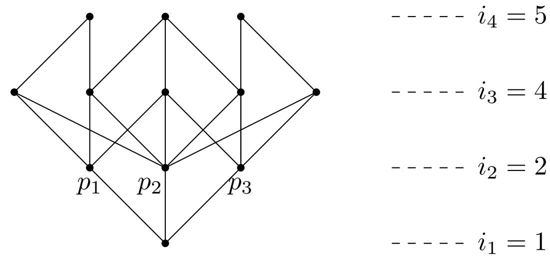

FIGURE 5. $\{1,2,4,5\}$-rank-selected subposet of the Rhomb poset

Remark: Let $S = \{i, i+1, \ldots, i+t\}$ and $S_m = \{i, i+1, \ldots, i+t-m\}$, where $i \geq 1$. Then

$$\tilde{f}(S) = 4\tilde{f}(S_2).$$

In particular, if $S = \{1, 2, \ldots, n\}$, then $\tilde{f}(S) = (5 \times 2^{n-1} - (-2)^{n-1})/4$.

Corollary 2.3. *The number of saturated chains of the Rhomb poset from rank* 0 *to rank* n $(n \geq 1)$ *is*

$$\frac{5 \times 2^{n-1} - (-2)^{n-1}}{4}.$$

Now we apply the formula of flag f-vector to establish the formula of flag h-vector.

Theorem 2.4. *Let* $S = \{i_1, \ldots, i_t\}$ *and* $S_m = \{i_1, i_2, \ldots, i_{t-m}\}$ *be the sets of rank numbers of the Rhomb poset. Define*

$$o(i_k) = \begin{cases} 1, & i_k \text{ odd} \\ 0, & i_k \text{ even} \end{cases},$$

where $k = 1, \ldots, t$. *Then*

$$\tilde{h}(i_j) = \frac{i_j - o(i_j)}{2^{o(i_j)}} \text{ and } \tilde{h}(S) = \frac{\tilde{h}(S_1)(i_t - i_{t-1} - o(i_{t-1}) - o(i_t))}{2^{o(i_t)}}.$$

Proof. By the definition of flag h-vector,

$$\tilde{h}(i_j) = -1 + \tilde{f}(i_j) = \frac{i_j + 1 - 2^{o(i_j)}}{2^{o(i_j)}} = \frac{i_j - o(i_j)}{2^{o(i_j)}}. \tag{2.1}$$

$$\begin{aligned} \tilde{h}(i_j, i_k) &= 1 - \tilde{f}(i_j) - \tilde{f}(i_k) + \tilde{f}(i_j, i_k) \\ &= -\tilde{h}(i_j) - \frac{i_k + 1}{2^{o(i_k)}} + \frac{\tilde{f}(i_j)(i_k - i_j + 2) + o(i_j) - 1}{2^{o(i_k)}}. \end{aligned}$$

In view of the parity of i_j and i_k, we get that

$$\tilde{h}(i_j, i_k) = \frac{\tilde{h}\,(i_j)(i_k - i_j - o(i_j) - o(i_k))}{2^{o(i_k)}}.$$

Now the formula of $\tilde{h}(S)$ is true for $t = 1, 2$. We then use induction on $t-1$, i.e.,

$$\tilde{h}(S_1) = \frac{\tilde{h}(S_2)\,(i_{t-1} - i_{t-2} - o(i_{t-2}) - o(i_{t-1}))}{2^{o(i_{t-1})}}, \tag{2.2}$$

and consider the induction step from $t-1$ to t. By the definition of flag h-vector,

$$\begin{aligned}\tilde{h}(S) &= (-1)^t + (-1)^{t-1} \sum_{1\le j\le t} \tilde{f}(i_j) + (-1)^{t-2} \sum_{j<p} \tilde{f}(i_j, i_p) + \cdots + \tilde{f}(S)\\ &= -\tilde{h}(S_1) + (-1)^{t-1}\tilde{f}(i_t) + (-1)^{t-2} \sum_{j=1}^{t-1} \tilde{f}(i_j, i_t) + \cdots + \tilde{f}(S).\end{aligned}$$

If we could prove that

$$\begin{aligned}&(-1)^{t-1}\tilde{f}(i_t) + (-1)^{t-2} \sum_{1\le j\le t-1} \tilde{f}(i_j, i_t) + \cdots + \tilde{f}(S)\\ &= \tilde{h}(S_1)\frac{i_t - i_{t-1} + 1 - o(i_{t-1})}{2^{o(i_t)}},\end{aligned} \tag{2.3}$$

then the the theorem follows from the fact that

$$\begin{aligned}\tilde{h}(S) &= -\tilde{h}(S_1) + \tilde{h}(S_1)\frac{i_t - i_{t-1} + 1 - o(i_{t-1})}{2^{o(i_t)}}\\ &= \tilde{h}(S_1)\frac{i_t - i_{t-1} - o(i_{t-1}) - o(i_t)}{2^{o(i_t)}}.\end{aligned}$$

To prove (2.3), we first need to ready the following two necessary identities.

Lemma 2.5.

$$\tilde{f}(i_t) - \tilde{f}(i_m, i_t) = -\tilde{h}(i_m)\frac{i_t - i_m + 1 - o(i_m)}{2^{o(i_t)}}, \tag{2.4}$$

where $1 \le m \le t-1$ *and*

$$\begin{aligned}&\tilde{f}(i_1, \ldots, i_r, i_t) - \tilde{f}(i_1, \ldots, i_r, i_{t-1}, i_t)\\ &= -\frac{\tilde{f}(S_{t-r})(i_{t-1} - i_r + 1 - o(i_{t-1})) + \tilde{f}(S_{t-r+1})(o(i_r) - 1)}{2^{o(i_{t-1})}2^{o(i_t)}}\\ &\quad\times \frac{i_t - i_{t-1} + 1 - o(i_{t-1})}{2^{o(i_{t-1})}2^{o(i_t)}}.\end{aligned} \tag{2.5}$$

Proof. By Theorem 2.1,

$$\begin{aligned}\tilde{f}(i_t) - \tilde{f}(i_m, i_t) &= \frac{i_t + 1}{2^{o(i_t)}} - \frac{\tilde{f}(i_m)(i_t - i_m + 2) + o(i_m) - 1}{2^{o(i_t)}} \\ &= \frac{i_t(1 - \tilde{f}(i_m)) + 2(1 - \tilde{f}(i_m)) + i_m\tilde{f}(i_m) - o(i_m)}{2^{o(i_t)}} \\ &= \frac{-\tilde{h}(i_m)(i_t - i_m + 2) + i_m - o(i_m)}{2^{o(i_t)}},\end{aligned}$$

where the last equation invokes the fact that $\tilde{h}(i_m) = -1 + \tilde{f}(i_m)$. By the equation (2.1), we have $i_m - o(i_m) = \tilde{h}(i_m)2^{o(i_m)}$. Note that $2^{o(i_m)} = 1 + o(i_m)$. Hence we have

$$\begin{aligned}\tilde{f}(i_t) - \tilde{f}(i_m, i_t) &= \frac{-\tilde{h}(i_m)(i_t - i_m + 2) + \tilde{h}(i_m)(1 + o(i_m))}{2^{o(i_t)}} \\ &= \frac{-\tilde{h}(i_m)(i_t - i_m + 1 - o(i_m))}{2^{o(i_t)}}.\end{aligned}$$

The proof of (2.4) is complete.

By the formula of flag f-vector, we have

$$\tilde{f}(i_1, \ldots, i_r, i_t) = \frac{\tilde{f}(S_{t-r})(i_t - i_r + 2) + (o(i_r) - 1)\tilde{f}(S_{t-r+1})}{2^{o(i_t)}}$$

and

$$\begin{aligned}&\tilde{f}(i_1, \ldots, i_r, i_{t-1}, i_t) \\ =& \frac{\tilde{f}(i_1, \ldots, i_r, i_{t-1})(i_t - i_{t-1} + 2) + (o(i_{t-1}) - 1)\tilde{f}(S_{t-r})}{2^{o(i_t)}} \\ =& \frac{\left(\tilde{f}(S_{t-r})(i_{t-1} - i_r + 2) + (o(i_r) - 1)\tilde{f}(S_{t-r+1})\right)(i_t - i_{t-1} + 2)}{2^{o(i_{t-1})}2^{o(i_t)}} \\ &+ \frac{2^{o(i_{t-1})}(o(i_{t-1}) - 1)\tilde{f}(S_{t-r})}{2^{o(i_{t-1})}2^{o(i_t)}}.\end{aligned}$$

Then the coefficient of $\tilde{f}(S_{t-r+1})$ in

$$2^{o(i_{t-1})}2^{o(i_t)}\left(\tilde{f}(i_1, \ldots, i_r, i_t) - \tilde{f}(i_1, \ldots, i_r, i_{t-1}, i_t)\right)$$

is

$$2^{o(i_{t-1})}(o(i_r) - 1) - (o(i_r) - 1)(i_t - i_{t-1} + 2) = (1 - o(i_r))(i_t - i_{t-1} + 1 - o(i_{t-1})),$$

where we invoke the fact that $2^{o(i_{t-1})} = 1 + o(i_{t-1})$.

The coefficient of $\tilde{f}(S_{t-r})$ in

$$2^{o(i_{t-1})}2^{o(i_t)}\left(\tilde{f}(i_1, \ldots, i_r, i_t) - \tilde{f}(i_1, \ldots, i_r, i_{t-1}, i_t)\right)$$

is

$$\begin{aligned}
&2^{o(i_{t-1})}(i_t - i_r + 2) - (i_{t-1} - i_r + 2)(i_t - i_{t-1} + 2) - 2^{o(i_{t-1})}(o(i_{t-1}) - 1) \\
&= (1 + o(i_{t-1}))(i_t - i_r + 3 - o(i_{t-1})) - (i_{t-1} - i_r + 2)(i_t - i_{t-1} + 2) \\
&= -(i_{t-1} - i_r + 1 - o(i_{t-1}))(i_t - i_{t-1} + 1 - o(i_{t-1})).
\end{aligned}$$

This completes the proof of equation (2.5). □

We now turn to prove (2.3). By (2.4), it is easy to check that (2.3) holds for $t = 2$. We next consider the case $t > 2$. Through rearrangement of the sum, we have that the LHS of main (2.3) is

$$\begin{aligned}
&(-1)^{t-1}\tilde{f}(i_t) + (-1)^{t-2}\sum_{1\le j<t}\tilde{f}(i_j, i_t) \\
&+ (-1)^{t-3}\sum_{j<p}\tilde{f}(i_j, i_p, i_t) + \cdots + \tilde{f}(S) \\
=&(-1)^{t-1}\left(\tilde{f}(i_t) - \tilde{f}(i_{t-1}, i_t)\right) + (-1)^{t-2}\sum_{j<t-1}\left(\tilde{f}(i_j, i_t) - \tilde{f}(i_j, i_{t-1}, i_t)\right) \\
&+ (-1)^{t-3}\sum_{j<p<t-1}\left(\tilde{f}(i_j, i_p, i_t) - \tilde{f}(i_j, i_p, i_{t-1}, i_t)\right) \\
&+ \cdots + (-1)\left(\tilde{f}(i_1, \ldots, i_{t-2}, i_t) - \tilde{f}(S)\right).
\end{aligned}$$

By Lemma 2.5, it is easy to see that the difference between each two flag f-vectors in the above brackets has a common factor

$$\frac{i_t - i_{t-1} + 1 - o(i_{t-1})}{2^{o(i_t)}}.$$

Compared with the RHS of (2.3), we have only to prove the following equation.

$$\begin{aligned}
(2.6)\qquad &(-1)^t\frac{i_{t-1} - o(i_{t-1})}{2^{o(i_{t-1})}} + \\
&(-1)^{t-1}\sum_{j<t-1}\frac{\tilde{f}(i_j)(i_{t-1} - i_j + 1 - o(i_{t-1})) + o(i_j) - 1}{2^{o(i_{t-1})}} + \cdots + \\
(2.7)\qquad &\frac{\tilde{f}(S_2)(i_{t-1} - i_{t-2} + 1 - o(i_{t-1})) + (o(i_{t-2}) - 1)\tilde{f}(S_3)}{2^{o(i_{t-1})}} \\
&= \tilde{h}(S_1).
\end{aligned}$$

Note that

$$\tilde{h}(S_2) = (-1)^t + (-1)^{t-1}\sum_{1\le j\le t-2}\tilde{f}(i_j) + (-1)^{t-2}\sum_{j<p}\tilde{f}(i_j, i_p) + \cdots + \tilde{f}(S_2).$$

Then collecting the coefficient of $(i_{t-1}-o(i_{t-1}))/2^{o(i_{t-1})}$ we find the LHS of (2.6) is

$$\frac{i_{t-1}-o(i_{t-1})}{2^{o(i_{t-1})}}\tilde{h}(S_2)+(-1)^t\sum_{j<t-1}\frac{\tilde{f}(i_j)(i_j-1)}{2^{o(i_{t-1})}}+\cdots+$$
$$(-1)\frac{\tilde{f}(S_2)(i_{t-2}-1)}{2^{o(i_{t-1})}}+(-1)^{t-1}\sum_{j<t-1}\frac{o(i_j)-1}{2^{o(i_{t-1})}}+\cdots+\frac{(o(i_{t-2})-1)\tilde{f}(S_3)}{2^{o(i_{t-1})}}.$$

Before proceeding further, we need to verify the following claim, thereby hoping to finish the proof of (2.6).

Lemma 2.6.

$$\begin{aligned}(-1)^t&\sum_{j<t-1}\tilde{f}(i_j)(i_j-1)+\cdots+(-1)\tilde{f}(S_2)(i_{t-2}-1)\\&+(-1)^{t-1}\sum_{j<t-1}(o(i_j)-1)+\cdots+(o(i_{t-2})-1)\tilde{f}(S_3)\\=&-(i_{t-2}+o(i_{t-2}))\tilde{h}(S_2).\end{aligned}\tag{2.8}$$

Proof. Our induction is on $t>2$. It is not hard to check that the case $t=3$ is ture. For the induction step, assume that the result is true for $t-1$, i.e.,

$$\begin{aligned}&(-1)^{t-1}\sum_{j<t-2}\tilde{f}(i_j)(i_j-1)+\cdots-\tilde{f}(S_3)(i_{t-3}-1)+(-1)^{t-2}\sum_{j<t-2}(o(i_j)-1)\\&+\cdots+(o(i_{t-3})-1)\tilde{f}(S_4)\\=&-(i_{t-3}+o(i_{t-3}))\tilde{h}(S_3),\end{aligned}$$

with which we rewrite the LHS of (2.8) as

$$\begin{aligned}&(i_{t-3}+o(i_{t-3}))\tilde{h}(S_3)\\&+(i_{t-2}-1)\left((-1)^t\tilde{f}(i_{t-2})+(-1)^{t-1}\sum_{j<t-2}\tilde{f}(i_j,i_{t-2})+\cdots-\tilde{f}(S_2)\right)\\&+(o(i_{t-2})-1)\left((-1)^{t-1}+(-1)^{t-2}\sum_{j<t-2}\tilde{f}(i_j)+\cdots+\tilde{f}(S_3)\right).\end{aligned}$$

Applying the induction hypothesis of (2.3), we have

$$\begin{aligned}&(-1)^t\tilde{f}(i_{t-2})+(-1)^{t-1}\sum_{j<t-2}\tilde{f}(i_j,i_{t-2})+\cdots-\tilde{f}(S_2)\\=&-\tilde{h}(S_3)\frac{i_{t-2}-i_{t-3}+1-o(i_{t-3})}{2^{o(i_{t-2})}}.\end{aligned}$$

Then the LHS of (2.8) is

$$\left((i_{t-3}+o(i_{t-3}))-(i_{t-2}-1)\frac{i_{t-2}-i_{t-3}+1-o(i_{t-3})}{2^{o(i_{t-2})}}+(o(i_{t-2})-1)\right)\tilde{h}(S_3)$$
$$=\tilde{h}(S_3)\frac{o(i_{t-2})(i_{t-3}+o(i_{t-3})+1)-i_{t-2}(i_{t-2}-i_{t-3}-o(i_{t-3}))}{2^{o(i_{t-2})}}$$
$$=\tilde{h}(S_3)\frac{-(i_{t-2}+o(i_{t-2}))(i_{t-2}-i_{t-3}-o(i_{t-2})-o(i_{t-3}))}{2^{o(i_{t-2})}}$$
$$=-(i_{t-2}+o(i_{t-2}))\tilde{h}(S_2),$$

where the last equation follows from the induction hypothesis of (2.2). □

By Lemma 2.6, the LHS of (2.6) is

$$\frac{i_{t-1}-o(i_{t-1})}{2^{o(i_{t-1})}}\tilde{h}(S_2)-\frac{i_{t-2}+o(i_{t-2})}{2^{o(i_{t-1})}}\tilde{h}(S_2)=\tilde{h}(S_1).$$

This completes the proof. □

Remark: By the recursive formula of flag h-vectors, it is easy to see that $\tilde{h}(S)\geq 0$ for any set S. Let $S=\{i_1,i_2,\ldots,i_t\}$. If there exists one $j\in\{1,2,\ldots,t\}$ such that $i_j=i_{j+1}-1$, then $\tilde{h}(S)=0$.

3. The sub-posets of the Square poset

In this section, we will consider certain sub-posets of the Square poset. The first kind is the *Tilt* poset which is pictorially defined by Figure 6. The Tilt poset is a sub-poset of the Square poset, in which only the points (a,b) with $a\equiv b \pmod 2$ are allowed. Specifically, its elements are

$$\{(a,b)\in\mathbb{N}\times\mathbb{N}|a\equiv b \pmod 2)\}$$

and $(a',b')\leq(a,b)$ if and only if $a'\leq a$ and $b'\leq b$. The second kind arises from considering the companion sub-poset of the Square poset where only the points (a,b) with $a\not\equiv b \pmod 2$ are allowed. This sub-poset is denoted by *Tilt* $^{(1)}$, pictorially shown in Figure 7. The last sub-poset to be considered is *Punc* as shown in Figure 8. The Punc poset is such a sub-poset of the Squre poset that defined by removing the points (a,b) for $a\equiv 1,\ b\equiv 1 \pmod 2$. In the remainder of this section, we will give the respective flag f- and h-vectors of these sub-posets.

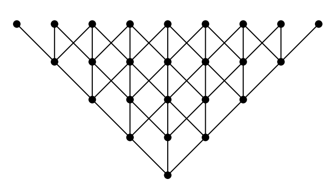

Figure 6. The Tilt poset

Theorem 3.1. *Let $S = \{i_1, i_2, \ldots, i_t\}$ be the set of rank numbers of Tilt poset. Then*

$$\tilde{f}(i_j) = 2i_j + 1,\ \tilde{f}(S) = (2i_1 + 1)(2i_2 - 2i_1 + 1)\cdots(2i_t - 2i_{t-1} + 1),$$

and

$$\tilde{h}(i_j) = 2i_j,\ \tilde{h}(S) = 2i_1(2i_2 - 2i_1 - 1)\cdots(2i_t - 2i_{t-1} - 1).$$

Proof. By definition of the Tilt poset, it is easy to see that the number of elements of rank i_j is $2i_j + 1$. Then $\tilde{f}(i_j) = 2i_j + 1$. Now the formula of $\tilde{f}(S)$ is true for $t = 1$. We then use induction on $t-1$, i.e., $\tilde{f}(i_1, \ldots, i_{t-1}) = (2i_1+1)(2i_2-2i_1+1)\cdots(2i_{t-1}-2i_{t-2}+1)$, and consider the induction step from $t-1$ to t. For every element A of rank i_{t-1}, there are $(2i_t - 2i_{t-1} + 1)$ elements of rank i_t, which are comparable with A. Hence, we get the formula of $\tilde{f}(S)$ by the induction hypothesis.

Since

$$\tilde{h}(i_j) = \tilde{f}(i_j) - 1 = 2i_j,$$

and

$$\tilde{h}(i_j, i_k) = 1 - \tilde{f}(i_j) - \tilde{f}(i_k) + \tilde{f}(i_j, i_k) = 2i_j(2i_k - 2i_j - 1),$$

it remains to show that the formula of $\tilde{h}(S)$ holds for $t > 2$. Let $S = S_0 = \{i_1, \ldots, i_t\}$ and $S_m = \{i_1, \ldots, i_{t-m}\}$, where $t > 2$. We use induction on m, assuming that

$$\tilde{h}(S_m) = 2i_1(2i_2 - 2i_1 - 1)\cdots(2i_{t-m} - 2i_{t-m-1} - 1),\ \text{for } m > 0. \tag{3.1}$$

Then we need to prove that this equality is correct for $m = 0$. By the definition of flag h-vector,

$$\begin{aligned}\tilde{h}(S) &= (-1)^t + (-1)^{t-1}\sum_{1\le j\le t}\tilde{f}(i_j) + (-1)^{t-2}\sum_{j<p}\tilde{f}(i_j, i_p) + \cdots + \tilde{f}(S)\\ &= -\tilde{h}(S_1) + (-1)^{t-1}\tilde{f}(i_t) + (-1)^{t-2}\sum_{j=1}^{t-1}\tilde{f}(i_j, i_t) + \cdots + \tilde{f}(S).\end{aligned}$$

If we could prove the following identity

$$\begin{aligned}&(-1)^{t-1}\tilde{f}(i_t) + (-1)^{t-2}\sum_{1\le j\le t-1}\tilde{f}(i_j, i_t) + (-1)^{t-3}\sum_{j<p}\tilde{f}(i_j, i_p, i_t)\\ &\quad + \cdots + \tilde{f}(S) = \tilde{h}(S_1)(2i_t - 2i_{t-1}),\end{aligned} \tag{3.2}$$

then the theorem follows from the fact that

$$\begin{aligned}\tilde{h}(S) &= -\tilde{h}(S_1) + \tilde{h}(S_1)(2i_t - 2i_{t-1})\\ &= \tilde{h}(S_1)(2i_t - 2i_{t-1} - 1).\end{aligned}$$

Before proving (3.2), we need to establish the following two necessary identities.

Lemma 3.2.

$$\tilde{f}(i_t) - \tilde{f}(i_j, i_t) = -\tilde{h}(i_j)(2i_t - 2i_j)$$

and

$$\tilde{f}(i_1, \dots, i_r, i_t) - \tilde{f}(i_1, \dots, i_r, i_{t-1}, i_t) = -\tilde{f}(S_{t-r})(2i_{t-1} - 2i_r)(2i_t - 2i_{t-1}).$$

Proof. By the formulae of flag f-vectors, it is easy to check that

$$\begin{aligned}\tilde{f}(i_t) - \tilde{f}(i_j, i_t) =& 2i_t + 1 - (2i_j + 1)(2i_t - 2i_j + 1)\\ =& - 2i_j(2i_t - 2i_j)\\ =& - \tilde{h}(i_j)(2i_t - 2i_j),\end{aligned}$$

and

$$\begin{aligned}&\tilde{f}(i_1, \dots, i_r, i_t) - \tilde{f}(i_1, \dots, i_r, i_{t-1}, i_t)\\ =&\tilde{f}(i_1, \dots, i_r)\left((2i_t - 2i_r + 1) - (2i_{t-1} - 2i_r + 1)(2i_t - 2i_{t-1} + 1)\right)\\ =& - \tilde{f}(S_{t-r})(2i_{t-1} - 2i_r)(2i_t - 2i_{t-1}).\end{aligned}$$

Lemma 3.2 is proved. □

We now turn to prove (3.2). By Lemma 3.2, it is easy to check that (3.2) holds for $t = 2$. Next we consider the case $t > 2$. By a rearrangement of the sum of, the LHS of (3.2) is

$$\begin{aligned}&(-1)^{t-1}\tilde{f}(i_t) + (-1)^{t-2}\sum_{j=1}^{t-1}\tilde{f}(i_j, i_t)\\ &+ (-1)^{t-3}\sum_{j<p\le t-1}\tilde{f}(i_j, i_p, i_t) + \cdots + \tilde{f}(S)\\ =&(-1)^{t-1}\left(\tilde{f}(i_t) - \tilde{f}(i_{t-1}, i_t)\right) + (-1)^{t-2}\sum_{j<t-1}\left(\tilde{f}(i_j, i_t) - \tilde{f}(i_j, i_{t-1}, i_t)\right)\\ &+ (-1)^{t-3}\sum_{j<p<t-1}\left(\tilde{f}(i_j, i_p, i_t) - \tilde{f}(i_j, i_p, i_{t-1}, i_t)\right) + \cdots\\ &- \left(\tilde{f}(i_1, \dots, i_{t-2}, i_t) - \tilde{f}(S)\right).\end{aligned}$$

By Lemma 3.2, it is easy to see that the difference between two flag f-vectors in each of the above bracket has a common factor $2i_t - 2i_{t-1}$. Compared with the RHS of (3.2), obviously it remains to prove the following equation

$$\begin{aligned}\tilde{h}(S_1) =&(-1)^t \times 2i_{t-1} + (-1)^{t-1}\sum_{j<t-1}\tilde{f}(i_j)(2i_{t-1} - 2i_j) + \cdots\\ &+ \tilde{f}(S_2)(2i_{t-1} - 2i_{t-2}).\end{aligned} \tag{3.3}$$

Collecting the coefficients of $2i_{t-1}$, the RHS of (3.3) is

$$\begin{aligned} & 2i_{t-1}\tilde{h}(S_2) + (-1)^t \sum_{j<t-1} \tilde{f}(i_j)2i_j \\ & + (-1)^{t-1} \sum_{j<p<t-1} \tilde{f}(i_j, i_p)2i_p + \cdots + (-1)\tilde{f}(S_2)2i_{t-2}. \end{aligned}$$

We now turn to prove our second claim.

Lemma 3.3.

$$\begin{aligned} & (-1)^t \sum_{j<t-1} \tilde{f}(i_j)2i_j + (-1)^{t-1} \sum_{j<p<t-1} \tilde{f}(i_j, i_p)2i_p \\ & + \cdots + (-1)\tilde{f}(S_2)2i_{t-2} = -\tilde{h}(S_2)(2i_{t-2} + 1). \end{aligned} \tag{3.4}$$

Proof. We use induction on $t > 2$. It is not hard to check that the case $t = 3$ is ture. For the induction step, assume that the result is true for $t - 1$, i.e.,

$$\begin{aligned} & (-1)^{t-1} \sum_{j<t-2} \tilde{f}(i_j)2i_j \\ & + (-1)^{t-2} \sum_{j<p<t-2} \tilde{f}(i_j, i_p)2i_p + \cdots + (-1)\tilde{f}(S_3)2i_{t-3} \\ = & - \tilde{h}(S_3)(2i_{t-3} + 1) \end{aligned}$$

Then the LHS of (3.4) is

$$\begin{aligned} & \tilde{h}(S_3)(2i_{t-3} + 1) \\ & + 2i_{t-2} \left((-1)^t \tilde{f}(i_{t-2}) + (-1)^{t-1} \sum_{j<t-2} \tilde{f}(i_j, i_{t-2}) + \cdots - \tilde{f}(S_2) \right). \end{aligned}$$

Applying the induction hypothesis of (3.2), we have

$$\begin{aligned} & (-1)^t \tilde{f}(i_{t-2}) + (-1)^{t-1} \sum_{j<t-2} \tilde{f}(i_j, i_{t-2}) + \cdots - \tilde{f}(i_1, \ldots, i_{t-2}) \\ = & - \tilde{h}(S_3)(2i_{t-2} - 2i_{t-3}). \end{aligned}$$

Then the LHS of (3.4) is

$$\begin{aligned} & \tilde{h}(S_3)(2i_{t-3} + 1) - 2i_{t-2}\tilde{h}(S_3)(2i_{t-2} - 2i_{t-3}) \\ = & - \tilde{h}(S_2)(2i_{t-2} + 1), \end{aligned}$$

where the last equation follows from (3.1). □

Hence the RHS of (3.3) is

$$2i_{t-1}\tilde{h}(S_2) - \tilde{h}(S_2)(2i_{t-2} + 1) = \tilde{h}(S_2)(2i_{t-1} - 2i_{t-2} - 1) = \tilde{h}(S_1).$$

This completes the proof. □

Corollary 3.4. *The number of saturated chains of the Tilt poset from* $(0,0)$ *to level* n *is* 3^n.

The flag f- and h-vectors of the poset Tilt $^{(1)}$ can be obtained in the similar way with the Tilt poset and, likewise, those of the Punc poset can be obtained in the similar way with the Rhomb poset. Hence we omit the proofs of the following two theorems.

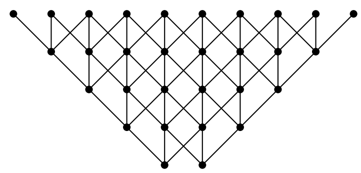

FIGURE 7. The Tilt$^{(1)}$ poset

Theorem 3.5. *Let* $S = \{i_1, i_2, \ldots, i_t\}$ *be the set of rank numbers of the Tilt* $^{(1)}$ *poset. Then*

$$\tilde{f}(i_j) = 2i_j + 2,\ \tilde{f}(S) = (2i_1 + 2)(2i_2 - 2i_1 + 1)\cdots(2i_t - 2i_{t-1} + 1),$$

and

$$\tilde{h}(i_j) = 2i_j + 1,\ \tilde{h}(S) = (2i_1 + 1)(2i_2 - 2i_1 - 1)\cdots(2i_t - 2i_{t-1} - 1).$$

Corollary 3.6. *The number of saturated chains of the Tilt* $^{(1)}$ *poset from level* 0 *(the bottom elements) to level* n *is* $2 \cdot 3^n$.

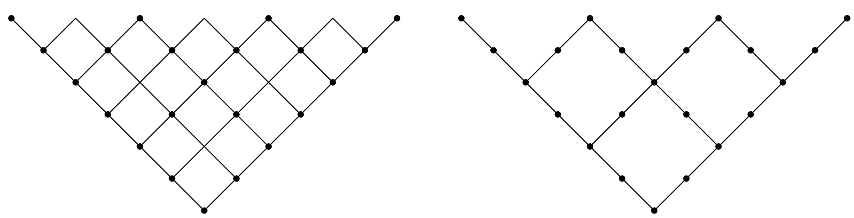

FIGURE 8. The Punc poset and its Hasse diagram

Theorem 3.7. *Let* $S = \{i_1, \ldots, i_t\}$ *and* $S_m = \{i_1, i_2, \ldots, i_{t-m}\}$ *be the sets of rank numbers of the Punc poset. Define*

$$o(i_k) = \begin{cases} 1, & i_k \text{ odd} \\ 0, & i_k \text{ even} \end{cases}$$

where $k = 1, \ldots, t$. *Then*

$$\tilde{f}(i_j) = \frac{i_j + 2 - o(i_j)}{2^{1-o(i_j)}}, \quad \tilde{f}(S) = \frac{\tilde{f}(S_1)(i_t - i_{t-1} + 2 - o(i_{t-1}) - o(i_t))}{2^{1-o(i_t)}},$$

and

$$\tilde{h}(i_j) = \frac{i_j}{2^{1-o(i_j)}}, \quad \tilde{h}(S) = \frac{\tilde{h}(S_1)(i_t - i_{t-1} - 2) - o(i_{t-1})\tilde{h}(S_2)}{2^{1-o(i_t)}}.$$

Corollary 3.8. *The number of saturated chains of the Punc poset from level* 0 *(the bottom element) to level* n *is* $2^{\lceil \frac{n}{2} \rceil}$.

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