

A NOTE ON TOTAL CO-INDEPENDENT DOMINATION
IN TREES

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ABSTRACT. A set D of vertices of a graph G is a total dominating set if every vertex of G is adjacent to at least one vertex in D . The total dominating set D is called a total co-independent dominating set if $V(G) \setminus D$ is an independent set and has at least one vertex. The minimum cardinality of any total co-independent dominating set is denoted by $\gamma_{t,coi}(G)$. In this paper, we show that, for any tree T of order n and diameter at least three, $n - \beta(T) \leq \gamma_{t,coi}(T) \leq n - |L(T)|$ where $\beta(T)$ is the maximum cardinality of any independent set and $L(T)$ is the set of leaves of T . We also characterize the families of trees attaining the extremal bounds above and show that the differences between the value of $\gamma_{t,coi}(T)$ and these bounds can be arbitrarily large for some classes of trees.

1. INTRODUCTION

Throughout this work we consider a finite simple graph $G = (V(G), E(G))$ of order $n = |V(G)|$. Given a vertex v of G , $N_G(v)$ represents the *open neighborhood* of v , *i.e.*, the set of all neighbors of v in G . The *degree* of v is $\deg_G(v) = |N_G(v)|$. If $S \subseteq V(G)$, then the *open neighborhood* of S is $N_G(S) = \cup_{v \in S} N_G(v)$. The *minimum* and *maximum degrees* of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For any two vertices u and v , the *distance* $d(u, v)$ between u and v is the length of a shortest $u - v$ path. The *diameter* of G is the largest possible distance between any two vertices of G , and is denoted by $diam(G)$. For any other graph theory terminology we follow the book [27].

A *dominating set* of a graph G is a set of vertices $D \subseteq V(G)$ such that every vertex of G not in D is adjacent to at least one vertex in D . The *domination number* of G is the minimum cardinality of a dominating set of G and is denoted by $\gamma(G)$. Domination in graphs is a classical topic, and nowadays one of the most active areas, of research in graph theory. This

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Received by the editors March 28, 2022, and in revised form August 11, 2022.

2000 *Mathematics Subject Classification.* 05C69, 05C05.

Key words and phrases. total domination number, independence number, total co-independent domination number, trees.

fact can be seen for instance through the more than 1620 articles published on the topic (more than 1070 of them in the last 10 years), according to the MathSciNet database with the queries: “domination number” or “dominating sets”. The two books [16, 17] contain a significant number of the most important results in the topic before this new century.

One interesting research activity on domination in graphs concerns its relationship with other graph parameters. A remarkable case regards vertex independence, *i.e.*, sets of vertices inducing an edgeless graph. The most natural relationship in this direction is clearly the mixing of both concepts which gives rise to the independent domination number, *i.e.*, the minimum cardinality of a dominating set which is also an independent set. Independent domination in graphs was formally introduced in [3] and [23] in the early 1960s (a fairly complete survey on this topic was recently published in [12]). On the other hand, some other investigations connecting domination and independence in graphs can easily be found in the literature. We remark here on two of them. The first one is searching for two disjoint sets in a graph, in which one of the sets is a maximal independent set and the second one a minimal dominating set (a particular case appears whether both sets form a partition of the vertex set of the graph). Several studies on this topic have been developed in the last few years. The Ph.D. thesis [21] contains several results and citations on the subject. The second one is related to finding the minimum cardinality of a dominating set which intersects every maximal independent set of a graph: independent transversal domination (see [13], where the theme was introduced, and also [1].)

Other popular studies on domination in graphs deal with modifying the domination property. One of the most popular parameters in this way is total domination, which differs from standard domination by the added property of dominating all the vertices of the graph, instead of only those outside of the set. It is then not surprising that total domination has been also related to independence in graphs. For instance, the total domination version of independent transversal domination is already known from [8, 7]. However, not much exists concerning finding two disjoint sets in which one of the sets is a total dominating set, and the other one, an independent set. A closely related idea to this one above was recently published in [25], where the concept of total co-independent dominating set was introduced. Such work contains just a few results which are not very deep. However, the concept is interesting and deserves a more detailed study.

More formally, a set $D \subseteq V(G)$ is a *total dominating set* of G if every vertex in $V(G)$ is adjacent to at least one vertex in D . The *total domination number* of G is the minimum cardinality of any total dominating set of G and is denoted by $\gamma_t(G)$. A $\gamma_t(G)$ -*set* is a total dominating set of cardinality $\gamma_t(G)$. For more information on total domination see the survey [18]. On the other hand, a set S of vertices is *independent* if S induces an empty graph. The *independence number* of G is the largest cardinality of an independent

set of G and is denoted by $\beta(G)$. An independent set of cardinality $\beta(G)$ is called a $\beta(G)$ -set.

A total dominating set D of a graph G is called a *total co-independent dominating set* if the set of vertices of the subgraph induced by $V(G) \setminus D$ is independent and not empty. The minimum cardinality among all total co-independent dominating sets of G is denoted by $\gamma_{t,coi}(G)$. A total co-independent dominating set of cardinality $\gamma_{t,coi}(G)$ is a $\gamma_{t,coi}(G)$ -set. These concepts were previously introduced and barely studied in [25], and also recently and more deeply, in [4, 5]. A slightly different version of this parameter was introduced in [20], and also recently studied in [22], where the condition of $V(G) \setminus D$ being not empty was not required. In these latter works, the total co-independent domination number is called the total outer-independent domination number. Even so, both parameters almost always behave in the same manner, we prefer to continue using the terminology of [25]. The results on this parameter would lead one to deduce some conclusions about the existence of partitions of the vertex set of a graph into a total dominating set and an independent set, as already mentioned.

Let T be a tree. A *leaf* of T is a vertex of degree one. A *support vertex* of T is a vertex adjacent to a leaf and not itself a leaf, and a *semi-support vertex* is a vertex adjacent to a support vertex that is neither a leaf nor a support vertex. By an *isolated support vertex* of T we mean an isolated vertex of the subgraph induced by the support vertices of T . The set of leaves is denoted by $L(T)$, the set of support vertices is denoted by $S(T)$, and the set of semi-support vertices is denoted by $SS(T)$. Moreover, $S^*(T)$ is the set of isolated support vertices of T . Also, a *double star* is a tree with exactly two adjacent vertices of degree larger than or equal to two and the remaining vertices are leaves.

Studies on characterizing domination related parameters in trees have been very popular in the last decade. One can find in the literature several works showing all the trees satisfying diverse properties. For instance, to just name a few of them, we remark some on examples, which are not likely an exhaustive list of the most remarkable and/or recent cases.

- In [26] it was proved that for any tree T of order n and l leaves, $\gamma_t(T) \geq (n - l + 2)/2$, and all the trees achieving such bound are given.
- In [24], a characterization of the family of trees with equal total domination and paired-domination numbers was given.
- In [14], a characterization of trees with equal domination and independent domination numbers was presented.
- In [10] it was proved that the restrained domination number of a tree T of order n is bounded below by $\lceil (n + 2)/3 \rceil$ and all the extremal trees achieving this lower bound were constructively characterized.

- In [2], a constructive characterization of the trees for which the Roman domination number strongly equals the weak Roman domination number was given.
- In [19], trees with equal total domination and game total domination number were characterized.
- In [6], a constructive characterization of vertex cover Roman trees was given, that is, trees whose outer independent Roman domination number equals twice its vertex cover number.
- In [9], three different characterizations concerning weak Roman domination in trees were presented.

Other styles of characterizations of domination parameters in trees were presented in [11, 15]. In this sense, in the present work we propose to improve the visibility of this new parameter, namely the total co-independent domination number, throughout characterizing several families of trees achieving some specific values of this mentioned parameter.

The total co-independent domination number of a graph G has been introduced in [25], where a few of its combinatorial properties were determined. Among them, a couple of almost trivial bounds in terms of $\beta(G)$ and the order of G were proved for $\gamma_{t,coi}(G)$. As an example, for any graph G of order n , $n - \beta(G) \leq \gamma_{t,coi}(G) \leq n - 1$. It is readily seen that such bounds could be improved in a number of situations. For instance, in the case of a tree T , this can be done for the upper bound as we next show.

Theorem 1.1. *For any tree T of order n and $\text{diam}(T) \geq 3$,*

$$n - \beta(T) \leq \gamma_{t,coi}(T) \leq n - |L(T)|.$$

Proof. The lower bound was already given in [25]. Moreover, the upper bound follows by the fact that the set $V(T) \setminus L(T)$ is a total co-independent dominating set, since T has diameter at least three. \square

One can immediately think about characterizing the trees achieving the bounds for the total co-independent domination number given above. This is accomplished in the next section. In order to easily proceed with our exposition, we say that a tree T belongs to the family \mathcal{T}_β , if $\gamma_{t,coi}(T) = n - \beta(T)$ and that T is in the family \mathcal{T}_L , if $\gamma_{t,coi}(T) = n - |L(T)|$. Note that $\mathcal{T}_L = \mathcal{T}_\beta$ if and only if $L(T)$ is a $\beta(T)$ -set.

2. THE CHARACTERIZATIONS

In order to provide a constructive characterization of the trees belonging to the family \mathcal{T}_β we need to introduce some operations to be made over a tree. In this regard, by attaching a path P to a vertex v of T we mean adding the path P and joining v to a vertex of P .

Operation O_1 : Attach a path P_1 to a vertex of T which is in some $\gamma_{t,coi}(T)$ -set.

Operation O_2 : Attach a path P_2 to a vertex of T which is in some $\gamma_{t,coi}(T)$ -set.

Operation O_3 : Attach a path P_4 to a vertex v of T which is in some $\gamma_{t,coi}(T)$ -set, by joining v to a leaf of P_4 .

Operation O_4 : Attach a path P_4 to a vertex v of T which is in some $\beta(T)$ -set, by joining v to a support vertex of P_4 .

Let \mathcal{T} be the family of trees defined as $\mathcal{T} = \{T \mid T \text{ is } P_4 \text{ or is a tree obtained from } P_4 \text{ by a finite sequence of operations } O_1, O_2, O_3, O_4\}$. We first show that every tree in the family \mathcal{T} belongs to the family \mathcal{T}_β .

Lemma 2.1. *If $T \in \mathcal{T}$, then $T \in \mathcal{T}_\beta$.*

Proof. We proceed by induction on the number $r(T)$ of operations required to construct the tree T . If $r(T) = 0$, then $T = P_4$ and $T \in \mathcal{T}_\beta$. This establishes the base case. Hence, we now assume that $k \geq 1$ is an integer and that each tree $T' \in \mathcal{T}$ with $r(T') < k$ satisfies that $T' \in \mathcal{T}_\beta$. Let $T \in \mathcal{T}$ be a tree with $r(T) = k$. Then, T can be obtained from a tree $T' \in \mathcal{T}$ with $r(T') = k - 1$ by one of the operations O_1, O_2, O_3 or O_4 . We shall prove that $T \in \mathcal{T}_\beta$. To this end, as $T' \in \mathcal{T}_\beta$, let D' be a $\gamma_{t,coi}(T')$ -set containing no leaves and let $B' = V(T') \setminus D'$ be a $\beta(T')$ -set containing all leaves. We consider the following situations.

Case 1. T is obtained from T' by operation O_1 . Assume T is obtained from T' by adding the vertex u and the edge uv , where $v \in D'$. Notice that u is a leaf of T and that $B' \cup \{u\}$ is a $\beta(T)$ -set, otherwise there would be an independent set in T' with one vertex more than in B' , since only one vertex has been added to T' (to obtain T) that would increase the value for $\beta(T)$ by more than one. On the other hand, D' is a total co-independent dominating set of T . Thus, $\gamma_{t,coi}(T) \leq |D'| = |V(T)| - (\beta(T') + 1) = |V(T)| - \beta(T)$, and by Theorem 1.1, $\gamma_{t,coi}(T) = |V(T)| - \beta(T)$, which means $T \in \mathcal{T}_\beta$.

Case 2. T is obtained from T' by operation O_2 . Assume T is obtained from T' by adding the path u_1u_2 and the edge u_1v where $v \in D'$. Note that the vertex u_2 is a leaf of T , and that u_1 is its support vertex. So, $B' \cup \{u_2\}$ is a $\beta(T)$ -set, otherwise there would be an independent set in T' with one vertex more than in B' , since only one vertex from the pair u_1, u_2 could be added to any independent set of T' to get an independent set of T of larger cardinality than that of B' . It is now not difficult to see that $D = D' \cup \{u_1\}$ is a total co-independent dominating set of T . Hence, $\gamma_{t,coi}(T) \leq |D| = |D'| + 1 = |V(T')| - \beta(T') + 1 = |V(T)| - 2 - (\beta(T) - 1) + 1 = |V(T)| - \beta(T)$. Therefore, by Theorem 1.1, $\gamma_{t,coi}(T) = |V(T)| - \beta(T)$ and $T \in \mathcal{T}_\beta$.

Case 3. T is obtained from T' by operation O_3 . Assume T is obtained from T' by adding the path $P_4 = h_1u_1u_2h_2$ to a vertex $v \in D'$ by the edge vh_1 . By using some similar reasons as in the case above, it is easily seen that $B = B' \cup \{h_1, h_2\}$ is a $\beta(T)$ -set and that $D = D' \cup \{u_1, u_2\}$ is a total co-independent dominating set of T . So, $\gamma_{t,coi}(T) \leq |D| = |D'| + 2 = |V(T')| - \beta(T') + 2 = (|V(T)| - 4) - (\beta(T) - 2) + 2 = |V(T)| - \beta(T)$. Therefore, by Theorem 1.1, $\gamma_{t,coi}(T) = |V(T)| - \beta(T)$ and $T \in \mathcal{T}_\beta$.

Case 4. T is obtained from T' by operation O_4 . Assume T is obtained from T' by adding a path $P_4 = h_1u_1u_2h_2$ to a vertex v of T' , which is in some

$\beta(T')$ -set, throughout the edge vu_1 . Note that the set $B = B' \cup \{h_1, h_2\}$ is an independent set of T . Moreover, since there can be at most two vertices of the path P_4 in any $\beta(T)$ -set, it must happen that B is a $\beta(T)$ -set, otherwise there would be an independent set in T' of cardinality larger than $\beta(T')$, which is not possible. Now, it is readily seen that $D = D' \cup \{u_1, u_2\}$ is a total co-independent dominating set in T . Thus, $\gamma_{t,coi}(T) \leq |D| = |D'| + 2 = |V(T')| - \beta(T') + 2 = (|V(T)| - 4) - (\beta(T) - 2) + 2 = |V(T)| - \beta(T)$. Therefore, again as above, by Theorem 1.1, $\gamma_{t,coi}(T) = |V(T)| - \beta(T)$ and $T \in \mathcal{T}_\beta$. \square

We now turn our attention to the opposite direction concerning the lemma above. In this sense, from now on we shall need the following terminology and notation in our results. Given a tree T and a set $S \subsetneq V(T)$, by $T - S$ we denote a tree obtained from T by removing from T all the vertices in S and all its incident edges (if $S = \{v\}$ for some vertex v , then we simply write $T - v$). For an integer $r \geq 2$, by Q_r we mean a graph which is obtained from a path $P_{r+2} = vs_1s_2 \dots s_r$ by attaching a path P_1 to every vertex of $P_{r+2} - v$. In Figure 1 we show the example of Q_5 .

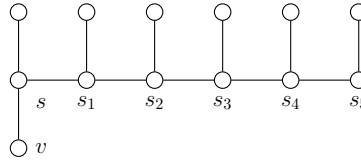


FIGURE 1. The structure of the tree Q_5 .

We next show that every tree of the family \mathcal{T}_β belongs to the family \mathcal{T} . To this end, we need the following observation.

Observation 2.2. *Let T be a tree of order at least three. If B is a $\beta(T)$ -set, then for every $v \in B$ there exists a vertex $u \in B$ such that $d(v, u) \leq 3$.*

Lemma 2.3. *If $T \in \mathcal{T}_\beta$, then $T \in \mathcal{T}$.*

Proof. We proceed by induction on the order $n \geq 4$ of the trees $T \in \mathcal{T}_\beta$. If T is a double star, then T can be obtained from P_4 by repeatedly applying operation O_1 . This establishes the base case. We assume that $k > 4$ is an integer and that each tree $T' \in \mathcal{T}_\beta$ with $|V(T')| < k$ satisfies that $T' \in \mathcal{T}$.

Let T be a tree such that $T \in \mathcal{T}_\beta$ and $|V(T)| = k$. Let (D, B) be a partition of $V(T)$ where D is a $\gamma_{t,coi}(T)$ -set containing no leaves and B is a $\beta(T)$ -set containing all leaves. We analyze the following situations.

Case 1: $|S(T)| < |L(T)|$. We consider a support vertex v that is adjacent to at least two leaves. Let $h \in N(v) \cap L(T)$ and $T' = T - h$. Since $h \in B$, it follows $B' = B \setminus \{h\}$ is an independent set in T' , which is moreover a $\beta(T')$ -set, since otherwise we would get an independent set of larger cardinality in T than $\beta(T)$, which is not possible. Moreover, it is easily seen that D is also a total co-independent dominating set of T' . Thus, $\gamma_{t,coi}(T') \leq |D| = \gamma_{t,coi}(T) = |V(T)| - \beta(T) = |V(T')| + 1 - (\beta(T') + 1) = |V(T')| - \beta(T')$.

So, by Theorem 1.1, $\gamma_{t,coi}(T') = |V(T')| - \beta(T')$ and $T' \in \mathcal{T}_\beta$. Now, by the induction hypothesis $T' \in \mathcal{T}$. Therefore, since T can be obtained from T' by operation O_1 , it follows $T \in \mathcal{T}$.

Case 2: $|S(T)| = |L(T)|$ and $|SS(T)| = 0$. In this case we note that $V(T) = S(T) \cup L(T)$ and it is easily observed that $|S(T)| \geq 3$, $S(T)$ is a $\gamma_{t,coi}(T)$ -set and $L(T)$ is a $\beta(T)$ -set. Let $s \in S(T)$ such that $|N(s) \cap S(T)| = 1$ (note that such s always exists) and let $h \in L(T)$ be the leaf adjacent to s . First notice that the vertex $x \in N(s) \cap S(T)$ must have at least two neighbors in $S(T)$, otherwise T is P_4 , which is not possible. Hence, we observe that $B' = B \setminus \{h\}$ is an independent set of $T' = T - \{h, s\}$. Moreover, by reasoning similar to the case above, B' is also a $\beta(T')$ -set. On the other hand, $S(T') = S(T) \setminus \{s\}$ is clearly a total co-independent dominating set of T' . So, $\gamma_{t,coi}(T') \leq |S(T')| = |S(T)| - 1 = |V(T)| - |L(T)| - 1 = |V(T')| + 2 - (|L(T')| + 1) - 1 = |V(T')| - \beta(T')$. Thus, by Theorem 1.1, $\gamma_{t,coi}(T') = |V(T')| - \beta(T')$ and $T' \in \mathcal{T}_\beta$. By the induction hypothesis $T' \in \mathcal{T}$. Since T can be obtained from T' by operation O_2 , we get $T \in \mathcal{T}$.

Case 3: $|S(T)| = |L(T)|$ and $|SS(T)| > 0$. Herein we denote by $P(x, y)$ the set of vertices on the unique path between x and y , including x and y . Let h, h' be two leaves at the maximum possible distance in T such that there is $v \in SS(T) \cap P(h, h')$ with $d(v, h) = 2$ or $d(v, h') = 2$. Without loss of generality assume that $d(v, h) = 2$ and let s be the support adjacent to h . Since $|S(T)| = |L(T)|$, we observe that $N(s) \subseteq S(T) \cup \{h, v\}$ and also every support vertex is adjacent to exactly one leaf. We have now some possible scenarios.

Case 3.1 $|N(s) \cap S(T)| = 1$. Hence, by the maximality of the path $P(h, h')$, it must happen that T has an induced subgraph isomorphic to a graph Q_r , as previously described, obtained from the vertices v, s, h and some supports, say $s_1, s_2, \dots, s_r \in S(T)$, with the leaves h_1, h_2, \dots, h_r , adjacent to the supports s_1, s_2, \dots, s_r , respectively.

Assume $r = 1$. Note that $s, s_1 \in D$ and $h, h_1 \in B$. We first consider $v \in D$. Let $T' = T - \{h_1, s_1\}$, $D' = D \setminus \{s_1\}$ and $B' = B \setminus \{h_1\}$. Clearly, B' is a $\beta(T')$ -set. Also, we note that D' is a total co-independent dominating set of T' . So, $\gamma_{t,coi}(T') \leq |D'| = |D| - 1 = (|V(T)| - |B|) - 1 = (|V(T')| + 2) - (\beta(T') + 1) - 1 = |V(T')| - \beta(T')$. Thus, by Theorem 1.1, $\gamma_{t,coi}(T') = |V(T')| - \beta(T')$ and $T' \in \mathcal{T}_\beta$. By inductive hypothesis $T' \in \mathcal{T}$, and since T can be obtained from T' by operation O_2 , we obtain $T \in \mathcal{T}$.

We now consider $v \in B$. Let $T' = T - \{s, h, s_1, h_1\}$, $D' = D \setminus \{s, s_1\}$ and $B' = B \setminus \{h, h_1\}$. We can again deduce B' is a $\beta(T')$ -set since at most two vertices of $\{s, h, s_1, h_1\}$ can belong to any independent set of T . Moreover, D' is a total co-independent dominating set of T' . So, $\gamma_{t,coi}(T') \leq |D'| = |D| - 2 = (|V(T)| - |B|) - 2 = (|V(T')| + 4) - (\beta(T') + 2) - 2 = |V(T')| - \beta(T')$ and, by Theorem 1.1, we get $\gamma_{t,coi}(T') = |V(T')| - \beta(T')$, which leads to

$T' \in \mathcal{T}_\beta$. By inductive hypothesis $T' \in \mathcal{T}$, and together with the fact that T can be obtained from T' by operation O_4 , the required result $T \in \mathcal{T}$ follows.

Assume now that $r \geq 2$. Since $s, s_1, \dots, s_r \in D$ and $h, h_1, \dots, h_r \in B$, if $T' = T - \{h_r, s_r\}$, then it is readily seen that $B' = B \setminus \{h_r\}$ is a $\beta(T')$ -set and that $D' = D \setminus \{s_r\}$ is a total co-independent dominating set of T' . By using a similar procedure as above ($r = 1$) we obtain $T' \in \mathcal{T}$ and, due to that T can be obtained from T' by operation O_2 , it follows $T \in \mathcal{T}$.

Case 3.2 $|N(s) \cap S(T)| > 1$. We note that this case is analogous to that one above whether $|N(s) \cap S(T)| = 1$ and $r \geq 2$.

Case 3.3: $|N(s) \cap S(T)| = 0$. Clearly, s has degree two. Also, it must happen $v, s \in D$ and $h \in B$. Assume the subgraph induced by $P(h, h')$ is $h s v u_1 u_2 \dots s' h'$, where $h, h' \in L(T)$ and $s, s' \in S(T)$. By the choice of $P(h, h')$, it follows that $N(v) \subseteq S(T) \cup \{u_1\}$. We consider again some possible scenarios.

Case 3.3.1: $|N(v) \cap S(T)| > 1$. In this case, the vertex v is also totally dominated by another support different from s . We note that $u_1 \in B$, because $B \cup \{v\}$ would be an independent set of cardinality larger than that of B . Hence, we consider the tree $T' = T - \{h, s\}$ and the sets $D' = D \setminus \{s\}$, $B' = B \setminus \{h\}$. It can be deduced as above that B' is a $\beta(T')$ -set. Moreover, D' is a total co-independent dominating set of T' . So, $\gamma_{t,coi}(T') \leq |D'| = |D| - 1 = (|V(T)| - |B|) - 1 = (|V(T')| + 2) - (\beta(T') - 1) - 1 = |V(T')| - \beta(T')$. Thus, by Theorem 1.1, we obtain $\gamma_{t,coi}(T') = |V(T')| - \beta(T')$, which means $T' \in \mathcal{T}_\beta$, and by inductive hypothesis $T' \in \mathcal{T}$. Furthermore T can be obtained from T' by operation O_2 , which allows to claim $T \in \mathcal{T}$.

Case 3.3.2: $|N(v) \cap S(T)| = 1$. Clearly s, v have degree two. We firstly consider the case when $N(u_1) = \{v, u_2\}$. By Observation 2.2 we note that $u_1 \in B$, and consequently $u_2 \in D$ since $T \in \mathcal{T}_\beta$. Let $T' = T - \{h, s, v, u_1\}$, $D' = D \setminus \{v, s\}$ and $B' = B \setminus \{h, u_1\}$. Now, similarly to Case 3.1 (whether $r = 1$ and $v \in B$) we can deduce B' is a $\beta(T')$ -set. Also, D' is a total co-independent dominating set of T' . So, $\gamma_{t,coi}(T') \leq |D'| = |D| - 2 = (|V(T)| - |B|) - 2 = (|V(T')| + 4) - (\beta(T') + 2) - 2 = |V(T')| - \beta(T')$, and, by Theorem 1.1, $\gamma_{t,coi}(T') = |V(T')| - \beta(T')$. Thus, $T' \in \mathcal{T}_\beta$. Hence, the inductive hypothesis $T' \in \mathcal{T}$ together with the fact that T can be obtained from T' by operation O_3 , we deduce $T \in \mathcal{T}$.

On the other hand, assume that there is a vertex $w \in N(u_1) \setminus \{v, u_2\}$. Since $u_1 \in B$, it must be also that $w \in D$, and consequently w has a neighbor belonging to D (which is clearly not u_1). Moreover, by the maximality of $P(h', h)$, we note that $w \notin SS(T)$ because $B' = B \setminus \{u_1\} \cup \{v, w\}$ is an independent set of cardinality larger than that of B . Hence, it follows $w \in S(T) \setminus S^*(T)$.

Let $x \in V(T)$ be a leaf adjacent to the support w . We note that T have an induced subgraph isomorphic to a graph Q_r , obtained from the vertices u_1, w, x and some supports, say $w_1, w_2, \dots, w_r \in S(T)$, with the leaves x_1, x_2, \dots, x_r , adjacent to the supports w_1, w_2, \dots, w_r , respectively.

By using a similar procedure to that of Case 3.1, it follows $T \in \mathcal{T}$ which completes the proof. \square

As an immediate consequence of Lemma 2.1 and Lemma 2.3 we have the following characterization.

Theorem 2.4. *Let T be a tree. Then $T \in \mathcal{T}_\beta$ if and only if $T \in \mathcal{T}$.*

We next see that all the operations O_1 to O_4 are required in the characterization above. First, it is easy to see that operation O_1 is required to obtain a double star from the path P_4 . The examples given in Figure 2 show that operations O_2 , O_3 and O_4 are also required.

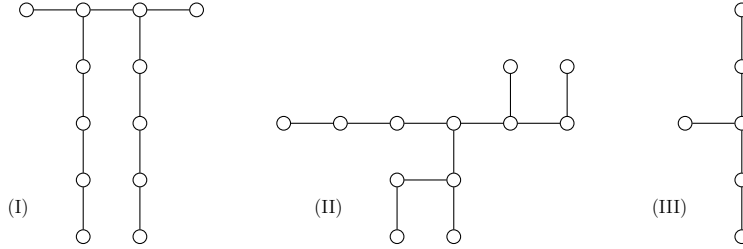


FIGURE 2. The tree (I) can only be obtained from P_4 by a sequence of operations O_3, O_3 or O_4, O_3 ; the tree (II) can only be obtained from P_4 by a sequence of operations O_4, O_4 or O_3, O_4 and the tree (III) can only be obtained from P_4 by the operation O_2 .

We next characterize those trees T satisfying that $T \in \mathcal{T}_L$. Before, we need to introduce the following result.

Theorem 2.5. [25] *A total co-independent dominating set D of a graph G is minimal if and only if for each vertex $v \in D$, one of the following conditions is satisfied.*

- (i) *There exists a vertex $u \in V(G)$ such that $N(u) \cap D = \{v\}$.*
- (ii) *There exists $w \in V(G) \setminus D$ adjacent to v .*

Theorem 2.6. *Let T be any tree with $\text{diam}(T) \geq 3$. Then $T \in \mathcal{T}_L$ if and only if the following conditions are satisfied.*

- (i) $V(T) = L(T) \cup S(T) \cup SS(T)$.
- (ii) *Every vertex $v \in SS(T)$ has a neighbor $u \in S^*(T)$ such that $N(u) \setminus L(T) = \{v\}$.*

Proof. We first assume that $T \in \mathcal{T}_L$. Let $D = V(T) \setminus L(T)$ be a $\gamma_{t,coi}(T)$ -set. If there exists a vertex $v \in D \setminus (SS(T) \cup S(T))$, then $D \setminus \{v\}$ is a total co-independent dominating set of T . Hence, $\gamma_{t,coi}(T) \leq |D \setminus \{v\}| = |D| - 1 = |V(T)| - |L(T)| - 1$, which contradicts the fact that $T \in \mathcal{T}_L$. Therefore, $D = S(T) \cup SS(T)$ and as a consequence, it follows that $V(T) =$

$L(T) \cup S(T) \cup SS(T)$, as desired. Now, let $v \in SS(T)$. Since D is a $\gamma_{t,coi}(T)$ -set and v is not adjacent to a leaf, by Theorem 2.5, there exists a vertex $u \in D$ such that $N(u) \cap D = \{v\}$. As a consequence, $u \in S^*(T)$ and $N(u) \setminus L(T) = \{v\}$, as desired.

On the other hand, we assume that conditions (i) and (ii) hold. Let D' be a $\gamma_{t,coi}(T)$ -set containing no leaves. By definition, it follows that $S(T) \subseteq D'$. Now, we proceed to prove that $SS(T) \subseteq D'$. Let $v \in SS(T)$. By condition (ii) there exists a vertex $u \in N(v) \cap S^*(T)$ such that $N(u) \setminus L(T) = \{v\}$. As $u \in D'$, $N(u) \cap D' \neq \emptyset$ and $D' \subseteq V(T) \setminus L(T)$, it follows that $v \in D'$, as required. Therefore, $S(T) \cup SS(T) \subseteq D'$, which implies that $|V(T)| - |L(T)| = |S(T)| + |SS(T)| \leq |D'| = \gamma_{t,coi}(T)$. By the upper bound given in Theorem 1.1 we obtain that $\gamma_{t,coi}(T) = |V(T)| - |L(T)|$, i.e., $T \in \mathcal{T}_L$, which completes the proof. \square

An interesting question that arises from the Theorems 2.4 and 2.6 is the following. Can the differences $\gamma_{t,coi}(T) - (|V(T)| - \beta(T))$ and $(|V(T)| - |L(T)|) - \gamma_{t,coi}(T)$ be as large as possible? We next give an affirmative answer to that question. In this sense, the following family of trees \mathcal{F} is required. Given two integers b, d , a tree $T_{b,d} \in \mathcal{F}$ is defined as follows.

- We begin with a tree T of order $b + d$ with vertex set $V(T) = \{u_1, \dots, u_b, v_1, \dots, v_d\}$.
- Attach two paths P_1 to every vertex of T .
- Attach a star S_3 to every vertex $u_i \in V(T)$, $i \in \{1, \dots, b\}$, by adding an edge between u_i and a leaf of the star S_3 .
- Attach a star S_3 with a subdivided edge to every vertex $v_i \in V(T)$, $i \in \{1, \dots, d\}$, by adding an edge between v_i and the leaf corresponding to the subdivided edge.

An example of a tree of the family \mathcal{F} is given in Figure 3.

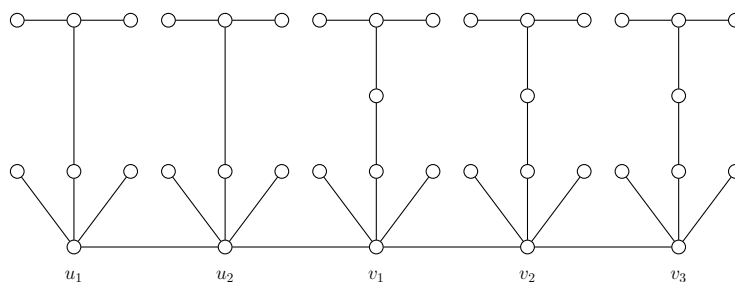


FIGURE 3. The tree $T_{2,3}$ by taking T as the path P_5 .

We next give several properties of the trees of the family \mathcal{F} , which are straightforward to observe and, according to this fact, the proofs are left to the reader.

Observation 2.7. *Let b, d be any two positive integers. Then,*

- (i) $T_{b,d}$ has order $7b + 8d$,

- (ii) $T_{b,d}$ has $4b + 4d$ leaves,
- (iii) $\beta(T_{b,d}) = 5b + 5d$,
- (iv) $\gamma_{t,coi}(T_{b,d}) = 3b + 3d$.

According to the results above, for any positive integers b, d we see that the tree $T_{b,d} \in \mathcal{F}$ satisfies the following relationships, which gives an answer to our previous question.

- $\gamma_{t,coi}(T_{b,d}) - (|V(T_{b,d})| - \beta(T_{b,d})) = b$.
- $(|V(T_{b,d})| - |L(T_{b,d})|) - \gamma_{t,coi}(T_{b,d}) = d$.

ACKNOWLEDGEMENT

The authors would like to thank the anonymous reviewers for the careful reading of the manuscript, and for all the suggestions which highly contributed to improving the quality and presentation of the article. The last author (Ismael G. Yero) has been partially supported by “Junta de Andalucía”, FEDER-UPO Research and Development Call, reference number UPO-1263769.

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