



## TILINGS OF A HONEYCOMB STRIP AND HIGHER ORDER FIBONACCI NUMBERS

TOMISLAV DOŠLIĆ AND LUKA PODRUG

**ABSTRACT.** In this paper, we explore two types of tilings of a honeycomb strip and derive some closed-form formulas for the number of tilings. Furthermore, we obtain some new identities involving tribonacci numbers, Padovan numbers and Narayana’s cow sequence and provide combinatorial proofs for several known identities about those numbers.

### 1. INTRODUCTION

Tilings or tessellations appear as natural solutions of many practical problems and their aesthetic appeal motivates the interest that goes way beyond the limits of their practical relevance. In mathematics, tiling-related problems appear in almost all areas, ranging from purely recreational settings of plane geometry all the way to the deep questions of eigenvalue count asymptotics for boundary-value problems in higher-dimensional spaces [12, 13]. Many of those problems, formulated in simple and intuitive terms and seemingly innocuous, quickly turn out to be quite intractable in their generality. That motivates interest in their restricted versions that might be more accessible. In this paper, we look at several such restricted problems when the area being tiled has a given structure and the allowed tiles belong to a small set of given shapes. In particular, we consider various problems involving tiling a narrow strip of the hexagonal lattice in the plane with several types of tiles made of regular hexagons. Similar problems for strips in square and triangular lattices have been considered in several recent papers [1, 4, 5, 9].

The substrate (i.e., the area to be tiled) is a honeycomb strip  $H_n$  composed of  $n$  regular hexagons arranged in two rows in which the hexagons are numbered starting from the bottom left corner, as shown in Figure 1 for  $n = 12$ . The number of hexagons in the strip is called its length. The choice

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of the substrate might seem arbitrary, but it provides a neat visual model for a linear array of locally interacting units with additional longer-range connections: the inner dual of a strip of length  $n$  is, in fact,  $P_n^2$ , the path on  $n$  vertices with edges between all vertices at distance less than or equal to 2 in  $P_n$ . Another way to look at it is as the ladder graph with descending diagonals, another familiar structure. Clearly, tilings with monomers and dimers in the strip correspond to matchings in its inner dual, thus enabling us to directly transfer known results about matchings into our context. We refer the reader to the classical monograph by Lovász and Plummer [10] for all necessary details on matchings.

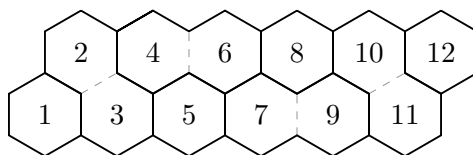


FIGURE 1. A tiling of a honeycomb strip of length 12 using 4 dimers.

We start by examining the tilings of such strips by monomers (i.e., single hexagons) and dimers made of two hexagons joined along an edge. Such tilings have been considered recently by Dresden and Jin [4], who found that the total number of such tilings is given by the tetranacci numbers. We refine their results in several ways. First, in Section 2, we obtain the formula for the number of such tilings with a specified number of dimers. Then we consider tilings with colored monomers and dimers in Section 3. Along the way we obtain combinatorial proofs for generalizations of several identities involving tetranacci numbers from the paper by Dresden and Jin; we present them in Section 4. Section 5 is devoted to another type of restricted tilings of the honeycomb strip. There we prohibit horizontal dimers but allow trimers of consecutive hexagons. The total number of such tilings is given in terms of tribonacci numbers, and Padovan and Narayana's cow numbers appear as special cases. Combinatorial proofs of some related identities are presented in Section 6. The paper is closed with some remarks listing some open problems and indicating several possible directions for future work.

## 2. TILING A HONEYCOMB STRIP WITH EXACTLY $k$ DIMERS

In this section, we consider a honeycomb strip of length  $n$  and its tilings by hexagonal monomers and dimers shown in Figure 2. We are interested in the number of such tilings with a given number of dimers. The dimers can be in any position; in Figure 2 we see a descending, a horizontal and an ascending dimer, from left to right, respectively. Ascending and descending dimers will be both called slanted when their exact orientation is not important. We denote the number of all possible tilings of a honeycomb strip of length  $n$  using exactly  $k$  dimers by  $c_{n,k}$ .

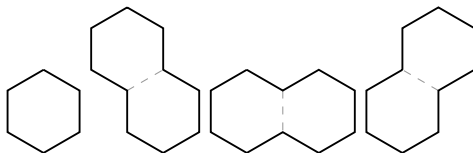


FIGURE 2. Monomer and three possible positions of a dimer tile.

Dresden and Jin [4] proved that the total number of all possible ways to tile a strip with monomers and dimers  $h_n$  satisfies recursion

$$(2.1) \quad h_n = h_{n-1} + h_{n-2} + h_{n-3} + h_{n-4}$$

with initial values  $h_1 = 1$ ,  $h_2 = 2$ , and  $h_3 = 4$ . It makes sense to define  $h_0 = 1$ , accounting for the only possible tiling (the empty one) of the empty honeycomb strip. Their recurrence is the same as the recurrence for the tetranacci numbers  $Q_n$  (A000078 in [11]) with shifted initial values. Hence,  $h_n = Q_{n+3}$ . We wish to determine  $c_{n,k}$ , the number of such tilings using exactly  $k$  dimers, and hence  $n - 2k$  monomers. It is easy to see that  $c_{n,k} = 0$  for  $k > \lfloor \frac{n}{2} \rfloor$ , since the strip with  $n$  hexagons can contain at most  $\lfloor \frac{n}{2} \rfloor$  dimers. On the lower end, there is only one tiling without dimers, so  $c_{n,0} = 1$  for all  $n$ . By stacking  $k$  dimers at the beginning of the strip, it is always possible to tile the remainder by monomers, so it follows that all  $c_{n,k}$  for  $k$  between 1 and  $\lfloor \frac{n}{2} \rfloor$  will be strictly positive. Hence the numbers  $c_{n,k}$  will be arranged in a triangular array without internal zeros. In Table 1 we give the list of initial values that can be easily verified.

$$\begin{aligned} c_{0,0} &= 1 \\ c_{1,0} &= 1 \\ c_{2,0} &= 1 & c_{2,1} &= 1 \\ c_{3,0} &= 1 & c_{3,1} &= 3 \end{aligned}$$

TABLE 1. Initial values of  $c_{n,k}$ .

In the next theorem, we give a recurrence relation for  $c_{n,k}$ .

**Theorem 2.1.** *Let  $n \geq 4$  be an integer and  $c_{n,k}$  be the number of ways to tile a honeycomb strip of length  $n$  by using exactly  $k$  dimers and  $n - 2k$  monomers. Then the numbers  $c_{n,k}$  satisfy the recurrence relation*

$$(2.2) \quad c_{n,k} = c_{n-1,k} + c_{n-2,k-1} + c_{n-3,k-1} + c_{n-4,k-2}$$

with the initial conditions given in Table 1.

*Proof.* We consider an arbitrary tiling which uses  $k$  dimers and note that the  $n$ -th hexagon can be tiled either by a dimer or by a monomer. The number of all such tilings with the last hexagon tiled by a monomer is  $c_{n-1,k}$ , since the number of dimers  $k$  remains the same. If the last hexagon is part of a dimer, then we distinguish two possible situations: either the dimer is slanted or it

is horizontal. The number of tilings ending with a slanted dimer is  $c_{n-2,k-1}$  since the last dimer increases the length of a strip by two and the number of dimers by one. If the dimer is horizontal, it means that it must cover the  $(n-2)$ -nd and the  $n$ -th hexagon. In that case, we have two subcases: either the  $(n-1)$ -st hexagon is tiled by monomer, and the rest of the strip can be tiled in  $c_{n-3,k-1}$  ways, or  $(n-1)$ -st hexagon forms a dimer with  $(n-3)$ -rd hexagon, and rest of the strip can be tiled in  $c_{n-4,k-2}$  ways. Described cases are illustrated in Figure 3 from left to right, respectively.

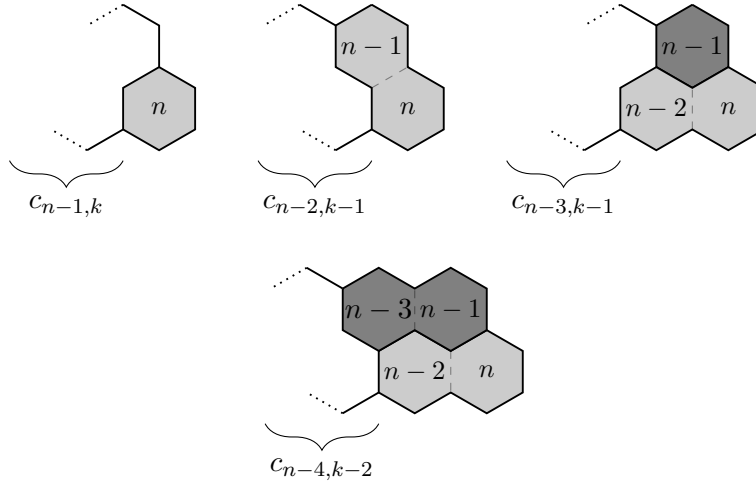


FIGURE 3. All possible endings of a tiled honeycomb strip of length  $n$ .

Since the listed cases and subcases are disjoint and describe all possible situations, the total number of tilings is the sum of the respective counting numbers i.e.,  $c_{n,k} = c_{n-1,k} + c_{n-2,k-1} + c_{n-3,k-1} + c_{n-4,k-2}$ , which proves our theorem.  $\square$

We are now able to list the initial rows of the triangle of  $c_{n,k}$ , which we do in Table 2 below.

The triangle of Table 2 appears as the sequence A101350 in [11]. Its left-most column consists of all 1's, counting the unique tilings without dimers. The second column seems to be given by  $c_{n,1} = 2n - 3$ . Indeed, the only dimer in the tiling can cover either hexagons  $(i, i + 1)$  for  $1 \leq i \leq n - 1$  or hexagons  $(i, i + 2)$  for  $1 \leq i \leq n - 2$ , resulting in  $2n - 3$  possible tilings. As expected, the rows of the triangle sum to the (shifted) tetranacci numbers,

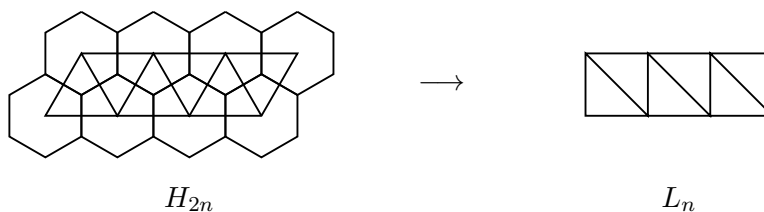
$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} c_{n,k} = Q_{n+3},$$

since by disregarding values  $k$ , recurrence (2.2) becomes the defining recurrence for the tetranacci numbers. The appearance of the Fibonacci numbers

| $n/k$ | 0 | 1  | 2  | 3  | 4 | 5 | 6 | ... | $Q_n$ |
|-------|---|----|----|----|---|---|---|-----|-------|
| 0     | 1 |    |    |    |   |   |   |     | 1     |
| 1     | 1 |    |    |    |   |   |   |     | 1     |
| 2     | 1 | 1  |    |    |   |   |   |     | 2     |
| 3     | 1 | 3  |    |    |   |   |   |     | 4     |
| 4     | 1 | 5  | 2  |    |   |   |   |     | 8     |
| 5     | 1 | 7  | 7  |    |   |   |   |     | 15    |
| 6     | 1 | 9  | 16 | 3  |   |   |   |     | 29    |
| 7     | 1 | 11 | 29 | 15 |   |   |   |     | 56    |
| 8     | 1 | 13 | 46 | 43 | 5 |   |   |     | 108   |

TABLE 2. The initial values of  $c_{n,k}$ .

as the rightmost diagonal,  $c_{2n,n} = F_{n+1}$ , can be readily explained by looking at the inner dual of the strip. As mentioned before, it is the ladder graph with the descending diagonal in each square, as shown in Figure 4. Clearly, tilings with  $n$  dimers correspond to perfect matchings in the inner dual.

FIGURE 4. Hexagonal strip of a length  $2n$  and its inner dual

simple parity argument dictates that no diagonal can participate in such a perfect matching. By omitting the diagonals we are left with a ladder graph and it is a well-known folklore result that perfect matchings in ladder graphs are counted by Fibonacci numbers. Somewhat less obvious is the appearance of the convolution of Fibonacci numbers and shifted Fibonacci numbers as the first descending subdiagonal,

$$c_{2n+1,n} = \sum_{k=0}^n F_{k+1} F_{n+2-k} = A023610(n),$$

but it follows by observing that the only monomer breaks the strip into two pieces each of which can be tiled by dimers only, and the number of such tilings is obtained by summing the corresponding products, hence leading to convolution. There are no formulas in the OEIS for other columns or diagonals. In the rest of this section, we determine formulas for all elements of the triangle  $c_{n,k}$ .

It is well known that for the Fibonacci numbers, one has

$$c_{2n,n} = \sum_{m=0}^n \binom{n-m}{m}.$$

By writing this as

$$c_{2n,n} = \sum_{m=0}^n \binom{n-m}{m} \binom{n-m}{0},$$

and by noting that a similar formula

$$c_{2n+1,n} = \sum_{m=0}^{n-1} \binom{n-m}{m} \binom{n-m}{1},$$

can be readily verified by induction, it becomes natural to consider

$$\sum_{m=0}^{n-k} \binom{n-m}{m} \binom{n-m}{k},$$

as the formula for the elements on descending diagonals. By shifting the indices  $n \rightarrow n - k$  and  $k \rightarrow n - 2k$  one arrives at expression for  $c_{n,k}$ .

**Theorem 2.2.** *The number of ways to tile a honeycomb strip of length  $n$  using  $k$  dimers and  $n - 2k$  monomers is equal to*

$$(2.3) \quad c_{n,k} = \sum_{m=0}^k \binom{n-k-m}{m} \binom{n-k-m}{n-2k}.$$

Theorem 2.2 can be proven by induction, but we prefer to present a combinatorial proof. To do that we need some new terms and one lemma.

We say that a tiling of a honeycomb strip is *breakable* at the position  $k$  if a given tiling can be divided into two tiled strips, the first strip of length  $k$  and the second of length  $n - k$ . Note that breaking the strip is only allowed along the edge of the tile. If no such  $k$  exists, we say that tiling is *unbreakable*.

For example, if the first two hexagons form a dimer, the tiling is unbreakable at position 1, since it is not allowed to break a tiling through the dimer. As an example, Figure 5 illustrates all breakable positions of a given tiling.

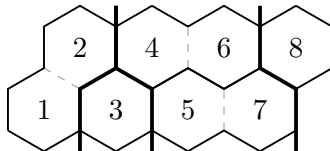


FIGURE 5. Tiling of a honeycomb strip that is breakable at positions 2, 3 and 7.

**Lemma 2.3.** *For  $n > 4$ , every tiled strip of length  $n$  is breakable into four types of unbreakable tiled strips: length-one strip tiled with a single monomer, length-two strip tiled with a single dimer, length-three strip tiled with a horizontal dimer and a monomer, and length-four strip tiled with two horizontal dimers.*

*Proof.* Every left or right slanted dimer forms a strip of length two. When removed, we are left with smaller strips, each of them tiled with hexagons and horizontal dimers. Every horizontal dimer occupies positions in the form  $\{i, i + 2\}$ . If position  $i + 1$  is occupied by a monomer, hexagons in position  $i, i + 1$  and  $i + 2$  form a length-three tiled strip. If position  $i + 1$  is occupied by another horizontal dimer, that dimer can occupy positions  $i - 1, i + 1$  or  $i + 1, i + 3$ . Either way, those two horizontal dimers form a length-four tiled strip. After they are removed, we are left with only monomers, where each monomer forms a simple tiled strip of length one. Those are the only four types of unbreakable tilings. They are illustrated in Figure 6.  $\square$

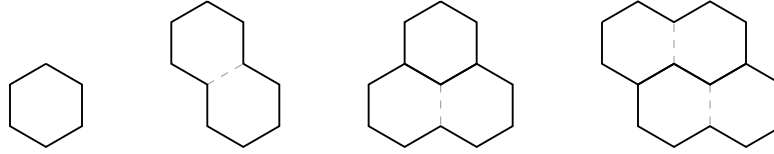
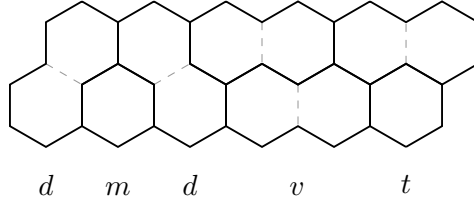


FIGURE 6. All unbreakable types of tiled strip. The second and the fourth can be left or right-slanted, and the third can be upside down, depending on the parity of the position.

*Proof of Theorem 2.2.* We denote types of tiled strip from Figure 6 by  $M$ ,  $D$ ,  $T$  and  $V$ , from left to right, respectively. By Lemma 2.3, an arbitrary tiling of a strip  $H_n$  of length  $n > 4$  can be broken into those four types of unbreakable tilings. Breaking a given tiled strip into unbreakable strips produces a unique number of tiled strips of each type. So, let  $k_1$  denote the number of strips of  $D$ ,  $k_2$  the number of strips of  $T$ ,  $k_3$  the number of strips of  $V$ , and since the strip has length  $n$ , what remains are  $n - 2k_1 - 3k_2 - 4k_3$  strips of  $M$ . Now we establish 1-1 correspondence between two sets: the first set, that contains all tilings of a strip  $H_n$  which by breaking produce  $k_1$  strips of type  $D$ ,  $k_2$  strips of type  $T$ ,  $k_3$  strips of type  $V$  and  $n - 2k_1 - 3k_2 - 4k_3$  strips of type  $M$ , and the second set that contains all permutations with repetition of a set with  $n - k_1 - 2k_2 - 3k_3$  elements, where there are  $k_1$  elements of type  $d$ ,  $k_2$  elements of type  $t$ ,  $k_3$  elements of type  $v$  and  $n - 2k_1 - 3k_2 - 4k_3$  elements of type  $m$ . From an arbitrary permutation, we obtain the corresponding tiling as follows: we replace elements  $v$ ,  $t$ ,  $d$  and  $m$  with tiled strips of type  $V$ ,  $T$ ,  $D$  and  $M$ , respectively. For example, permutation  $dmdvt$  yields the tiling shown in Figure 7. The way to obtain a permutation from a given tiling is obvious.

The number of all permutation in the second set is

FIGURE 7. Tiling that corresponds to permutation  $dmdvt$ .

$$\frac{(n-k_1-2k_2-3k_3)!}{k_1!k_2!k_3!(n-2k_1-3k_2-4k_3)!}.$$

One can easily verify that this expression can be written as

$$\binom{n-k_1-2k_2-3k_3}{k_1} \binom{n-2k_1-2k_2-3k_3}{k_3} \binom{n-2k_1-2k_2-4k_3}{k_2}.$$

Since we are interested in the number  $c_{n,k}$  which denotes the number of ways to tile a length- $n$  strip that contains exactly  $k$  dimers, note that  $k_1 + k_2 + 2k_3$  must be equal to  $k$ . Hence, we have

$$c_{n,k} = \sum_{k_1+k_2+2k_3=k} \binom{n-k_1-2k_2-3k_3}{k_1} \binom{n-2k_1-2k_2-3k_3}{k_3} \binom{n-2k_1-2k_2-4k_3}{k_2}.$$

By introducing a new index of summation  $m = k_2 + k_3$  and by substitutions  $k_2 = m - k_3$  and  $k_1 = k - k_2 - 2k_3 = k - m - k_3$  we obtain:

$$\begin{aligned} c_{n,k} &= \sum_{m=0}^k \sum_{k_3=0}^m \binom{n-k-m}{k-m-k_3} \binom{n-2k+k_3}{k_3} \binom{n-2k}{m-k_3} \\ &= \sum_{m=0}^k \sum_{k_3=0}^m \binom{n-k-m}{n-2k+k_3} \binom{n-2k+k_3}{n-2k} \binom{n-2k}{m-k_3}. \end{aligned}$$

Finally, by using identity  $\binom{n}{k} \binom{k}{m} = \binom{n}{m} \binom{n-m}{k-m}$  on the first two binomial coefficients and Vandermonde's convolution  $\sum_k \binom{n}{k} \binom{r}{m-k} = \binom{n+r}{m}$  (see [8]), we arrive to

$$\begin{aligned} c_{n,k} &= \sum_{m=0}^k \binom{n-k-m}{n-2k} \left( \sum_{k_3=0}^m \binom{k-m}{k_3} \binom{n-2k}{m-k_3} \right) \\ &= \sum_{m=0}^k \binom{n-k-m}{n-2k} \binom{n-k-m}{m}, \end{aligned}$$

which concludes our proof.  $\square$

Since the row sums in Table 2 are tetranacci numbers, Theorem 2.2 gives us identity

$$Q_{n+3} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{m=0}^k \binom{n-k-m}{m} \binom{n-k-m}{n-2k}.$$



### 3. TILINGS OF HONEYCOMB STRIP WITH COLORED DIMERS AND MONOMERS

Katz and Stenson [9] used colored squares and dominos to tile  $(2 \times n)$ -rectangular board and obtained a recursive relation for the number of all ways to tile a board. They also proved some combinatorial identities involving the number of such tilings. In this section, we do a honeycomb strip analog. We continue to count tilings of a hexagon strip with dimers and monomers, but we allow  $a$  different colors for monomers and  $b$  different colors for dimers. Let  $h_n^{a,b}$  denotes the number of all different tilings of a strip with  $n$  hexagons. It is convenient to define  $h_0^{a,b} = 1$ . We start with initial values illustrated in Figure 8. One can easily see that  $h_1^{a,b} = a$  since we have  $a$  colors to choose from for a monomer. Similarly,  $h_2^{a,b} = a^2 + b$ , since we can tile a strip with two monomers in  $a^2$  ways or with one dimer in  $b$  ways. For  $n = 3$ , note that if we use only monomers, we can choose colors in  $a^3$  ways, and if we use one dimer and one monomer, we can put the dimer in 3 different positions, and for each of those positions we can choose colors for tiles in  $ab$  ways. Hence,  $h_3^{a,b} = a^3 + 3ab$ .

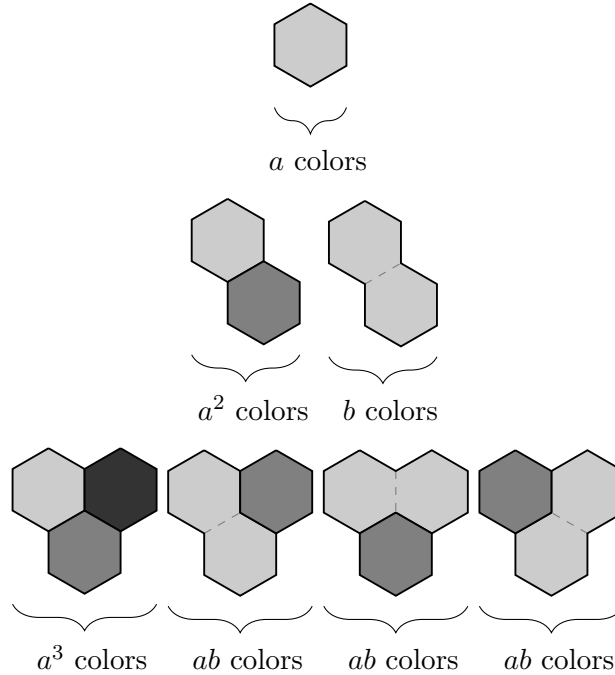


FIGURE 8. All possible tilings for  $n = 1, 2, 3$ .

In the next theorem, we give a recursive relation for  $h_n^{a,b}$ .

**Theorem 3.1.** *For  $n \geq 4$ , the number of all possible tilings of the honeycomb strip containing  $n$  hexagons with  $a$  different kinds of monomers and  $b$*

different kinds of dimers satisfies the recursive relation

$$h_n^{a,b} = a \cdot h_{n-1}^{a,b} + b \cdot h_{n-2}^{a,b} + ab \cdot h_{n-3}^{a,b} + b^2 \cdot h_{n-4}^{a,b}$$

with the initial conditions  $h_0^{a,b} = 1$ ,  $h_1^{a,b} = a$ ,  $h_2^{a,b} = a^2 + b$ , and  $h_3^{a,b} = a^3 + 3ab$ .

*Proof.* The proof is similar to the proof of Theorem 2.1, but here we must also pay attention to the colors. We consider an arbitrary tiling and note that  $n$ -th hexagon can either be tiled by a monomer or a dimer. In the case when  $n$ -th hexagon is tiled by a monomer, the rest of the strip can be tiled in  $h_{n-1}^{a,b}$  ways, but the monomer can be colored in  $a$  different ways, which gives us the total of  $a \cdot h_n^{a,b}$  possible ways. If the last hexagon is part of a dimer, then we distinguish two possible situations: either the dimer is slanted, or the dimer is horizontal. The number of tilings ending in a slanted dimer is  $h_{n-2}^{a,b}$ , and the last dimer can be colored in  $b$  ways. So there are  $b \cdot h_{n-2}^{a,b}$  such tilings. As in the proof of Theorem 2.1, if the dimer is horizontal, it means that it covers the  $(n - 2)$ -nd and the  $n$ -th hexagon. In that case, the  $(n - 1)$ -st hexagon can be tiled by monomer, we can choose colors in  $ab$  ways, and the rest of the strip can be tiled in  $h_{n-3}^{a,b}$  ways. This gives us the  $ab \cdot h_{n-3}^{a,b}$  possible tiling in this case. The last case is if the  $(n - 1)$ -st hexagon forms a dimer with  $(n - 3)$ -rd hexagon. There are  $b^2 \cdot h_{n-4}^{a,b}$  such tilings. All cases are illustrated in Figure 9. This gives us the relation

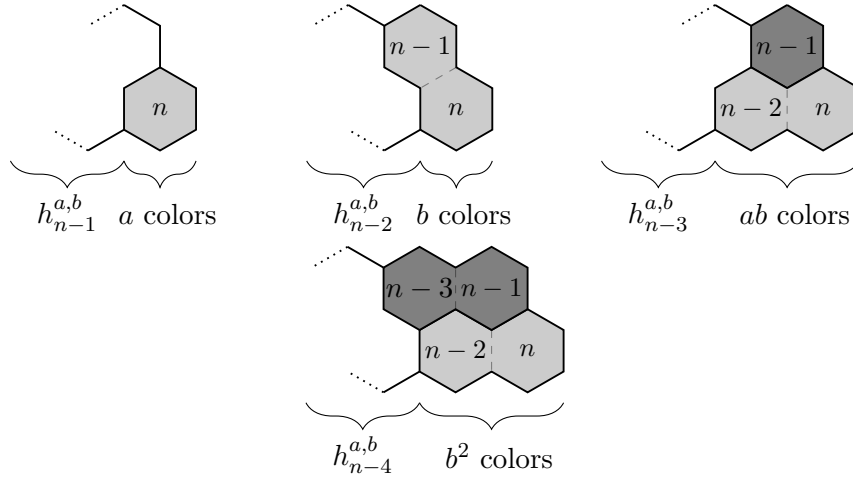


FIGURE 9. All possible endings of a colored tiling of a strip with  $n$  hexagons.

$h_n^{a,b} = a \cdot h_{n-1}^{a,b} + b \cdot h_{n-2}^{a,b} + ab \cdot h_{n-3}^{a,b} + b^2 \cdot h_{n-4}^{a,b}$ , which proves our theorem.  $\square$

We can now list some first values of  $h_n^{a,b}$ . We can notice that the values  $c_{n,k}$  from the last section appear in every row as coefficients of a bivariate polynomial. Connection between these values is given in the next theorem:

| $n$ | $h_n^{a,b}$                     |
|-----|---------------------------------|
| 0   | 1                               |
| 1   | $a$                             |
| 2   | $a^2 + b$                       |
| 3   | $a^3 + 3ab$                     |
| 4   | $a^4 + 5a^2b + 2b^2$            |
| 5   | $a^5 + 7a^3b + 7ab^2$           |
| 6   | $a^6 + 9a^4b + 16a^2b^2 + 3b^3$ |

TABLE 3. Some first values of  $h_n^{a,b}$ .

**Theorem 3.2.** *The number  $h_n^{a,b}$  of all possible tilings of the honeycomb strip of length  $n$  with monomers of  $a$  different colors and dimers of  $b$  different colors is given by*

$$h_n^{a,b} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} c_{n,k} a^{n-2k} b^k = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{m=0}^k \binom{n-k-m}{m} \binom{n-k-m}{k-m} a^{n-2k} b^k.$$

*Proof.* We could prove the theorem by induction, but again we present a simple combinatorial proof. The number  $h_n^{a,b}$  denotes the number of all possible tilings of the strip of a length  $n$ . For a fixed  $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$ , there are  $c_{n,k}$  possible ways to tile a strip with exactly  $k$  dimers, and since this tiling has  $k$  dimers and  $n - 2k$  monomers, the colors can be selected in  $a^{n-2k} b^k$  ways which gives a total of  $c_{n,k} a^{n-2k} b^k$  possible tilings. Since every tiling of the strip can contain  $0, 1, \dots, \lfloor \frac{n}{2} \rfloor - 1$  or  $\lfloor \frac{n}{2} \rfloor$  dimers, the overall number of tilings is the sum of these cases, that is

$$h_n^{a,b} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} c_{n,k} a^{n-2k} b^k.$$

□

#### 4. SOME (GENERALIZED) COMBINATORIAL IDENTITIES INVOLVING TETRANACCI NUMBERS

In this section, we generalize several of the identities obtained by Dresden and Jin [4] to the case of colored tilings of a honeycomb strip. All of the following identities reduce to the mentioned identities of Dresden and Jin by setting  $a = b = 1$ .

**Theorem 4.1.** *For every  $m, n \geq 0$*

$$\begin{aligned} h_{m+n}^{a,b} &= h_m^{a,b} h_n^{a,b} + h_{m-1}^{a,b} \left( b h_{n-1}^{a,b} + a b h_{n-2}^{a,b} + b^2 h_{n-3}^{a,b} \right) \\ &\quad + h_{m-2}^{a,b} \left( a b h_{n-1}^{a,b} + b^2 h_{n-2}^{a,b} \right) + b^2 h_{n-1}^{a,b} h_{m-3}^{a,b}. \end{aligned}$$

*Proof.* We consider a tiling of a honeycomb strip containing  $m+n$  hexagons. We have  $h_{m+n}^{a,b}$  such tilings. On the other hand, there are  $h_m^{a,b} \cdot h_n^{a,b}$  tilings that are breakable at position  $m$ , as shown in the Figure 10. All other tilings

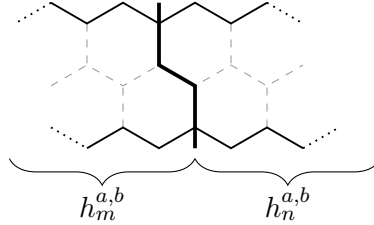


FIGURE 10. Breakable tiling at position  $m$ .

are unbreakable at position  $m$ . If that is the case, unbreakability can occur because of the right-inclined, left-inclined or horizontal dimer crossing the line of the break. Figure 11 shows all possible situations that can occur if tiling is not breakable at position  $m$ . Note that any tiling of a honeycomb strip is breakable if  $n > 4$ . Summing all these cases gives us the proof of the

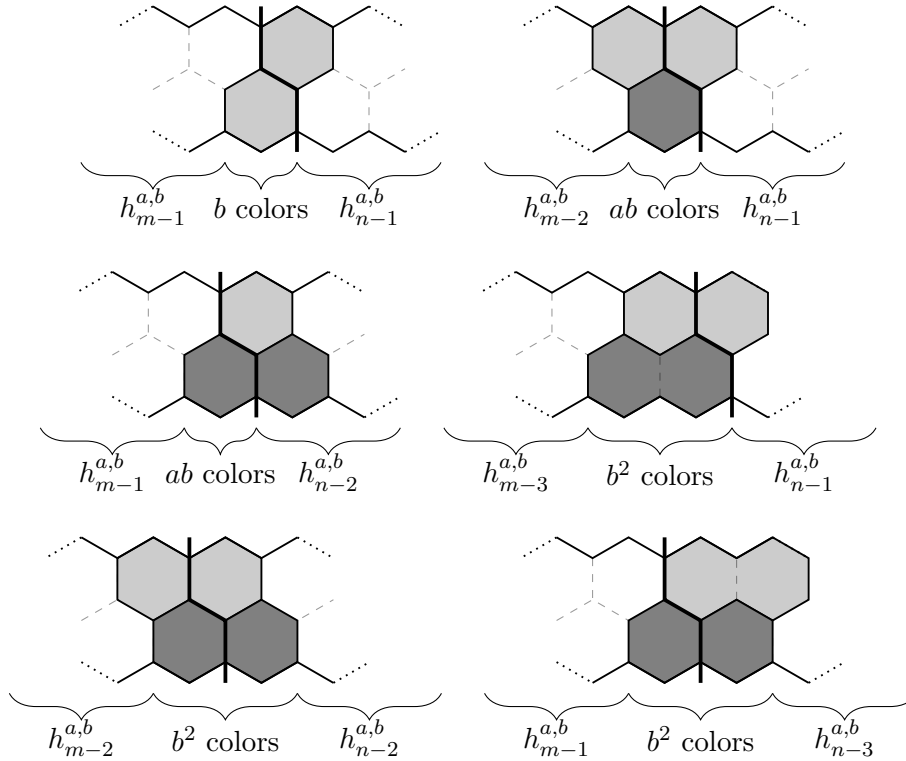


FIGURE 11. Layouts that can occur if tiling is not breakable at position  $m$ .

theorem. □

Our second identity counts tilings of the strip containing at least one dimer.

**Theorem 4.2.** *For every integer  $n \geq 1$ ,*

$$(4.1) \quad h_n^{a,b} - a^n = bh_{n-2}^{a,b} + 2b \sum_{k=3}^n a^{k-2} h_{n-k}^{a,b} + b^2 \sum_{k=3}^n a^{k-3} h_{n-k-1}^{a,b}.$$

*Proof.* We prove the result by double-counting all the ways to tile a strip using at least one dimer. On one hand, there are  $h_n^{a,b} - a^n$  such tilings, since the only tiling without dimers uses only monomers, and we can choose colors in  $a^n$  ways. The other way to count such tilings keeps track of the position where the first dimer occurs. Since the dimer covers two positions in the strip, we use the larger number to determine its position. For example, a dimer occupying hexagons 1 (or 2) and 3 has position 3. First, we start with slanted dimers. If the position of the first dimer is  $k$  for  $k \geq 2$ , the first part of the strip consists of  $k - 2$  monomers and the rest of the strip can be tiled in  $h_{n-k}^{a,b}$  ways, which gives us total of

$$b \sum_{k=2}^n a^{k-2} h_{n-k}^{a,b}$$

ways. We must now consider the horizontal dimer case. Note that the horizontal dimer cannot have positions 1 and 2. If the position of a dimer is  $k$  for  $k \geq 3$ , then the dimer occupies hexagons  $k - 2$  and  $k$ . We have two subcases, depending on whether the  $(k - 1)$ -st hexagon is tiled by a monomer or by a dimer. In the second case, it must be paired with  $(k + 1)$ -st hexagon, since position  $k$  is first to occur. In the first subcase, dimer and monomer can be colored in  $ab$  ways, the first part of the strip consisting of  $k - 3$  monomers can be colored in  $a^{k-3}$  ways, and the rest of the strip can be tiled in  $h_{n-k}^{a,b}$  ways, which gives us the total of

$$b \sum_{k=3}^n a^{k-2} h_{n-k}^{a,b}$$

ways. The latter subcase involves two dimers, the first occupying hexagons  $k - 2$  and  $k$ , and the second covering  $k - 1$  and  $k + 1$ . These dimers can be colored in  $b^2$  ways, the first part of the strip consisting of  $k - 3$  monomers can be colored in  $a^{k-3}$  ways, and the rest can be tiled in  $h_{n-k-1}^{a,b}$  ways. Since all the cases are disjoint, the overall number is the sum of the respective counting numbers, which proves our theorem. □

We conclude this section with a pair of identities counting tilings of the strip containing at least one monomer.

**Theorem 4.3.** *For every integer  $n \geq 0$  we have*

$$(4.2) \quad h_{2n}^{a,b} - b^n F_{n+1} = a \sum_{k=0}^n b^k h_{2n-2k-1}^{a,b} F_{k+2}$$

and

$$(4.3) \quad h_{2n-1}^{a,b} = a \sum_{k=0}^n b^k h_{n-2k-1}^{a,b} F_{k+2}.$$

*Proof.* The number of tilings of the  $2n$ -strip only by dimers is  $b^n F_{n+1}$ . Hence, the number of tilings containing at least one monomer is  $h_{2n}^{a,b} - b^n F_{n+1}$ . On the other hand, we can count such tilings based on the position of the first monomer. First, we consider the odd positions in the strip. If the first monomer occurs at position  $2k + 1$ , for some  $0 \leq k \leq n - 1$ , the first part of the strip is tiled only by dimers, and that can be done in  $b^k F_{k+1}$  ways, the monomer can be colored in  $a$  ways, and the rest of the strip can be tiled in  $h_{2n-2k-1}^{a,b}$  ways. Figure 12 illustrates this case. Since the monomer can

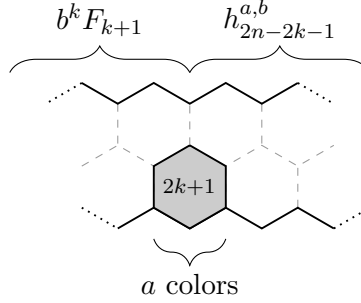
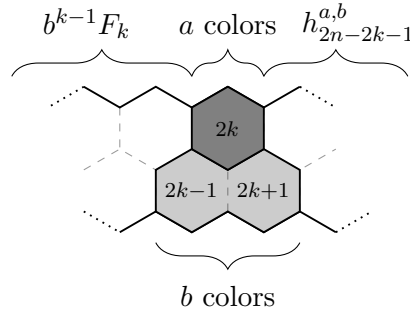


FIGURE 12. The hexagon occurs at position  $2k + 1$ .

occur at any position  $2k + 1$  for  $0 \leq k \leq n - 1$ , the total number of ways that monomer occurs at odd position is

$$a \sum_{k=0}^{n-1} b^k h_{2n-2k-1}^{a,b} F_{k+1}.$$

Now we consider the even positions. The case is similar, but there are some different details. If the first monomer occurs at position  $2k$  for  $1 \leq k \leq n$ , then all  $2k - 1$  hexagons must be tiled with dimers. For this to be possible, the  $(2k + 1)$ -st hexagon must be tiled by the same dimer as  $(2k - 1)$ -st. This dimer and monomer can be colored in  $ab$  ways. The first part of the strip containing  $2k - 2$  hexagons can be tiled only by dimers and in  $b^{k-1} F_k$  ways, and the rest of the strip in  $h_{2n-2k-1}^{a,b}$  ways. This case is illustrated in Figure 13.

FIGURE 13. The hexagon occurs at position  $2k$ .

The overall number of ways that a monomer occurs at even position is

$$ab \sum_{k=0}^{n-1} b^{k-1} h_{2n-2k-1}^{a,b} F_k,$$

and the total number of tilings is the sum of these two cases. Since  $h_{-1}^{a,b} = 0$  and  $F_0 = 0$ , the first sum can be extended to  $k = n$  and the second to  $k = 0$ .

$$\begin{aligned} h_{2n}^{a,b} - b^n F_{n+1} &= a \sum_{k=0}^{n-1} b^k h_{2n-2k-1}^{a,b} F_{k+1} + ab \sum_{k=1}^n b^{k-1} h_{2n-2k-1}^{a,b} F_k \\ &= a \sum_{k=0}^n b^k h_{2n-2k-1}^{a,b} F_{k+1} + a \sum_{k=0}^n b^k h_{2n-2k-1}^{a,b} F_k \\ &= a \sum_{k=0}^n b^k h_{2n-2k-1}^{a,b} F_{k+2}. \end{aligned}$$

The proof of the second identity is similar. When the length of the strip is odd, i.e.  $2n - 1$ , the left hand side is  $h_n^{a,b}$ , since it cannot be tiled only by dimers, and the proof for the right-hand side is the same, hence the theorem follows.  $\square$

## 5. TILING OF A HONEYCOMB STRIP AND TRIBONACCI NUMBERS

The tribonacci numbers (sequence A000073 in OEIS [11]) are the sequence of integers starting with  $T_0 = 0$ ,  $T_1 = 0$  and  $T_2 = 1$  and defined by recursive relation

$$(5.1) \quad T_n = T_{n-1} + T_{n-2} + T_{n-3}, \text{ for } n \geq 3.$$

For the reader's convenience, we list the first few values of the sequence in Table 4.

In this section we are still interested in counting all tilings of a honeycomb strip of a given length, but now by using different types of tiles. We still allow monomers and slanted dimers, but we prohibit horizontal dimers. In

|       |   |   |   |   |   |   |   |    |    |    |    |
|-------|---|---|---|---|---|---|---|----|----|----|----|
| $n$   | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7  | 8  | 9  | 10 |
| $T_n$ | 0 | 0 | 1 | 1 | 2 | 4 | 7 | 13 | 27 | 44 | 81 |

TABLE 4. The first few values of tribonacci numbers.

addition, we allow trimers of consecutively numbered hexagons. By prohibiting horizontal dimers we effectively suppress longer-range connections represented by horizontal edges in the inner dual. Also, by allowing trimers of the form  $\{i - 1, i, i + 1\}$  we abandon the context of matchings and instead work with packings in the inner dual. The allowed tiles are illustrated in Figure 14.

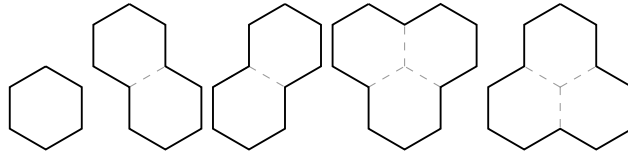


FIGURE 14. The allowed types of tiles.

Let  $g_n$  denote the number of ways to tile a hexagonal strip of length  $n$  by using only the allowed tiles. It is convenient to define  $g_0 = 1$ , and it is immediately clear that  $g_1 = 1$ ,  $g_2 = 2$ .

**Theorem 5.1.** *Let  $g_n$  denote the number of all ways to tile a honeycomb strip of length  $n$  by using only the allowed type of tiles. Then*

$$g_n = T_{n+2},$$

where  $T_n$  denotes  $n$ -th tribonacci number.

*Proof.* We start with an arbitrary tiling of a strip. There are three disjoint cases involving the  $n$ -th hexagon. If the hexagon is tiled by a monomer, then the rest of the strip can be tiled in  $g_{n-1}$  ways. If it is covered by a dimer, there are  $g_{n-2}$  such tilings, and finally, if the rightmost hexagon is covered by a trimer, there are  $g_{n-3}$  such tilings. By summing the respective numbers we obtain a recurrence that is the same as the defining recurrence for the tribonacci numbers, and the initial values determine the value of the shift.  $\square$

In the next part, we refine our results by counting the number of tilings with a fixed number of trimers, dimers or monomers. We denote these numbers by  $t_{n,k}$ ,  $u_{n,k}$  and  $v_{n,k}$ , respectively, where  $n$ , as usual, denotes the length of a strip, and  $k$  the number of tiles of a certain kind. We can also fix the number of all types of tiles. Let  $g_n^{k,l}$  denotes the number of all ways to tile a strip of a length  $n$  using exactly  $k$  trimers,  $l$  dimers and  $n - 3k - 2l$  monomers. We list some first values in the Table 5. From the definition it is clear that  $t_{n,k} = 0$  for  $k > \lfloor \frac{n}{3} \rfloor$ ,  $u_{n,k} = 0$  for  $k > \lfloor \frac{n}{2} \rfloor$  and  $v_{n,k} = 0$  for  $k > n$ . It is also convenient to define  $t_{0,0} = u_{0,0} = v_{0,0} = 1$ . For these sequences we



can obtain recursive relations in the obvious way, by considering the state of the last hexagon to see whether it is covered by a trimer, by a dimer, or by a monomer. The recursive relations are:

$$(5.2) \quad t_{n,k} = t_{n-1,k} + t_{n-2,k} + t_{n-3,k-1},$$

$$(5.3) \quad u_{n,k} = u_{n-1,k} + u_{n-2,k-1} + u_{n-3,k}$$

and

$$(5.4) \quad v_{n,k} = v_{n-1,k-1} + v_{n-2,k} + v_{n-3,k}.$$

We can now list some first values of the corresponding triangles: The

|       |           |    |   |       |           |    |    |    |   |
|-------|-----------|----|---|-------|-----------|----|----|----|---|
| $n/k$ | 0         | 1  | 2 | $n/k$ | 0         | 1  | 2  | 3  | 4 |
| 0     | 1         |    |   | 0     | 1         |    |    |    |   |
| 1     | 1         |    |   | 1     | 1         |    |    |    |   |
| 2     | 2         |    |   | 2     | 1         | 1  |    |    |   |
| 3     | 3         | 1  |   | 3     | 2         | 2  |    |    |   |
| 4     | 5         | 2  |   | 4     | 3         | 3  | 1  |    |   |
| 5     | 8         | 5  |   | 5     | 4         | 6  | 3  |    |   |
| 6     | 13        | 10 | 1 | 6     | 6         | 11 | 6  | 1  |   |
| 7     | 21        | 20 | 3 | 7     | 9         | 18 | 13 | 4  |   |
| 8     | 34        | 38 | 9 | 8     | 13        | 30 | 27 | 10 | 1 |
|       | $t_{n,k}$ |    |   |       | $u_{n,k}$ |    |    |    |   |

|       |           |    |    |    |    |   |   |   |   |
|-------|-----------|----|----|----|----|---|---|---|---|
| $n/k$ | 0         | 1  | 2  | 3  | 4  | 5 | 6 | 7 | 8 |
| 0     | 1         |    |    |    |    |   |   |   |   |
| 1     | 0         | 1  |    |    |    |   |   |   |   |
| 2     | 1         | 0  | 1  |    |    |   |   |   |   |
| 3     | 1         | 2  | 0  | 1  |    |   |   |   |   |
| 4     | 1         | 2  | 3  | 0  | 1  |   |   |   |   |
| 5     | 2         | 3  | 3  | 4  | 0  | 1 |   |   |   |
| 6     | 2         | 6  | 6  | 4  | 5  | 0 | 1 |   |   |
| 7     | 3         | 7  | 12 | 10 | 5  | 6 | 0 | 1 |   |
| 8     | 4         | 12 | 16 | 20 | 15 | 6 | 7 | 0 | 1 |
|       | $v_{n,k}$ |    |    |    |    |   |   |   |   |

TABLE 5. Initial values for  $t_{n,k}$ ,  $u_{n,k}$  and  $v_{n,k}$ .

first and the second triangle of Table 5 are not in the OEIS, while the third one appears as A104578 [11], the Padovan convolution triangle. The same arguments as the ones used on  $c_{n,k}$  show that the rows of those triangles do not have internal zeros, with the obvious exception of the zeros appearing in the first descending subdiagonal of  $v_{n,k}$ .

Before we go any further, we introduce two closely related sequences defined by Fibonacci-like recurrences of length three, the Narayana's cows sequence (A000930) and the Padovan sequence (A000931). We denote the

$n$ -th element of these sequences by  $N_n$  and  $P_n$ , respectively. The initial values are  $N_0 = N_1 = N_2 = 1$  and  $P_0 = 1, P_1 = P_2 = 0$ , and for  $n \geq 3$  we have recursive relations  $N_n = N_{n-1} + N_{n-3}$  and  $P_n = P_{n-2} + P_{n-3}$ . We refer the reader to [11] for more details about those sequences. In particular, we draw the reader's attention to the fact that there are several other sequences referred to as the Narayana numbers, for example A001263, a very important triangle of numbers refining the Catalan numbers and appearing in many different contexts. In the rest of this paper, when we refer to Narayana's numbers, we always mean A000930.

We now take a closer look at sequences  $t_{n,0}$ ,  $u_{n,0}$  and  $v_{n,0}$ , i.e., at the number of tilings where one type of tile is omitted. The sequence  $t_{n,0} = F_{n+1}$ , since such tilings contain only slanted dimers and monomers; since such tilings correspond to matchings in the path on  $n$  vertices, they are counted by Fibonacci numbers.

The sequence  $u_{n,0}$  counts the number of all ways to tile a length- $n$  strip by using only monomers and trimers, hence its elements satisfy the defining recurrence for the Narayana's cow sequence. Similarly, since the elements of the sequence  $v_{n,0}$  are the numbers of all different tilings where monomers are omitted, they satisfy Padovan's recursion. We have  $u_{n,0} = N_n$  and  $v_{n,0} = P_{n+3}$ . In the next three theorems, we present connections between the number of tilings and the above listed sequences. It turns out that the elements of the three triangles of Table 5 can be expressed by convolution-like formulas involving the Fibonacci, the Narayana's and the Padovan numbers. Such formulas could have been anticipated from the second column of triangle  $t_{n,k}$  which seems to be the (shifted) self-convolution of Fibonacci numbers and also from the name of the entry A104578 in OEIS.

**Theorem 5.2.** *For  $n \geq 0$ , the number of ways to tile a strip with  $n$  hexagons using exactly  $k$  trimers is*

$$(5.5) \quad t_{n,k} = \sum_{\substack{i_0, \dots, i_k \geq 0 \\ i_0 + \dots + i_k = n - 2k + 1}} F_{i_0} \cdots F_{i_k}.$$

*Proof.* If there are no trimers in the tiling, one can only use dimers or monomers to tile a strip and the number of ways to do that is  $t_{n,0} = F_{n+1}$ . If we use exactly  $k$  trimers, those trimers divide our strip into  $k + 1$  smaller strips. In this sense we allow the strip to be of a length 0 if two trimers are adjacent; the sub-strips of length 0 can also appear at the beginning or at the end of a strip. We have a strip with  $n$  hexagons which is tiled with  $k$  trimers, so there are  $n - 3k$  hexagons left to tile. Since the position of each trimer is arbitrary, the lengths of strips between and around them can vary from 0 to  $n - 3k$ , but the sum of the lengths must be constant, that is  $i_0 + i_1 + \dots + i_k = n - 3k$ . Each of those smaller strips can be tiled only by dimers or monomers, hence in  $t_{i_j,0}$  ways, where  $0 \leq j \leq k$ . Summing this

over all positions of the trimers we have:

$$\begin{aligned}
t_{n,k} &= \sum_{\substack{i_0, \dots, i_k \geq 0 \\ i_0 + \dots + i_k = n-3k}} t_{i_0,0} \cdots t_{i_k,0} \\
&= \sum_{\substack{i_0, \dots, i_k \geq 0 \\ i_0 + \dots + i_k = n-3k}} F_{i_0+1} \cdots F_{i_k+1} \\
&= \sum_{\substack{i_0, \dots, i_k \geq 0 \\ i_0 + \dots + i_k = n-2k+1}} F_{i_0} \cdots F_{i_k}.
\end{aligned}$$

□

Note that Theorem 5.2 allows us to express the tribonacci numbers as a double sum

$$(5.6) \quad T_{n+2} = \sum_{k=0}^n t_{n,k} = \sum_{k=0}^n \sum_{\substack{i_0, \dots, i_k \geq 0 \\ i_0 + \dots + i_k = n-2k+1}} F_{i_0} \cdots F_{i_k}.$$

**Theorem 5.3.** *For  $n \geq 0$ , the number of ways to tile a strip with  $n$  hexagons using exactly  $k$  dimers is*

$$(5.7) \quad u_{n,k} = \sum_{\substack{i_0, \dots, i_k \geq 0 \\ i_0 + \dots + i_k = n-2k}} N_{i_0} \cdots N_{i_k}.$$

*Proof.* We already know that the number of tilings with no dimers is  $u_{n,0} = N_n$ . Now we look at the tilings of the strip with  $n$  hexagons that have exactly  $k$  dimers. That leaves us with  $n - 2k$  hexagons to be tiled by monomers and trimers. As in the proof of Theorem 5.2, we note that  $k$  dimers divide the strip into  $k + 1$  smaller strips, each of the length  $0 \leq i_j \leq n - 2k$ . Each smaller strip can be tiled in  $N_{i_j}$  ways, and after summing over all possible positions of  $k$  dimers we have

$$u_{n,k} = \sum_{\substack{i_0, \dots, i_k \geq 0 \\ i_0 + \dots + i_k = n-2k}} N_{i_0} \cdots N_{i_k}.$$

□

The next result gives a new combinatorial interpretation of the sequence A104578 of [11].

**Theorem 5.4.** *For  $n \geq 0$ , the number of ways to tile a strip with  $n$  hexagons using exactly  $k$  monomers is*

$$(5.8) \quad v_{n,k} = \sum_{\substack{i_0, \dots, i_k > 0 \\ i_0 + \dots + i_k = n+2k+3}} P_{i_0} \cdots P_{i_k}.$$

*Proof.* The proof will be analogous to the two previous proofs. The number of tilings with no monomers is  $v_{n,0} = P_{n+3}$ . A monomer does not divide our strip, but if it first appears in position  $i$ , we will consider strips left and right from it. We count the number of tilings of the strip  $H_n$  that have exactly  $k$  monomers. That leaves us  $n - k$  untiled hexagons. Omitting  $k$  hexagons leaves  $k + 1$  smaller strips, each of the length  $0 \leq i_j \leq n - k$ . Each smaller strip can be tiled in  $P_{i_j+3}$  ways, and after summing over all possible positions of  $k$  monomers we have

$$\begin{aligned} v_{n,k} &= \sum_{\substack{i_0, \dots, i_k \geq 0 \\ i_0 + \dots + i_k = n - k}} P_{i_0+3} \cdots P_{i_k+3} \\ &= \sum_{\substack{i_0, \dots, i_k > 0 \\ i_0 + \dots + i_k = n + 2k + 3}} P_{i_0} \cdots P_{i_k}. \end{aligned}$$

□

Now we turn our attention to the number of tilings of a strip of length  $n$  with a fixed number of tiles of each type. Recall that the number of tilings consisting of  $k$  trimers,  $l$  dimers and  $n - 3k - 2l$  monomers is denoted by  $g_n^{k,l}$ . In the next theorem, we give a closed-form formula for  $g_n^{k,l}$ .

**Theorem 5.5.** *For  $n \geq 0$ , the number of ways to tile a strip with  $n$  hexagons using exactly  $k$  trimers,  $l$  dimers and  $n - 2k - l$  monomers is*

$$(5.9) \quad g_n^{k,l} = \binom{n - 3k - l}{l} \binom{n - 2k - l}{k}.$$

*Proof.* Consider a set consisting of all arbitrary tilings of a length- $n$  strip that have exactly  $k$  trimers,  $l$  dimers and  $n - 3k - 2l$  monomers. To prove this theorem, we establish a 1-1 correspondence between that set and the set of all permutations of  $n - 2k - l$  elements where we have  $k$  elements  $t$ ,  $l$  elements  $d$  and  $n - 3k - 2l$  elements  $m$ . From an arbitrary permutation, we obtain the corresponding tiling as follows: we replace each element  $t$  with a trimer, each element  $d$  with a dimer, and each element  $m$  with a monomer. In this manner, we obtained a tiling of a strip of length  $n$  with the prescribed number of tiles of each type. For example, the permutation  $tmdmt$  corresponds to the tiling shown in the Figure 15. In an obvious way,

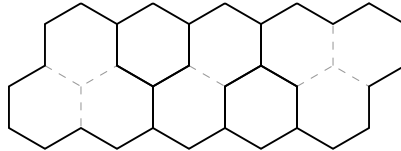


FIGURE 15. Tiling corresponding to the permutation  $tmdmt$ .

we can also obtain a permutation from a given tiling. Since the total number

of permutations of this set is  $\frac{(n-2k-l)!}{k!l!(n-3k-2l)!}$ , we arrive to:

$$\begin{aligned} g_n^{k,l} &= \frac{(n-2k-l)!}{k!l!(n-3k-2l)!} \cdot \frac{(n-3k-l)!}{(n-3k-l)!} \\ &= \frac{(n-3k-l)!}{l!(n-3k-2l)!} \cdot \frac{(n-2k-l)!}{k!(n-3k-l)!} \\ &= \binom{n-3k-l}{l} \binom{n-2k-l}{k}. \end{aligned}$$

□

From Theorem 5.5 we arrive to yet another identity for tribonacci numbers:

$$(5.10) \quad T_{n+2} = \sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} \sum_{l=0}^{\lfloor \frac{n-3k}{2} \rfloor} \binom{n-3k-l}{l} \binom{n-2k-l}{k}.$$

Specially, if we set  $k = 0$  in the first equation we have

$$\begin{aligned} t_{n,0} &= \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-l}{l} \binom{n-l}{0} \\ &= \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-l}{l} \\ &= F_{n+1}. \end{aligned}$$

Since Theorem 5.5 gives us the number of all tilings using the prescribed number of tiles of each type, we can express values  $t_{n,k}$  and  $u_{n,k}$  in a new way by summing over  $l$  and  $k$ , respectively.

**Corollary 5.6.** *For  $n \geq 0$ ,*

$$(5.11) \quad t_{n,k} = \sum_{l=0}^{\lfloor \frac{n-3k}{2} \rfloor} \binom{n-3k-l}{l} \binom{n-2k-l}{k},$$

and

$$(5.12) \quad u_{n,l} = \sum_{k=0}^{\lfloor \frac{n-2l}{3} \rfloor} \binom{n-3k-l}{l} \binom{n-2k-l}{k}.$$

## 6. SOME COMBINATORIAL IDENTITIES INVOLVING TRIBONACCI NUMBERS

In this section, we prove in a combinatorial way, several identities involving the tribonacci, Narayana's, Padovan and Fibonacci numbers. We begin with a well-known identity for tribonacci numbers and we give it a new combinatorial interpretation:

**Theorem 6.1.** For  $n \geq 4$ ,

$$T_n + T_{n-4} = 2T_{n-1}.$$

*Proof.* Let  $\mathcal{G}_n$  denotes the set of all tilings of a length- $n$  strip,  $\mathcal{M}_n$ ,  $\mathcal{D}_n$  and  $\mathcal{T}_n$  the tilings ending with a monomer, dimer or trimer, respectively. As before, the cardinal number of the set  $\mathcal{G}_n$  is  $g_n$ . It is clear that  $\mathcal{G}_n = \mathcal{M}_n \dot{\cup} \mathcal{D}_n \dot{\cup} \mathcal{T}_n$ . To prove the theorem we have to establish 1-1 correspondence between sets  $\mathcal{G}_{n-2} \cup \mathcal{G}_{n-6}$  and  $\mathcal{G}_{n-3} \times \{0, 1\}$ .

To each tiling from the set  $\mathcal{G}_{n-3}$  we add a monomer at the end to obtain an element of  $\mathcal{M}_{n-2}$ . Thus, we obtained bijection between the sets  $\mathcal{G}_{n-3}$  and  $\mathcal{M}_{n-2}$ . In this way, we have used all the tilings of the set  $\mathcal{G}_{n-3}$  once. Now we take the tilings from the set  $\mathcal{G}_{n-3}$  again, and if it ends with a trimer, i.e. if it is an element of  $\mathcal{T}_{n-3}$ , we remove it to obtain a tiling of length  $n - 6$ , i.e., element of a set  $\mathcal{G}_{n-6}$ . If it ends with a dimer (an element of  $\mathcal{D}_{n-3}$ ), we remove it and replace it with a trimer to obtain an element from  $\mathcal{T}_{n-2}$ . Finally, if the tiling is an element of  $\mathcal{M}_{n-3}$ , we replace the last monomer with a dimer, to obtain an element of  $\mathcal{D}_{n-2}$ . In this way, we have used every tiling of a length  $n - 3$  twice and obtained all tilings of a length  $n - 2$  and  $n - 6$  exactly once. The diagram that visualizes 1-1 correspondence between the two sets is shown in Figure 16.

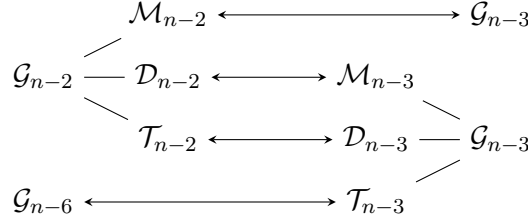


FIGURE 16. 1-1 correspondence between sets  $\mathcal{G}_{n-2} \cup \mathcal{G}_{n-6}$  and  $\mathcal{G}_{n-3} \times \{0, 1\}$ .

It follows that  $g_{n-2} + g_{n-6} = 2g_{n-3}$ , and since  $g_n = T_{n+2}$ , the theorem follows.  $\square$

For the next few identities, it is useful to recall the definition of breakability. We say that a tiling of a honeycomb strip is breakable at the position  $k$  if given tiling can be divided into two tiled strips, the first one containing the leftmost  $k$  hexagons and the second one containing the rest.

Our next identity differentiates tilings based on the breakability.

**Theorem 6.2.** For any integers  $m, n \geq 1$  we have the identity

$$T_{m+n} = T_m T_n + T_{m+1} T_{n+1} + T_{m-1} T_n + T_m T_{n-1}.$$

*Proof.* We consider an arbitrary tiling of a strip of length  $m + n - 2$ . If the tiling is breakable at position  $m - 1$ , we divide it into two strips of a length  $m - 1$  and  $n - 1$ . Hence, the total number of tiling in this case is  $g_{m-1} g_{n-1}$ . If

the tiling is not breakable at position  $m - 1$ , that means that either a dimer or a trimer is blocking it. If the dimer is preventing the tiling from breaking, there are strips of lengths  $m - 2$  and  $n - 2$  on each side, so the total number of tilings in this case is  $g_{m-2}g_{n-2}$ . If the trimer is blocking it, it can reduce the length of the left or of the right strip by two. So the total number of tilings in this case is  $g_{m-3}g_{n-2} + g_{m-2}g_{n-3}$ . By summing the contributions of all these cases we obtain  $g_{m+n-2} = g_{m-1}g_{n-1} + g_{m-2}g_{n-2} + g_{m-3}g_{n-2} + g_{m-2}g_{n-3}$ , and by using the equality  $g_n = T_{n+2}$  we have  $T_{m+n} = T_m T_n + T_{m+1} T_{n+1} + T_{m-1} T_n + T_m T_{n-1}$ .  $\square$

The next identity was proved by Frontczak [7] by using generating functions. Here we provide a combinatorial interpretation.

**Theorem 6.3.** *For any integer  $n \geq 0$  we have the identity*

$$T_{n+2} = \sum_{k=0}^{n+1} F_k T_{n-k}.$$

*Proof.* We prove this theorem by counting all ways to tile a strip by using at least one trimer. The total number of ways to tile a length- $n$  strip without trimers is  $t_{n,0} = F_{n+1}$ , hence the number of tilings having at least one trimer is  $T_{n+2} - F_{n+1}$ . On the other hand, we can count the same tilings by observing where the first trimer appears. If the leftmost trimer occupies hexagons  $\{i, i + 1, i + 2\}$ , we say that the position of trimer is  $i$ . So, all possible positions range from 1 to  $n - 2$ . If a trimer first appears at position  $k$ , the leftmost  $k - 1$  hexagons are tiled only by monomers and dimers, and the number of all ways to do that is  $F_k$ . The rest of the strip, of length  $n - k - 2$ , can be tiled in  $T_{n-k}$  ways. By summing over all possible positions of the leftmost trimer, we have

$$T_{n+2} - F_{n+1} = \sum_{k=1}^{n-2} F_k T_{n-k}.$$

By using  $T_{n-3} = T_n - T_{n-1} - T_{n-2}$ , one can extend the tribonacci numbers to negative integers and obtain  $T_{-1} = 1$ . Since  $T_1 = T_0 = 0$ , the sum above can be extended to obtain

$$T_{n+2} = \sum_{k=1}^{n+1} F_k T_{n-k},$$

which concludes our proof.  $\square$

In their recent paper, Dresden and Tulsikh [6] proved a generalized formula for the collection of convolution formulas involving sequences that satisfy similar recurrences. Our last two identities can be derived from that formula but again here we present a combinatorial interpretation.

**Theorem 6.4.** *For any integer  $n \geq 0$  we have the identity*

$$T_{n+2} = \sum_{k=0}^n N_k T_{n-k} + N_n.$$

*Proof.* We prove this theorem by counting all ways to tile a strip by using at least one dimer. The proof is analogous to the previous one. The total number of ways to tile a strip of length  $n$  without dimers is  $u_{n,0} = N_n$ , hence the number of tilings having at least one dimer is  $T_{n+2} - N_n$ . Similarly as before, we can count the same thing by observing where the leftmost dimer appears. If the leftmost dimer occupies hexagons  $\{i, i+1\}$ , we say that its position is  $i$ . So, all possible positions range from 1 to  $n-1$ . If a dimer first appears at position  $k$ , the leftmost  $k-1$  hexagons can be tiled in  $N_{k-1}$  ways. The rest of the strip is of length  $n-k-1$  and it can be tiled in  $T_{n-k+1}$  ways. By summing over all possible positions of the leftmost dimer we have

$$T_{n+2} - N_n = \sum_{k=1}^{n-1} N_{k-1} T_{n-k+1}.$$

Some rearranging of indices and fact that  $T_0 = T_1 = 0$  bring us to

$$T_{n+2} - N_n = \sum_{k=0}^n N_k T_{n-k}$$

and our proof is over.  $\square$

**Theorem 6.5.** *For any integer  $n \geq 0$  we have the identity*

$$T_{n+2} = \sum_{k=1}^n P_{k+2} T_{n-k+2} + P_{n+3}.$$

*Proof.* Analogously as in two previous theorems, we prove this theorem by counting all ways to tile a strip by using at least one monomer. The number of ways to tile a strip of length  $n$  with at least one monomer is  $g_n - v_{n,0} = T_{n+2} - P_{n+3}$ . Now we can count the same thing by observing the position of the leftmost monomer. All possible positions for the first monomer range from 1 to  $n$ . If it first appears at position  $k$ , the first part of the strip, i.e., the leftmost  $k-1$  hexagons, can be tiled in  $P_{k+2}$  ways. The rest of the strip is of length  $n-k$  and can be tiled in  $T_{n-k+2}$  ways. By summing over all possible positions of the leftmost monomer we have

$$T_{n+2} - P_{n+3} = \sum_{k=1}^n P_{k+2} T_{n-k+2},$$

which concludes our proof.  $\square$



## 7. CONCLUDING REMARKS

In this paper, we have considered various ways of tiling a narrow honeycomb strip of a given length with different types of tiles. We have refined some previously known results for the total number of tilings of a given type by deriving formulas for the number of such tilings with a prescribed number of tiles of a given type. We have also considered tilings with colored tiles and obtained the corresponding formulas. Along the way, we have provided combinatorial interpretations for some known identities and established a number of new ones. Also, we have provided closed-form expressions for several triangles of numbers appearing in the OEIS.

In order to keep this contribution at a reasonable length, we have omitted many interesting problems related to the considered ones. In particular, we have not considered any jamming-related scenarios, i.e., the tilings that are suboptimal with respect to the number of large( $r$ ) tiles. The existence of connections of our tilings with such problems is indicated by the appearance of Padovan numbers in both contexts [3]. Further, we have not examined statistical properties such as the expected number of tiles in a random tiling of a strip of a given length in the way done in ref. [2]. We have not looked at the asymptotic behavior of the counting sequences. Each of the mentioned omissions could be an interesting topic for further research.

Another interesting direction would be to look in more detail at triangles  $c_{n,k}$ ,  $t_{n,k}$ ,  $u_{n,k}$ , and  $v_{n,k}$ . We have shown that their rows (with one trivial exception) do not have internal zeros. By inspection of the first few rows of  $c_{n,k}$ ,  $t_{n,k}$ , and  $u_{n,k}$  one can observe that the rows seem to be also unimodal and even log-concave. It would be interesting to investigate whether those properties hold for the whole triangles. Both properties are violated in rows of  $v_{n,k}$ , but the violations seem to be restricted to the right end. What happens if the rightmost three elements are omitted? Also, the position of the maximum presents an interesting challenge. Finally, more interesting identities could be derived by looking at ascending and descending diagonals of different slopes in those triangles. We hope that at least some of the listed problems will be addressed in our future work.

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