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ON THE DIRECTED HAMILTON-WATERLOO PROBLEM WITH TWO CYCLE SIZES

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ABSTRACT. The Directed Hamilton-Waterloo Problem asks for a directed 2-factorization of the complete symmetric digraph K_v^* where there are two non-isomorphic 2-factors. In the uniform version of the problem, factors consist of either directed m-cycles or n-cycles. In this paper, necessary conditions for a solution to this problem are given, and the problem is completely solved for the factors with $(m,n) \in \{(4,6),(4,8),(4,12),(4,16),(6,12),(8,16)\}$. Furthermore, the problem is solved for $(m,n) \in \{(3,5),(3,15),(5,15)\}$ when v is odd with a few possible exceptions.

1. INTRODUCTION

A decomposition of a graph G is a set $\mathcal{H} = \{H_1, H_2, \ldots, H_k\}$ of subgraphs of G such that $\bigcup_{i=1}^{k} E(H_i) = E(G)$ and $E(H_i) \cap E(H_j) = \emptyset$ for $i \neq j$. Such a decomposition is called an $\{H_1, H_2, \ldots, H_k\}$-decomposition of G. A factor in a graph G is a spanning (not necessarily connected) subgraph of G. If a graph G can be decomposed into r_i factors isomorphic to the factor F_i for $i \in [1,t]$, then we say that G has an $\{F_1^{r_1}, F_2^{r_2}, \ldots, F_t^{r_t}\}$-factorization. When each F_i factor consists of only n_i-cycles for $i \in [1,t]$, then we will call the F_i factor as a C_{n_i}-factor and call this factorization as a $\{C_{n_1}^{r_1}, C_{n_2}^{r_2}, \ldots, C_{n_t}^{r_t}\}$-factorization where each r_i is the number of C_{n_i}-factors.

Graph factorizations constitute an important part of graph decomposition problems, especially when each factor is of regular degree. A k-regular spanning subgraph of G is called a k-factor of G. It is easy to see that a 1-factor is a perfect matching in a graph and a 2-factor is either a Hamilton cycle or a union of cycles. When it comes to 2-factorizations, there are two well-known graph factorization problems. One problem is the Oberwolfach Problem, which is posed by Ringel (see [17]) as a seating arrangement problem at a meeting in Oberwolfach. Given a conference venue with k_i round

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tables, each of which has m_i seats for $i \in [1,t]$, it asks whether it is possible that each participant of the conference (say v many for odd v) sits next to (left or right) each other participant exactly once at the end of $\frac{v-1}{2}$ nights. In graph theory language, it asks whether the complete graph K_v (or $K_v - I$ in the spouse avoiding version for even v) decomposes into isomorphic 2-factors where each 2-factor consists of k_i m_i-cycles for each $i \in [1,t]$. This problem is denoted by $\mathrm{OP}(m_1^{k_1}, m_2^{k_2}, \ldots, m_t^{k_t})$. If there is only one type of cycle, say of length m, in the factor, it can be denoted as $\mathrm{OP}(m^k)$, and its solution gives a $\{C_m^{\frac{v-1}{2}}\}$-factorization (or in short, a C_m-factorization) of K_v.

The Hamilton-Waterloo Problem is a generalization of the Oberwolfach Problem where there are two conference venues (one in Hamilton and one in Waterloo as one may guess) with different seating arrangements. This time each 2-factor can be isomorphic to one of the given two 2-factors, say F_1 or F_2. If F_1 consists of only m-cycles and F_2 consists of only n-cycles, then the corresponding Hamilton-Waterloo Problem is called as the uniform version, and it is denoted by $\mathrm{HWP}(v; m^r, n^s)$ where r and s are the numbers of C_m and C_n-factors where $r + s = \frac{v-1}{2}$, respectively. Having a solution to $\mathrm{HWP}(v; m^r, n^s)$ means that K_v has a $\{C_m^r, C_n^s\}$-factorization. Solving the problem completely is to have a solution for all possible r and s.

The uniform versions of both problems are well-studied. In articles [3, 4, 20], authors solved the uniform version of the Oberwolfach Problem completely. But the general case of the Oberwolfach Problem is still open. It is known that $\mathrm{OP}(3^2)$, $\mathrm{OP}(3^4)$, $\mathrm{OP}(4,5)$ and $\mathrm{OP}(3^2,5)$ have no solution. In [6, 18, 25], it is shown that $\mathrm{OP}(m_1^{k_1}, m_2^{k_2}, \ldots, m_t^{k_t})$ has a solution for all $n \leq 60$ with the above exceptions.

As the first results on the uniform Hamilton-Waterloo Problem, Adams et al. [5] showed that $\mathrm{HWP}(v; m^r, n^s)$ has a solution for all $v \leq 16$ and gave solutions for the small cases where $(m,n) \in \{(4,6), (4,8), (4,16), (8,16), (3,5), (3,15), (5,15)\}$. Cycle sizes $(3,4)$ and in general $(4,m)$ for odd m has been studied by several authors (see [15], [16], [22], [27]).

When m and n are odd, the problem is almost completely solved in [13, 14] for odd v. In [7], the problem is solved in the case that both m and n are even and $v \equiv 0 \pmod 4$ except possibly when $r = 1$ or $s = 1$. When m and n are both even and $v \equiv 2 \pmod 4$, this problem is solved by R. Häggkvist in [19] whenever r and s are both even. Also, if m is even and $m|n$, the problem is completely solved in [8].

One generalization of these problems may be to consider sitting on the right and sitting on the left of a participant as separate entities. To represent such a sitting, one has to use directed cycles which led us to work on directed graphs. There are studies on the directed Oberwolfach Problem, and here we work on the directed version of the Hamilton-Waterloo Problem.

We will denote a digraph D as $D = (V(D), E(D))$, where $V(D)$ is the vertex set and $E(D)$ is the arc set. For clarity, edges and arcs are denoted by using curly braces and parentheses, respectively. For a simple graph G, we use G^* to denote the symmetric digraph with vertex set $V(G^*) = V(G)$ and arc set $E(G^*) = \bigcup_{\{x,y\}\in E(G)}\{(x,y),(y,x)\}$. Hence, K_v^* and $K_{(x:y)}^*$ respectively denote the complete symmetric digraph of order v and the complete symmetric equipartite digraph with y parts of size x. Also, $\vec{C}_n$ will denote the directed cycle of order n.

Similarly, a set $\{H_1, H_2, \ldots, H_k\}$ of arc-disjoint subdigraphs of a digraph D is called a decomposition of D if $\bigcup_{i=1}^{k} E(H_i) = E(D)$. If a symmetric digraph G^* has decomposition which consists of r_i factors having directed n_i cycles for $i \in [1,t]$, then we say G^* has a $\left\{\vec{C}_{n_1}^{r_1}, \vec{C}_{n_2}^{r_2}, \ldots, \vec{C}_{n_t}^{r_t}\right\}$-factorization.

In the Directed versions of the Oberwolfach and the Hamilton-Waterloo Problems, K_v^* is decomposed into factors of directed cycles. Hence, the seating arrangement is done over $v-1$ nights. If the sizes of directed cycles are $m_1, m_2, \ldots, m_t$ and the number of each directed cycle m_i is k_i for $i \in [1,t]$ where $\sum_{i=1}^{t} k_i m_i = v$, the Directed Oberwolfach Problem is denoted by $\mathrm{OP}^*(m_1^{k_1}, \ldots, m_t^{k_t})$. Similarly, $\mathrm{HWP}^*(v; m^r, n^s)$ denotes the uniform directed Hamilton-Waterloo Problem with directed cycle sizes m and n. Again, if $\mathrm{HWP}^*(v; m^r, n^s)$ has a solution, it means that K_v^* has a $\{\vec{C}_m^r, \vec{C}_n^s\}$-factorization.

So far, the Directed Oberwolfach Problem has only partial results, but the Directed Hamilton-Waterloo Problem has not been studied yet to the best of our knowledge.

As the first result on the Directed Oberwolfach Problem, $\mathrm{OP}^*(3^k)$ with an exception $v = 6$ is solved by Bermond et al. [9]. In [10], Bennett and Zhang solved $\mathrm{OP}^*(4^k)$ except for $v = 12$, and Adams and Bryant solved the remaining case $\mathrm{OP}^*(4^3)$ (in an unpublished paper "Resolvable directed cycle systems of all indices for cycle length 3 and 4").

In [2], Alspach et al. showed that K_v^* can be decomposed into $\vec{C}_m$ cycles with exceptions $(v,m) \neq (4,4), (6,3), (6,6)$ if and only if $m|v(v-1)$. They studied the problem in cases where v and m are even or odd, separately.

Burgess and Šajna [12] investigated the necessary and sufficient conditions for the Directed Oberwolfach Problem with cycles of length m. In case m is even, they obtained a complete solution and presented a partial solution for odd cycle size. Also, they conjectured that K_{2m}^* admits a directed m-cycle factorization for odd m if and only if $m \geq 5$. In [11], Burgess et al. proved this conjecture for $m \leq 49$.

The following theorem summarizes the results from [1, 9, 10, 11, 12, 23], and completely settles the directed Oberwolfach problem with uniform cycle length.

Theorem 1.1. [1, 9, 10, 11, 12, 23] $\mathrm{OP}^*(m^k)$ *has a solution if and only if* $(m,k) \notin \{(3,2), (4,1), (6,1)\}$.

In [26], Shabani and Šajna proved that K_v^* has a $\{\vec{C}_2, \vec{C}_{v-2}\}$-factorization for $v \geq 5$ and obtained the necessary and sufficient conditions for K_v^* to admit a $\{\vec{C}_m, \vec{C}_{v-m}\}$-factorization for $2 \leq m \leq v-2$ and for odd v. Also they showed that if $v \geq 5$ and $v \equiv 1, 3, \text{or } 7 \pmod 8$, then K_v^* has a $\{\vec{C}_2, \vec{C}_2, \ldots, \vec{C}_2, \vec{C}_3\}$-factorization.

In this paper, we follow the lead of the first results on the undirected Hamilton-Waterloo Problem and give solutions to the cases with directed cycle sizes $\{(4,6),(4,8),(4,12),(4,16),(6,12),(8,16),(3,5),(3,15),(5,15)\}$. We first give the necessary conditions for a solution to $\mathrm{HWP}^*(v; m^r, n^s)$ to exist. Second, we make the observation that for any given solution to $\mathrm{HWP}(v; m^r, n^s)$, one can construct a solution to $\mathrm{HWP}^*(v; m^{2r}, n^{2s})$ for odd v. Then, we give two different constructions depending on the parity of the cycle sizes. For even cycle sizes, using our construction in Lemma 3.2 and the preliminary lemmata required in the construction, $\mathrm{HWP}^*(v; m^r, n^s)$ is solved for $(m,n) \in \{(4,6),(4,8),(4,12),(4,16),(6,12),(8,16)\}$ with $r+s = v-1$. For odd cycle sizes, we give a new construction in Lemma 4.1 when v is odd. Using this construction and the results required for this construction, we state that $\mathrm{HWP}^*(v; m^r, n^s)$ has a solution for $(m,n) \in \{(3,5),(3,15),(5,15)\}$ for odd v with a few possible exceptions. Constructions given in Lemma 3.2 and Lemma 4.1 are general constructions and they can be used to solve the problem also for the other cycle sizes as long as the necessary small cases can be found.

Let us first start with the necessary conditions, and then move to the preliminary results.

Lemma 1.2. *If* $\mathrm{HWP}^*(v; m^r, n^s)$ *has a solution then the following statements hold:*

(1) if $r > 0$, $v \equiv 0 \pmod m$,
(2) if $s > 0$, $v \equiv 0 \pmod n$,
(3) $r + s = v - 1$.

2. Preliminary Results

If G_1 and G_2 are two edge disjoint graphs with $V(G_1) = V(G_2)$, then we use $G_1 \oplus G_2$ to denote the graph on the same vertex set with $E(G_1 \oplus G_2) = E(G_1) \cup E(G_2)$. We will denote the vertex disjoint union of α copies of G by αG. Finally, $\overline{K}_n$ denotes the empty graph on n vertices.

Let G and H be graphs. The wreath product of G and H, denoted by $G \wr H$, is the graph obtained by replacing each vertex x of G with a copy of H, say H_x, and replacing each edge $\{x, y\}$ of G with the edges joining every vertex of H_x to every vertex of H_y.

In the case G and H are both digraphs, then the $G \wr H$ is the digraph obtained by replacing each vertex x of G with a copy of H, say H_x, and replacing each arc (x, y) of G by an arc pointing from every vertex of H_x

to every vertex of H_y. For example, $K_x^* \wr \overline{K}_y \cong K_{(y:x)}^*$, $\overline{K}_x \wr K_y^* \cong xK_y^*$ and $\overline{K}_x \wr \overline{K}_y \cong \overline{K}_{xy}$.

If G has a $\{H_1, H_2, \ldots, H_k\}$-decomposition, then $G \wr \overline{K}_n$ has a $\{H_1 \wr \overline{K}_n, H_2 \wr \overline{K}_n, \ldots, H_k \wr \overline{K}_n\}$-decomposition (see [4]). Also, for given three graphs G, H, and J, $(G \wr H) \wr J = G \wr (H \wr J)$, that is, the wreath product is associative (see *p.* 185 of [21]). Note that, the above properties of the wreath product extend to digraphs.

Let A be a finite additive group and let S be a subset of A, where S does not contain the identity of A. The Directed Cayley graph $\overrightarrow{X}(A;S)$ on A with connection set S is digraph with $V(\overrightarrow{X}(A;S)) = A$ and $E(\overrightarrow{X}(A;S)) = \{(x,y) : x, y \in A, y - x \in S\}$.

The following observation is useful to reduce the number of cases when v is odd.

Observation 2.1. *If* HWP$(v; m^r, n^s)$ *has a solution for some r and s and v is odd, then* HWP$^*(v; m^{2r}, n^{2s})$ *has a solution for the same r and s.*

A solution for HWP$^*(v; m^{2r}, n^{2s})$ is obtained from a solution of HWP$(v; m^r, n^s)$ by taking two copies of each 2-factor and replacing each edge $\{x, y\}$ with the arcs (x, y) and (y, x) in the two 2-factors.

Similarly, we get an H^*-factorization of G^* from an H-factorization of G.

Lemma 2.2. *Let G be a graph and H be a subgraph of G. If G has an H-factorization, then G^* has an H^*-factorization.*

The following lemma and theorem will be used in the solutions of even and odd cases of HWP$^*(v; m^r, n^s)$, respectively.

Lemma 2.3. [12] *Let $m \geq 4$ be an even integer and x be a positive integer. Then $K^*_{(\frac{mx}{2}:2)}$ has a $\overrightarrow{C}_m$-factorization.*

Theorem 2.4. [24] *The complete equipartite graph $K_{(x:y)}$ has a C_m-factorization for $m \geq 3$ and $x \geq 2$ if and only if $m|xy$, $x(y-1)$ is even, m is even if $y = 2$ and $(x, y, m) \neq (2, 3, 3), (6, 3, 3), (2, 6, 3), (6, 2, 6)$.*

3. Even Cycle Sizes

We will make use of the following lemma in the first main construction of this paper.

Lemma 3.1. *$K_x^* \wr \overline{K}_2$ has a K_2^*-factorization for every integer $x \geq 2$.*

Proof. Notice that $K_x^* \wr \overline{K}_2 \cong K_{2x}^* - xK_2^*$. Using Kotzig's 1-factorization of K_{2x} and Lemma 2.2, a decomposition of $K_x^* \wr \overline{K}_2$ into $2x - 2$ K_2^*-factors is obtained. □

Here we give the main construction that is used to obtain solutions for the even cycle size cases.

Lemma 3.2. *Let $m \geq 4$ and $n \geq 4$ be even and $h = \mathrm{lcm}(m, n)$. If $\mathrm{HWP}^*(h; m^{r'}, n^{s'})$ has a solution for all nonnegative integers r', s' satisfying $r' + s' = h - 1$, then there is a solution to $\mathrm{HWP}^*(hx; m^r, n^s)$ for all nonnegative integers r, s, and x with $r + s = hx - 1$.*

Proof. We can decompose K^*_{hx} as follows:

$$K^*_{hx} \cong xK^*_h \oplus \left(K^*_x \wr \overline{K}_h\right) \tag{3.1}$$

Since $\overline{K}_h \cong \overline{K}_2 \wr \overline{K}_{\frac{h}{2}}$, $K^*_x \wr \overline{K}_h$ is isomorphic to $(K^*_x \wr \overline{K}_2) \wr \overline{K}_{\frac{h}{2}}$ by the associativity of the wreath product. Thus, by Lemma 3.1, $K^*_x \wr \overline{K}_h$ can be decomposed into factors each isomorphic to $K^*_2 \wr \overline{K}_{\frac{h}{2}}$, and since $K^*_2 \wr \overline{K}_{\frac{h}{2}} \cong K^*_{(\frac{h}{2}:2)}$, we have a decomposition of $K^*_x \wr \overline{K}_h$ into $2x - 2$ $K^*_{(\frac{h}{2}:2)}$-factors.

Now, let F_0 be the K^*_h-factor and $F_1, F_2, \ldots, F_{2x-2}$ be the $K^*_{(\frac{h}{2}:2)}$-factors of K^*_{hx}. Since $\mathrm{HWP}^*(h; m^{r'}, n^{s'})$ is assumed to have a solution for all nonnegative integers r' and s', F_0 has a $\{\vec{C}^{r'}_m, \vec{C}^{s'}_n\}$-factorization for all nonnegative integers r' and s' where $r' + s' = h - 1$. Also, by Lemma 2.3 $K^*_{(\frac{h}{2}:2)}$ has a $\vec{C}_m$- and a $\vec{C}_n$-factorization for $m, n \geq 4$, so each F_j has a $\{\vec{C}^{\frac{h}{2}r_j}_m, \vec{C}^{\frac{h}{2}s_j}_n\}$-factorization for $j \in \{1, 2, \ldots, 2x - 2\}$, where $r_j, s_j \in \{0, 1\}$ with $r_j + s_j = 1$. Those factorizations give us a $\{\vec{C}^r_m, \vec{C}^s_n\}$-factorization of K^*_{hx} where $r = r' + \frac{h}{2}\sum_{j=1}^{2x-2} r_j$ and $s = s' + \frac{h}{2}\sum_{j=1}^{2x-2} s_j$ with $r + s = r' + s' + \frac{h}{2}\sum_{j=1}^{2x-2}(r_j + s_j) = h - 1 + \frac{h}{2}(2x - 2) = hx - 1$.

Since any nonnegative integer $0 \leq r \leq hx - 1$ can be written as $r = r' + \frac{h}{2}a$ for integers $0 \leq r' \leq h - 1$, $0 \leq a \leq 2x - 2$ and even h, a solution to $\mathrm{HWP}^*(hx; m^r, n^s)$ exists for each $r \geq 0$ and $s \geq 0$ satisfying $r + s = hx - 1$. □

Lemma 3.3. *For every even integer $m \geq 2$, $\vec{C}_m \wr \overline{K}_2$ has a $\vec{C}_m$- or $\vec{C}_{2m}$-factorization.*

Proof. Let $m \geq 2$ be an integer. We can represent $\vec{C}_m \wr \overline{K}_2$ as $\vec{X}(\mathbb{Z}_2 \times \mathbb{Z}_m; S_1)$, the directed Cayley graph over $\mathbb{Z}_2 \times \mathbb{Z}_m$ with the connection set $S_1 = \{(0, 1), (1, 1)\}$. Let $\vec{C}_{(1)} = (v_0, v_1, \ldots, v_{m-1})$ be a cycle of $\vec{C}_m \wr \overline{K}_2$, where $v_i = (0, i)$ for $0 \leq i \leq m - 1$, and it can be checked that $F_1 = \vec{C}_{(1)} \cup (\vec{C}_{(1)} + (1, 0))$ is a directed m-cycle factor of $\vec{C}_m \wr \overline{K}_2$. Also, let $\vec{C}_{(2)} = (u_0, u_1, \ldots, u_{m-1})$ be a cycle of $\vec{C}_m \wr \overline{K}_2$, where

$$u_i = \begin{cases} (0, i) & if\ \ i\ \ is\ \ even \\ (1, i) & if\ \ i\ \ is\ \ odd \end{cases}$$

for $0 \leq i \leq m - 1$. It can be checked that $F_2 = \vec{C}_{(2)} \cup (\vec{C}_{(2)} + (1, 0))$ is a directed m-cycle factor of $\vec{C}_m \wr \overline{K}_2$. F_1 and F_2 are arc disjoint directed m-cycle factors of $\vec{C}_m \wr \overline{K}_2$. Thus $\{F_1, F_2\}$ is a $\vec{C}_m$-factorization of $\vec{C}_m \wr \overline{K}_2$.

Let $\vec{C}_{(3)} = (v_0, v_1, \ldots, v_{2m-1})$ and $\vec{C}_{(4)} = (w_0, w_1, \ldots, w_{2m-1})$ be cycles of $\vec{C}_m \wr \overline{K}_2$, where

$$v_i = \begin{cases} (0,i) & if \;\; 0 \le i \le m-1 \\ (1,i) & if \;\; m \le i \le 2m-1 \end{cases}$$

and

$$w_i = \begin{cases} v_i & if \;\; i \;\; is \;\; even, \\ v_i + (1,0) & if \;\; i \;\; is \;\; odd. \end{cases}$$

It can be checked that $\vec{C}_{(3)}$ and $\vec{C}_{(4)}$ are arc disjoint directed $2m$-cycle factors of $\vec{C}_m \wr \overline{K}_2$. Thus $\{\vec{C}_{(3)}, \vec{C}_{(4)}\}$ is a $\vec{C}_{2m}$-factorization of $\vec{C}_m \wr \overline{K}_2$. □

For $m \ge 2$, we can represent $(\vec{C}_m \wr \overline{K}_2) \oplus mK_2^*$ as the directed Cayley graph over $\mathbb{Z}_2 \times \mathbb{Z}_m$ with the connection set $S_2 = \{(0,1),(1,0),(1,1)\}$ where K_2^* consists of edges between $(0,i)$ and $(1,i)$ for $0 \le i \le m-1$. For brevity, we will denote $(\vec{C}_m \wr \overline{K}_2) \oplus mK_2^*$ by Γ_m.

Lemma 3.4. *For every integer $m \ge 2$, Γ_m has a $\{\vec{C}_m^1, \vec{C}_{2m}^2\}$-factorization.*

Proof. Let $\vec{C}_{(1)} = (v_0, v_1, \ldots, v_{m-1})$ be a cycle of Γ_m, where $v_i = (0,i)$ for $0 \le i \le m-1$, and it can be checked that $F_1 = \vec{C}_{(1)} \cup (\vec{C}_{(1)} + (1,0))$ is a directed m-cycle factor of Γ_m. Also, let $\vec{C}_{(2)} = (u_0, u_1, \ldots, u_{2m-1})$ be a cycle of Γ_m, where $u_{2i} = (0,i)$, and $u_{2i+1} = (1,i)$ for $0 \le i \le m-1$. Similarly, it can be checked that $F_2 = \vec{C}_{(2)}$ and $F_3 = \vec{C}_{(2)} + (1,0)$ are arc disjoint directed $2m$-cycle factors of Γ_m. Thus $\{F_1, F_2, F_3\}$ is a $\{\vec{C}_m^1, \vec{C}_{2m}^2\}$-factorization of Γ_m. □

The following lemmata give the base blocks of our main construction. The cases when $r = 0$ and $s = 0$ of the lemmata are obtained by Theorem 1.1 and the remaining factorizations for Lemma 3.5 and 3.6 are given in the Appendix.

Lemma 3.5. *For nonnegative integers r and s,* $\mathrm{HWP}^*(8; 4^r, 8^s)$ *has a solution if and only if $r + s = 7$.*

Lemma 3.6. *For nonnegative integers r and s,* $\mathrm{HWP}^*(12; m^r, n^s)$ *has a solution for $(m,n) \in \{(4,6),(4,12),(6,12)\}$ if and only if $r + s = 11$.*

Lemma 3.7. *For nonnegative integers r and s,* $\mathrm{HWP}^*(16; m^r, n^s)$ *has a solution for $(m,n) \in \{(4,16),(8,16)\}$ if and only if $r + s = 15$.*

Proof. By Theorem 1.1, the cases when $r = 0$ and $s = 0$ are obtained.

Case 1 : $(m,n) = (8,16)$:

We will first analyse when r is odd. We have that $K_{16}^* \cong (K_8^* \wr \overline{K}_2) \oplus 8K_2^*$ by (3.1), and K_8^* have a $\vec{C}_8$-factorization by Lemma 3.5. Then, we have a factorization of K_{16}^* into six $\vec{C}_8 \wr \overline{K}_2$ and a single

Γ_8 factor. Also, each $\vec{C}_8 \wr \overline{K}_2$ can be decomposed into two $\vec{C}_8$ or two $\vec{C}_{16}$-factors by Lemma 3.3. By Lemma 3.4, Γ_8 has a $\{\vec{C}_8^1, \vec{C}_{16}^2\}$-factorization. Now, let r_0 and s_0 be nonnegative integers with $r_0 + s_0 = 6$. Decomposing r_0 $\vec{C}_8 \wr \overline{K}_2$'s into $\vec{C}_8$-factors and remaining s_0 $\vec{C}_8 \wr \overline{K}_2$'s into $\vec{C}_{16}$-factors, as well as Γ_8 into a $\{\vec{C}_8^1, \vec{C}_{16}^2\}$-factor gives us a $\{\vec{C}_8^{2r_0+1}, \vec{C}_{16}^{2s_0+2}\}$-factorization of K_{16}^*.

Since any odd integer r can be written as $r = 2r_0 + 1$ for a nonnegative integer r_0, HWP$^*(16; 8^r, 16^s)$ has a solution for odd r with $r + s = 2r_0 + 1 + 2s_0 + 2 = 2(r_0 + s_0) + 3 = 15$.

We list the solutions to the remaining even cases in the Appendix.

Case 2 : For $(m, n) = (4, 16)$, solutions to all cases are given in the Appendix except for $r = 0$ and $s = 0$.

□

Theorem 3.8. *For nonnegative integers r and s,* HWP$^*(v; m^r, n^s)$ *has a solution for* $(m, n) \in \{(4, 6), (4, 8), (4, 12), (4, 16), (6, 12), (8, 16)\}$ *if and only if* $r + s = v - 1$ *and* $\text{lcm}(m, n)|v$.

Proof. If a solution to HWP$^*(v; m^r, n^s)$ exists for $(m, n) \in \{(4, 6), (4, 8), (4, 12), (4, 16), (6, 12), (8, 16)\}$, then by Lemma 1.2 we have $r + s = v - 1$, and since $m|v$ and $n|v$ we have $h = \text{lcm}(m, n)|v$.

For the sufficiency part, assume $h|v$ and $r + s = hx - 1 = v - 1$ where x is a nonnegative integer.

For $(m, n) = (4, 8)$, HWP$^*(8; 4^{r_0}, 8^{s_0})$ has a solution for all nonegative r_0 and s_0 with $r_0 + s_0 = 7$ by Lemma 3.5. Then, HWP$^*(v; 4^r, 8^s)$ has a solution for $r + s = 8x - 1 = v - 1$ by Lemma 3.2.

For $(m, n) \in \{(4, 6), (4, 12), (6, 12)\}$, HWP$^*(12; m^{r_1}, n^{s_1})$ has a solution for all nonegative r_1 and s_1 with $r_1 + s_1 = 11$ by Lemma 3.6. Then, HWP$^*(v; m^r, n^s)$ has a solution by Lemma 3.2 for $(m, n) \in \{(4, 6), (4, 12), (6, 12)\}$ with $r + s = 12x - 1 = v - 1$.

For $(m, n) \in \{(4, 16), (8, 16)\}$, HWP$^*(16; m^{r_2}, n^{s_2})$ has a solution for all nonegative r_2 and s_2 with $r_2 + s_2 = 15$ by Lemma 3.7. Then, by Lemma 3.2, HWP$^*(v; m^r, n^s)$ has a solution for $(m, n) \in \{(4, 16), (8, 16)\}$ with $r + s = 16x - 1 = v - 1$. □

4. Odd Cycle Sizes

Here we first give the following main construction for odd cycle sizes, and using this construction we prove that for $(m, n) \in \{(3, 5), (3, 15), (5, 15)\}$, HWP$^*(v; m^r, n^s)$ has a solution for all nonnegative integers r and s satisfying $r + s = v - 1$ with a few possible exceptions, where v is odd.

Lemma 4.1. *Let $m \geq 3$ and $n \geq 3$ be both odd, $h = \text{lcm}(m, n)$ and $3|h$. If* HWP$^*(h; m^{r'}, n^{s'})$ *has a solution for all r', s' satisfying $r' + s' = h - 1$, then there is a solution to* HWP$^*(hx; m^r, n^s)$ *for all nonnegative r, s and odd x satisfying $r + s = hx - 1$.*

Proof. By (3.1), we have a decomposition of K^*_{hx} into a K^*_h and a $\left(K^*_x \wr \overline{K}_h\right)$-factor. Since $\overline{K}_h \cong \overline{K}_3 \wr \overline{K}_{\frac{h}{3}}$, we have $K^*_x \wr \overline{K}_h \cong (K^*_x \wr \overline{K}_3) \wr \overline{K}_{\frac{h}{3}}$.

It is clear that $K^*_x \wr \overline{K}_3$ is isomorphic to $K^*_{3x} - xK^*_3$. Since a Kirkman triple system of order $3x$ exists, we have a C_3-factorization of K_{3x}. Then, a $C^*_3 \cong K^*_3$-factorization of $K^*_{3x} - xK^*_3$ is obtained by Lemma 2.2. So, $K^*_x \wr \overline{K}_3$ has a decomposition into $\frac{3x-3}{2}$ K^*_3-factors. In $K^*_x \wr \overline{K}_h$, these K^*_3-factors form $K^*_{(\frac{h}{3}:3)}$-factors since $K^*_3 \wr \overline{K}_{\frac{h}{3}} \cong K^*_{(\frac{h}{3}:3)}$.

Let F_0 be the K^*_h-factor and $F_1, F_2, \dots, F_{\frac{3x-3}{2}}$ be the $K^*_{(\frac{h}{3}:3)}$-factors of K^*_{hx}. Since HWP$^*(h; m^{r'}, n^{s'})$ is assumed to have a solution for all nonnegative integers r' and s' where $r' + s' = h-1$, F_0 has a $\{\vec{C}^{r'}_m, \vec{C}^{s'}_n\}$-factorization for all nonnegative integers r' and s' with $r' + s' = h - 1$. Also, $K^*_{(\frac{h}{3}:3)}$ has a $\vec{C}_m$-factorization and a $\vec{C}_n$-factorization by Lemma 2.2 and Theorem 2.4, so each F_j has a $\{\vec{C}^{\frac{2h}{3}r_j}_m, \vec{C}^{\frac{2h}{3}s_j}_n\}$-factorization for $j \in \{1, 2, \dots, \frac{3x-3}{2}\}$, where $r_j, s_j \in \{0, 1\}$ with $r_j + s_j = 1$. These factorizations give us a $\{\vec{C}^r_m, \vec{C}^s_n\}$-factorization of K^*_{hx} where $r = r' + \sum_{i=0}^{\frac{3x-3}{2}} \frac{2h}{3} r_i$ and $s = s' + \sum_{i=0}^{\frac{3x-3}{2}} \frac{2h}{3} s_i$ with $r + s = r' + s' + \sum_{i=0}^{\frac{3x-3}{2}} \frac{2h}{3}(r_i + s_i) = h - 1 + hx - h = hx - 1$.

Since any nonnegative integer $0 \le r \le hx-1$ can be written as $r = r' + \frac{2h}{3}a$ for integers $0 \le r' \le h-1$ and $0 \le a \le \frac{3x-3}{2}$, a solution to HWP$^*(hx; m^r, n^s)$ exists for each $r \ge 0$ and $s \ge 0$ satisfying $r + s = hx - 1$. □

Lemma 4.2. *For nonnegative integers r and s,* HWP$^*(15; m^r, n^s)$ *has a solution for $(m,n) \in \{(3,5), (3,15), (5,15)\}$ if and only if $r + s = 14$ except possibly for $r \in \{11, 12, 13\}$ when $(m,n) = (3,5)$ and for $r = 13$ when $(m,n) = (3,15)$.*

Proof. The cases when $r = 0$ and $s = 0$ can be obtained by Theorem 1.1. In [5], a solution to HWP$(15; m^{r_0}, n^{s_0})$ for $(m,n) \in \{(3,5), (3,15), (5,15)\}$ with the exception $(m, n, r_0, s_0) = (3, 5, 6, 1)$ is given by Theorem 4.1. Thus, by Observation 2.1, we have a solution to HWP$^*(15; m^r, n^s)$ for $(m,n) \in \{(3,5), (3,15), (5,15)\}$ with r and s are positive even integers except possibly when $(m, n, r, s) = (3, 5, 12, 2)$. We list the solutions for the odd cases in the Appendix. □

Using Lemma 4.1 and Lemma 4.2, we can give a solution to HWP$^*(v; m^r, n^s)$ for all nonnegative integers r and s when $(m,n) \in \{(3,5), (3,15), (5,15)\}$ satisfying $r + s = v - 1$ and odd $v > 15$ with a few possible exceptions.

Theorem 4.3. *For all nonnegative integers r, s and odd $v > 15$,* HWP$^*(v; m^r, n^s)$ *has a solution for $(m,n) \in \{(3,5), (3,15), (5,15)\}$ if and only if $r + s = v - 1$ and $15 | v$ except possibly $s \in \{1, 2, 3\}$ when $(m,n) = (3,5)$ and $s = 1$ when $(m,n) = (3,15)$.*

Proof. If a solution to HWP$^*(v; m^r, n^s)$ exists for $(m,n) \in \{(3,5), (3,15), (5,15)\}$, $r + s = v - 1$ and $\mathrm{lcm}(m,n) = 15 | v$ by Lemma 1.2.

For the sufficiency part, assume $v = 15x$ and $r+s = 15x-1$ where $x > 1$ is an odd integer.

For $(m,n) = (5,15)$, HWP$^*(15; 5^{r_0}, 15^{s_0})$ has a solution for all nonnegative r_0 and s_0 with $r_0 + s_0 = 14$ by Lemma 4.2. Then, HWP$^*(v; 5^r, 15^s)$ has a solution for all nonegative r and s with $r + s = 15x - 1 = v - 1$ by Lemma 4.1.

By (3.1), we have a decomposition of K^*_{15x} into a K^*_{15}-factor and $\frac{3x-3}{2}$ $K^*_{(5:3)}$-factors. By Lemma 4.2, K^*_{15} has a $\{\vec{C}^{r'}_m, \vec{C}^{s'}_n\}$-factorization for $(m,n) \in \{(3,5),(3,15)\}$ with $r' + s' = 14$ except possibly $r' \in \{11,12,13\}$ when $(m,n) = (3,5)$ and $r' = 13$ when $(m,n) = (3,15)$. Also, by Lemma 2.2 and Theorem 2.4, each $K^*_{(5:3)}$ has a $\{\vec{C}^{10r_j}_m, \vec{C}^{10s_j}_n\}$-factorization for $j \in \{1,2,\ldots,\frac{3x-3}{2}\}$, where $r_j, s_j \in \{0,1\}$ with $r_j + s_j = 1$.

Placing a $\vec{C}_m$-factorization on "a" of the $K^*_{(5:3)}$-factors, a $\vec{C}_n$-factorization on "b" of the $K^*_{(5:3)}$-factors for $0 \le a,b \le \frac{3x-3}{2}$ with $a+b = \frac{3x-3}{2}$, and taking a $\{\vec{C}^{r'}_m, \vec{C}^{s'}_n\}$-factorization of K^*_{15} give a $\{\vec{C}^{r'+10a}_m, \vec{C}^{s'+10b}_n\}$-factorization of K^*_{15x} for $(m,n) \in \{(3,5),(3,15)\}$. Let $r = r' + 10a$ and $s = s' + 10b$, then we have $r + s = r' + s' + 10(a+b) = 14 + 5(3x-3) = 15x - 1 = v - 1$ with $0 \le r,s \le 15x-1$.

For $(m,n) = (3,5)$, we can obtain the requested integer $r \in [0,15x-5] \cup \{15x-1\}$ from the sum of r' and $10a$ for $r' \in [0,14] \backslash \{11,12,13\}$ and $0 \le a \le \frac{3x-3}{2}$. Therefore, HWP$^*(15x; 3^r, 5^s)$ has a solution for all integers $0 \le r,s \le 15x-1$ with $r+s = 15x-1 = v-1$ except possibly when $s \in \{1,2,3\}$. Similarly, for $(m,n) = (3,15)$, since any nonnegative integer $r \in [0,15x-1] \backslash \{15x-2\}$ can be written as $r = r' + 10a$ for $r' \in [0,14] \backslash \{13\}$ and $0 \le a \le \frac{3x-3}{2}$, a solution to HWP$^*(15x; 3^r, 15^s)$ exists for all integers $0 \le r,s \le 15x-1$ with $r+s = 15x-1 = v-1$ except possibly when $s = 1$. □

According to our best knowledge, our results are the first findings for the directed version of the Hamilton-Waterloo Problem. We have first examined the cases $(m,n) \in \{(4,6),(4,8),(4,16),(8,16),(3,5),(3,15),(5,15)\}$, as done in the first paper on the undirected Hamilton-Waterloo Problem by Adams et al. [5]. We have also solved the problem for the cases $(m,n) \in \{(4,12),(6,12)\}$. In addition to studying odd cycle cases $\{(3,5),(3,15),(5,15)\}$, we have also observed that if HWP$(v; m^r, n^s)$ has a solution for odd v, then HWP$^*(v; m^{2r}, n^{2s})$ has a solution for the same r and s as well. Since there is no 2-factorizations of K_v for even v, we cannot arrive a similar observation when v is even and $m,n > 2$. Our constructions given in Lemma 3.2 and Lemma 4.1 can also be used to solve the problem for the other cycle sizes as long as the necessary small cases can be found.

Now we can combine our results in the following main theorem.

Theorem 4.4. *For nonnegative integers r and s,* HWP$^*(v; m^r, n^s)$ *has a solution for*

(1) $(m,n) \in \{(4,6),(4,8),(4,12),(4,16),(6,12),(8,16)\}$ *when* v *is even,*
(2) $(m,n) \in \{(3,5),(3,15),(5,15)\}$ *when* v *is odd*

if and only if $r+s=v-1$ *and* $\mathrm{lcm}(m,n)|v$ *except possibly* $s \in \{1,2,3\}$ *when* $(m,n)=(3,5)$ *and* $s=1$ *when* $(m,n)=(3,15)$.

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6. Appendix

Let $V(K_8^*)=\mathbb{Z}_8$, $V(K_{12}^*)=\mathbb{Z}_{12}$, $V(K_{16}^*)=\mathbb{Z}_{16}$ and $V(K_{15}^*)=\mathbb{Z}_{15}$,

(1) $\mathrm{HWP}^*(8;4^1,8^6)$,
[(0,1,2,3),(4,5,6,7)], [(0,3,2,4,6,5,7,1)], [(0,2,1,3,5,4,7,6)], [(0,7,5,3,6,1,4,2)], [(0,6,4,3,1,7,2,5)], [(0,5,1,6,2,7,3,4)], [(0,4,1,5,2,6,3,7)]

(2) $\mathrm{HWP}^*(8;4^2,8^5)$,
[(0,2,7,6),(1,4,5,3)], [(0,5,6,7),(1,3,2,4)], [(0,1,7,2,6,3,5,4)], [(0,3,7,4,6,5,1,2)],[(0,4,2,5,7,3,6,1)], [(0,6,2,3,4,7,1,5)], [(0,7,5,2,1,6,4,3)]

(3) $\mathrm{HWP}^*(8;4^3,8^4)$,
[(0,1,2,3),(4,5,6,7)], [(0,2,4,6),(1,3,5,7)], [(0,3,2,1),(4,7,6,5)], [(0,4,1,5,2,6,3,7)], [(0,5,1,6,2,7,3,4)], [(0,6,4,3,1,7,2,5)], [(0,7,5,3,6,1,4,2)]

(4) $\mathrm{HWP}^*(8;4^4,8^3)$,
[(0,3,7,4),(1,2,6,5)], [(0,1,7,2),(3,5,4,6)], [(0,2,7,6),(1,4,5,3)], [(0,5,6,7),(1,3,2,4)], [(0,4,2,5,7,3,6,1)], [(0,6,2,3,4,7,1,5)], [(0,7,5,2,1,6,4,3)]

(5) $\mathrm{HWP}^*(8;4^5,8^2)$,
[(0,1,2,3),(4,5,6,7)], [(0,2,4,6),(1,3,5,7)], [(0,3,2,1),(4,7,6,5)], [(0,6,4,2),(1,7,5,3)], [(0,7,2,5),(4,3,6,1)], [(0,4,1,5,2,6,3,7)], [(0,5,1,6,2,7,3,4)]

(6) $\mathrm{HWP}^*(8;4^6,8^1)$,
[(0,2,1,3),(4,7,6,5)], [(0,3,7,4),(1,5,2,6)], [(0,4,1,6),(2,7,5,3)], [(0,5,7,1),(2,4,3,6)], [(0,6,4,2),(1,7,3,5)], [(0,7,2,5),(1,4,6,3)], [(0,1,2,3,4,5,6,7)]

(7) $\mathrm{HWP}^*(12;4^1,6^{10})$,
[(0,2,1,3,4,6),(5,7,8,10,9,11)], [(0,3,1,4,2,5),(6,8,11,9,7,10)], [(0,4,1,5,2,7),(3,9,6,11,10,8)], [(0,5,1,6,2,8),(3,10,4,11,7,9)][(0,6,1,7,2,9),(3,8,4,10,5,11)], [(0,7,1,8,2,10),(3,11,6,5,9,4)], [(0,8,6,3,2,4),(1,9,5,10,7,11)], [(0,9,1,10,2,11),(3,7,6,4,8,5)], [(0,10,3,6,9,2), (1,11,4,7,5,8)], [(0,11,2,6,10,1),(3,5,4,9,8,7)], [(0,1,2,3),(4,5,6,7),(8,9,10,11)]

(8) $\mathrm{HWP}^*(12;4^2,6^9)$,
[(0,3,1,5,2,6),(4,7,10,8,11,9)], [(0,4,1,3,2,5),(6,8,10,9,7,11)], [0,5,1,6,2,8),(3,10,4,11,7,9)], [(0,6,1,7,2,9),(3,8,4,10,5,11)], [(0,7,1,9,6,10),(2,4,3,11,5,8)], [(0,8,1,11,2,7),(3,6,4,9,5,10)], [(0,9,1,10,2,11),(3,4,8,7,6,5)], [(0,10,7,3,9,2),(1,8,5,4,6,11)], [(0,11,4,2,10,1),(3,7,5,9,8,6)], [(0,1,2,3),(4,5,6,7),(8,9,10,11)], [(0,2,1,4),(3,5,7,8),(6,9,11,10)]

(9) $\mathrm{HWP}^*(12;4^3,6^8)$,
[(0,3,1,6,2,7),(4,8,10,5,11,9)], [(0,5,1,3,2,8),(4,6,10,9,7,11)], [(0,11,1,10,8,2),(3,4,7,6,5,9)], [(0,6,1,7,2,9),(3,8,4,11,5,10)], [(0,7,1,11,2,10),(3,9,6,8,5,4)], [(0,8,7,3,11,6),(1,9,5,2,4,10)], [(0,9,1,8,6,11),(2,5,3,7,10,4)], [(0,10,7,5,8,1),(2,11,3,6,4,9)], [(0,1,2,3),(4,5,6,7),(8,9,10,11)], [(0,2,1,4),(3,5,7,8),(6,9,11,10)], [(0,4,1,5),(2,6,3,10),(7,9,8,11)]

(10) $\mathrm{HWP}^*(12;4^4,6^7)$,
[(0,3,1,7,2,8),(4,6,10,5,11,9)], [(0,6,1,3,2,9),(4,11,5,8,7,10)], [(0,7,1,8,4,10),(2,11,6,5,3,9)], [(0,8,6,4,3,11),(1,9,7,5,2,10)], [(0,9,5,10,8,1),(2,7,6,11,3,4)], [(0,10,7,3,6,2),(1,11,4,8,5,9)], [(0,11,1,10,3,7),(2,5,4,9,6,8)], [(0,1,2,3),(4,5,6,7),(8,9,10,11)], [(0,2,1,4),(3,5,7,8),(6,9,11,10)], [(0,4,1,5),(2,6,3,10),(7,9,8,11)], [(0,5,1,6),(2,4,7,11),(3,8,10,9)]

(11) $\mathrm{HWP}^*(12;4^5,6^6)$,
[(0,3,1,8,4,2),(5,11,9,6,10,7)], [(0,7,1,9,2,8),(3,6,11,5,10,4)], [(0,8,5,9,1,10),(2,7,6,4,11,3)], [(0,9,7,10,1,11),(2,5,3,4,8,6)], [(0,10,5,2,11,1),(3,9,4,6,8,7)], [(0,11,6,5,4,9),(1,3,7,2,10,8)], [(0,1,2,3),(4,5,6,7),(8,9,10,11)], [(0,2,1,4),(3,5,7,8),(6,9,11,10)], [(0,4,1,5),(2,6,3,10),(7,9,8,11)], [(0,5,1,6),(2,4,7,11),(3,8,10,9)], [(0,6,1,7),(2,9,5,8),(3,11,4,10)]

(12) $\mathrm{HWP}^*(12;4^6,6^5)$,
[(0,3,1,9,6,8),(2,7,10,4,11,5)], [(0,8,7,6,11,9),(1,3,4,2,5,10)], [(0,9,7,5,11,1),(2,10,8,6,4,3)],
[(0,10,7,3,9,2),(1,11,6,5,4,8)], [(0,11,3,7,1,10),(2,8,5,9,4,6)], [(0,1,2,3),(4,5,6,7),(8,9,10,11)],
[(0,2,1,4),(3,5,7,8),(6,9,11,10)], [(0,4,1,5),(2,6,3,10),(7,9,8,11)], [(0,5,1,6),(2,4,7,11),(3,8,10,9)],
[(0,6,1,7),(2,9,5,8),(3,11,4,10)], [(0,7,2,11),(1,8,4,9),(3,6,10,5)]

(13) $\mathrm{HWP}^*(12;4^7,6^4)$,
[(0,3,9,2,7,1),(4,11,5,10,8,6)], [(0,9,4,3,2,8),(1,10,7,6,5,11)], [(0,10,4,2,5,9),(1,11,6,8,7,3)],
[(0,11,9,6,2,10),(1,3,7,5,4,8)], [(0,1,2,3),(4,5,6,7),(8,9,10,11)], [(0,2,1,4),(3,5,7,8),(6,9,11,10)],
[(0,4,1,5),(2,6,3,10),(7,9,8,11)], [(0,5,1,6),(2,4,7,11),(3,8,10,9)], [(0,6,1,7),(2,9,5,8),(3,11,4,10)],
[(0,7,2,11),(1,8,4,9),(3,6,10,5)], [0,8,5,2),(1,9,7,10),(3,4,6,11)]

(14) $\mathrm{HWP}^*(12;4^8,6^3)$,
[(0,3,9,2,7,1),(4,11,5,10,8,6)], [(0,9,4,3,2,8),(1,10,7,6,5,11)], [(0,10,4,2,5,9),(1,11,6,8,7,3)],
[(0,1,2,3),(4,5,6,7),(8,9,10,11)], [(0,2,1,4),(3,5,7,8),(6,11,10,9)], [(0,4,1,6),(2,11,9,7),(3,8,10,5)],
[(0,5,1,7),(2,6,9,8),(3,11,4,10)], [(0,6,2,10),(1,9,5,8),(3,4,7,11)], [(0,7,9,11),(1,3,6,10),(2,4,8,5)],
[(0,8,11,2),(1,5,4,9),(3,7,10,6)], [(0,11,7,5),(1,8,4,6),(2,9,3,10)]

(15) $\mathrm{HWP}^*(12;4^9,6^2)$,
[(0,3,9,2,7,1),(4,11,5,10,8,6)], [(0,9,4,3,2,8),(1,10,7,6,5,11)], [(0,1,2,3),(4,5,6,7),(8,9,10,11)],
[(0,2,1,4),(3,5,7,8),(6,9,11,10)], [(0,4,1,5),(2,6,3,10),(7,9,8,11)], [(0,5,1,7),(2,10,4,8),(3,6,11,9)],
[(0,6,2,11),(1,9,7,3),(4,10,5,8)], [(0,7,11,2),(1,8,5,9),(3,4,6,10)], [(0,8,7,10),(1,3,11,6),(2,9,5,4)],
[(0,10,9,6),(1,11,3,8),(2,4,7,5)], [(0,11,4,9),(1,6,8,10),(2,5,3,7)]

(16) $\mathrm{HWP}^*(12;4^{10},6^1)$,
[(0,3,9,2,7,1),(4,11,5,10,8,6)], [(0,1,2,3),(4,5,6,7),(8,9,10,11)], [(0,2,1,4),(3,5,7,6),(8,11,10,9)],
[(0,4,1,5),(2,6,8,10),(3,7,9,11)], [(0,5,1,6),(2,8,3,11),(4,9,7,10)], [(0,6,5,2),(1,9,3,10),(4,8,7,11)],
[(0,7,2,10),(1,8,5,3),(4,6,11,9)], [(0,8,1,11),(2,5,9,6),(3,4,10,7)], [(0,9,1,7),(2,4,3,8),(5,11,6,10)],
[(0,10,6,9),(1,3,2,11),(4,7,5,8)], [(0,11,7,8),(1,10,3,6),(2,9,5,4)]

(17) $\mathrm{HWP}^*(12;4^1,12^{10})$,
[(0,1,2,3),(4,5,6,7),(8,9,10,11)], [(0,2,1,3,4,6,5,7,8,10,9,11)], [(0,3,1,4,2,5,8,6,9,7,11,10)],
[(0,4,1,5,2,6,3,7,10,8,11,9)], [(0,5,1,6,2,4,10,3,11,7,9,8)], [(0,6,1,8,2,9,3,10,5,11,4,7)],
[(0,7,1,9,2,8,3,5,10,4,11,6)], [(0,8,1,7,3,9,6,10,2,11,5,4)], [(0,9,4,8,5,3,6,11,1,10,7,2)],
[(0,10,1,11,3,2,7,6,8,4,9,5)], [(0,11,2,10,6,4,3,8,7,5,9,1)]

(18) $\mathrm{HWP}^*(12;4^2,12^9)$,
[(0,1,2,3),(4,5,6,7),(8,9,10,11)], [(0,2,4,6),(8,10,1,3),(5,7,9,11)], [(0,3,1,4,2,5,8,6,9,7,11,10)],
[(0,4,1,5,2,6,3,7,10,8,11,9)], [(0,5,1,6,2,7,3,10,9,8,4,11)], [(0,6,1,7,2,9,5,10,3,11,4,8)],
[(0,7,1,8,2,10,4,9,6,11,3,5)], [(0,8,1,9,2,11,6,5,3,4,10,7)], [(0,9,1,11,7,8,5,4,3,6,10,2)],
[(0,10,6,8,3,9,4,7,5,11,2,1)], [(0,11,1,10,5,9,3,2,8,7,6,4)]

(19) $\mathrm{HWP}^*(12;4^3,12^8)$,
[(0,1,2,3),(4,5,6,7),(8,9,10,11)], [(0,2,4,6),(8,10,1,3),(5,7,9,11)], [(0,3,6,1),(7,10,2,8),(4,9,5,11)],
[(0,4,1,5,2,6,3,7,8,11,10,9)], [(0,5,1,4,2,7,3,10,6,11,9,8)], [(0,6,2,1,7,5,9,4,10,8,3,11)],
[(0,7,1,6,5,8,4,11,3,9,2,10)], [(0,8,1,9,3,2,5,10,7,11,6,4)], [(0,9,1,8,2,11,7,6,10,4,3,5)],
[(0,10,3,1,11,2,9,6,8,5,4,7)], [(0,11,1,10,5,3,4,8,6,9,7,2)]

(20) $\mathrm{HWP}^*(12;4^4,12^7)$,
[(0,1,2,3),(4,5,6,7),(8,9,10,11)], [(0,2,4,6),(8,10,1,3),(5,7,9,11)], [(0,3,6,1),(7,10,2,8),(4,9,5,11)],
[(0,4,1,10),(2,5,3,9),(11,7,6,8)], [(0,5,1,4,2,6,3,7,11,10,9,8)], [(0,6,2,1,5,4,10,7,8,3,11,9)],
[(0,7,1,6,4,11,2,9,3,10,8,5)], [(0,8,1,7,2,10,4,3,5,9,6,11)], [(0,9,1,11,6,10,5,8,4,7,3,2)],
[(0,10,3,4,8,6,5,2,11,1,9,7)], [(0,11,3,1,8,2,7,5,10,6,9,4)]

(21) $\mathrm{HWP}^*(12;4^5,12^6)$,
[(0,1,2,3),(4,5,6,7),(8,9,10,11)], [(0,2,4,6),(8,10,1,3),(5,7,9,11)], [(0,3,6,1),(7,10,2,8),(4,9,5,11)],
[(0,4,1,10),(2,5,3,9),(11,7,6,8)], [(0,8,3,7),(9,6,11,2),(1,4,10,5)], [(0,5,2,1,6,3,4,7,11,10,9,8)],
[(0,6,2,7,1,5,9,3,10,8,4,11)], [(0,7,2,6,4,8,5,10,3,11,1,9)], [(0,9,1,7,8,6,10,4,2,11,3,5)],
[(0,10,7,5,8,1,11,6,9,4,3,2)], [(0,11,9,7,3,1,8,2,10,6,5,4)]

(22) $\mathrm{HWP}^*(12;4^6,12^5)$,
[(0,1,2,3),(4,5,6,7),(8,9,10,11)], [(0,2,4,6),(8,10,1,3),(5,7,9,11)], [(0,3,6,1),(7,10,2,8),(4,9,5,11)],
[(0,4,1,10),(2,5,3,9),(11,7,6,8)], [(0,8,3,7),(9,6,11,2),(1,4,10,5)], [(0,9,3,2),(1,5,10,8),(11,6,4,7)],
[(0,5,2,1,6,3,4,11,10,9,7,8)], [(0,6,2,7,1,11,3,10,4,8,5,9)], [(0,7,2,10,3,11,1,9,8,6,5,4)],
[(0,10,6,9,1,7,3,5,8,4,2,11)], [(0,11,9,4,3,1,8,2,6,10,7,5)]

(23) $\mathrm{HWP}^*(12;4^7,12^4)$,
[(0,3,2,1),(4,7,6,9),(5,10,8,11)], [(0,4,8,3),(1,7,9,6),(2,5,11,10)], [(0,5,8,7),(1,9,2,11),(3,6,4,10)],
[(0,6,2,10),(1,11,8,5),(3,9,7,4)], [(0,7,5,4),(1,8,2,9),(3,10,6,11)], [(0,9,5,2),(1,6,3,8),(4,11,7,10)],

[(0,10,9,8,),(1,4,2,7),(3,11,6,5)], [(0,11,4,1,10,5,9,3,7,2,8,6)], [(0,8,4,9,11,2,6,10,7,3,1,5)], [(0,1,2,3,4,5,6,7,8,9,10,11)], [(0,2,4,6,8,10,1,3,5,7,11,9)]

(24) $\mathrm{HWP}^*(12;4^8,12^3)$,
[(0,3,2,1),(4,7,5,9),(6,11,10,8)], [(0,4,1,6),(2,5,3,10),(7,9,8,11)], [(0,5,2,7),(1,10,4,8),(3,11,6,9)], [(0,6,4,2),(1,11,8,7),(3,9,5,10)], [(0,7,4,3),(1,8,5,11),(2,10,9,6)], [(0,9,2,8),(1,4,11,5),(3,7,10,6)], [(0,10,5,4),(1,9,7,6),(2,11,3,8)], [(0,11,4,10),(1,7,2,9),(3,6,5,8)], [(0,8,4,9,11,2,6,10,7,3,1,5)], [(0,1,2,3,4,5,6,7,8,9,10,11)], [(0,2,4,6,8,10,1,3,5,7,11,9)]

(25) $\mathrm{HWP}^*(12;4^9,12^2)$,
[(0,3,1,4),(2,5,8,6),(7,9,11,10)], [(0,4,1,5),(2,6,3,9),(7,10,8,11)], [(0,5,1,6),(2,7,3,11),(4,10,9,8)], [(0,6,1,8),(2,10,3,7),(4,9,5,11)], [(0,7,5,10),(1,11,8,2),(3,6,9,4)], [(0,8,3,2),(1,7,4,11),(5,9,6,10)], [(0,9,7,1),(2,8,5,4),(3,10,6,11)], [(0,10,4,7),(1,9,3,8),(2,11,6,5)], [(0,11,5,3),(1,10,2,9),(4,8,7,6)], [(0,1,2,3,4,5,6,7,8,9,10,11)], [(0,2,4,6,8,10,1,3,5,7,11,9)]

(26) $\mathrm{HWP}^*(12;4^{10},12^1)$,
[(0,2,1,3),(4,6,5,7),(8,10,9,11)], [(0,3,1,4),(2,5,8,11),(6,9,7,10)], [(0,4,1,5),(2,6,3,9),(7,11,10,8)], [(0,5,1,6),(2,4,10,7),(3,11,9,8)], [(0,6,2,7),(1,9,4,11),(3,8,5,10)], [(0,7,1,8),(2,11,6,10),(3,5,4,9)], [(0,8,1,10),(2,9,5,3),(4,7,6,11)], [(0,9,6,1),(2,10,4,8),(3,7,5,11)], [(0,10,5,2),(1,11,7,9),(3,6,8,4)], [(0,11,5,9),(1,7,3,10),(2,8,6,4)], [(0,1,2,3,4,5,6,7,8,9,10,11)]

(27) $\mathrm{HWP}^*(12;6^1,12^{10})$,
[(0,3,9,4,11,1),(2,5,10,8,7,6)], [(0,1,2,3,4,5,6,7,8,9,10,11)], [(0,2,1,3,5,4,6,8,10,7,11,9)], [(0,4,1,5,2,6,3,7,9,8,11,10)], [(0,5,1,4,2,7,3,6,10,9,11,8)], [(0,6,1,7,10,2,8,3,11,4,9,5)], [(0,7,1,6,9,2,10,4,8,5,11,3)], [(0,8,1,9,3,2,4,10,6,11,5,7)], [(0,9,7,2,11,6,5,3,10,1,8,4)], [(0,10,3,8,6,4,7,5,9,1,11,2)], [(0,11,7,4,3,1,10,5,8,2,9,6)]

(28) $\mathrm{HWP}^*(12;6^2,12^9)$,
[(0,3,9,4,11,1),(2,5,10,8,7,6)], [(0,4,7,1,9,2),(3,10,6,11,8,5)], [(0,1,2,3,4,5,6,7,8,9,10,11)], [(0,2,1,3,5,4,6,8,10,7,11,9)], [(0,5,1,4,2,6,3,7,9,8,11,10)], [(0,6,1,5,2,4,9,11,7,10,3,8)], [(0,7,2,8,1,6,4,10,9,3,11,5)], [(0,8,2,7,3,6,9,1,10,5,11,4)], [(0,9,5,7,4,1,8,6,10,2,11,3)], [(0,10,1,11,6,5,8,4,3,2,9,7)], [(0,11,2,10,4,8,3,1,7,5,9,6)]

(29) $\mathrm{HWP}^*(12;6^3,12^8)$,
[(0,3,9,4,11,1),(2,5,10,8,7,6)], [(0,4,7,1,9,2),(3,10,6,11,8,5)], [(0,6,1,11,3,8),(2,7,10,9,5,4)], [(0,1,2,3,4,5,6,7,8,9,10,11)], [(0,2,1,3,5,7,4,6,9,8,11,10)], [(0,5,1,4,3,2,6,8,10,7,11,9)], [(0,7,2,4,8,1,5,9,11,6,10,3)], [(0,8,2,9,3,6,5,11,4,10,1,7)], [(0,9,1,8,6,3,11,7,5,2,10,4)], [(0,10,2,11,5,8,4,9,7,3,1,6)], [(0,11,2,8,3,7,9,6,4,1,10,5)]

(30) $\mathrm{HWP}^*(12;6^4,12^7)$,
[(0,3,9,4,11,1),(2,5,10,8,7,6)], [(0,4,7,1,9,2),(3,10,6,11,8,5)], [(0,6,1,11,3,8),(2,7,10,9,5,4)], [(0,9,1,8,2,10),(3,6,5,11,7,4)], [(0,1,2,3,4,5,6,7,8,9,10,11)], [(0,2,1,3,5,7,9,6,8,11,10,4)], [(0,5,1,4,6,3,2,8,10,7,11,9)], [(0,7,2,4,1,10,5,9,8,3,11,6)], [(0,8,4,9,11,2,6,10,3,1,7,5)], [(0,10,1,5,2,11,4,8,6,9,3,7)], [(0,11,5,8,1,6,4,10,2,9,7,3)]

(31) $\mathrm{HWP}^*(12;6^5,12^6)$,
[(0,3,6,1,9,2),(4,11,7,10,8,5)], [(0,4,2,5,10,3),(1,11,8,7,6,9)], [(0,6,11,1,4,10),(2,7,5,3,9,8)], [(0,7,4,3,8,1),(2,10,9,5,11,6)], [(0,9,4,7,1,8),(2,11,3,10,6,5)],[(0,1,2,3,4,5,6,7,8,9,10,11)], [(0,2,4,6,8,10,1,3,5,7,11,9)], [(0,5,8,3,11,10,2,1,7,9,6,4)], [(0,8,4,9,11,2,6,10,7,3,1,5)], [(0,10,4,8,11,5,1,6,3,2,9,7)], [(0,11,4,1,10,5,9,3,7,2,8,6)]

(32) $\mathrm{HWP}^*(12;6^6,12^5)$,
[(0,3,2,5,1,8),(4,11,6,9,7,10)], [(0,4,2,7,5,10),(1,6,11,3,9,8)], [(0,6,1,9,5,2),(3,8,11,7,4,10)], [(0,7,6,5,11,1),(2,9,4,3,10,8)], [(0,9,2,10,6,3),(1,11,5,4,8,7)], [(0,10,9,1,4,7),(2,11,8,5,3,6)], [(0,1,2,3,4,5,6,7,8,9,10,11)], [(0,2,4,6,8,10,1,3,5,7,11,9)], [(0,5,8,3,11,10,2,1,7,9,6,4)], [(0,8,4,9,11,2,6,10,7,3,1,5)], [(0,11,4,1,10,5,9,3,7,2,8,6)]

(33) $\mathrm{HWP}^*(12;6^7,12^4)$,
[(0,3,2,5,1,6),(4,7,10,9,8,11)], [(0,4,1,8,2,7),(3,9,5,10,6,11)], [(0,6,1,9,3,10),(2,8,5,4,11,7)], [(0,7,5,11,8,1),(2,9,4,10,3,6)], [(0,9,2,10,4,8),(1,11,6,5,3,7)], [(0,10,8,7,6,3),(1,4,2,11,5,9)], [(0,11,1,10,5,2),(3,8,6,9,7,4)], [(0,1,2,3,4,5,6,7,8,9,10,11)], [(0,2,4,6,8,10,1,3,5,7,11,9)], [(0,5,8,3,11,10,2,1,7,9,6,4)], [(0,8,4,9,11,2,6,10,7,3,1,5)]

(34) $\mathrm{HWP}^*(12;6^8,12^3)$,
[(0,3,1,4,2,5),(6,9,7,10,8,11)], [(0,4,1,5,2,6),(3,10,9,11,8,7)], [(0,6,1,8,2,7),(3,9,4,10,5,11)], [(0,7,1,6,10,3),(2,11,4,8,5,9)], [(0,8,4,7,2,10),(1,9,5,3,6,11)], [(0,9,3,8,6,2),(1,10,7,4,11,5)], [(0,10,6,3,2,8),(1,11,7,5,4,9)], [(0,11,2,9,8,1),(3,7,6,5,10,4)], [(0,1,2,3,4,5,6,7,8,9,10,11)], [(0,2,4,6,8,10,1,3,5,7,11,9)], [(0,5,8,3,11,10,2,1,7,9,6,4)]

(35) HWP*$(12; 6^9, 12^2)$,
[(0,3,1,4,2,5),(6,9,7,10,8,11)], [(0,4,1,5,2,6),(3,8,7,9,11,10)], [(0,5,1,6,2,7),(3,10,9,4,11,8)], [(0,6,1,7,2,8),(3,11,4,10,5,9)], [(0,7,1,9,6,3),(2,11,5,10,4,8)], [(0,8,5,3,6,10),(1,11,7,4,9,2)], [(0,9,8,6,11,1),(2,10,7,5,4,3)], [(0,10,6,5,11,2),(1,8,4,7,3,9)], [(0,11,3,7,6,4),(1,10,2,9,5,8)], [(0,1,2,3,4,5,6,7,8,9,10,11)], [(0,2,4,6,8,10,1,3,5,7,11,9)]

(36) HWP*$(12; 6^{10}, 12^1)$,
[(0,2,1,3,5,4),(6,8,7,9,11,10)], [(0,3,1,4,2,5),(6,9,7,10,8,11)], [(0,4,1,5,2,6),(3,7,11,8,10,9)], [(0,5,1,6,2,7),(3,10,4,11,9,8)], [(0,6,1,7,2,8),(3,9,4,10,5,11)], [(0,7,1,9,5,10),(2,11,4,8,6,3)], [(0,8,1,10,2,9),(3,11,7,4,6,5)], [(0,9,6,4,7,3),(1,11,5,8,2,10)], [(0,10,7,5,9,2),(1,8,4,3,6,11)], [(0,11,2,4,9,1),(3,8,5,7,6,10)], [(0,1,2,3,4,5,6,7,8,9,10,11)]

(37) HWP*$(16; 8^2, 16^{13})$,
[(0,1,2,3,4,5,6,7),(8,9,10,11,12,13,14,15)], [(0,2,4,6,8,10,12,14),(1,3,5,7,9,11,15,13)], [(0,3,8,6,15,10,1,14,5,4,9,7,13,2,12,11)], [(0,4,1,6,2,5,8,3,7,10,9,12,15,11,14,13)], [(0,5,1,7,3,2,6,4,8,11,9,14,10,13,15,12)], [(0,6,1,9,3,10,2,7,4,11,13,8,14,12,5,15)], [(0,7,1,10,3,6,5,2,15,14,8,13,11,4,12,9)], [(0,8,2,1,11,5,10,14,3,12,7,15,9,6,13,4)], [(0,9,4,2,8,12,3,13,6,14,1,15,7,11,10,5)], [(0,10,15,6,12,1,4,3,14,11,8,5,9,13,7,2)], [(0,11,3,15,2,13,5,12,4,14,9,1,8,7,6,10)], [(0,12,2,14,7,5,13,10,6,9,8,4,15,3,11,1)], [(0,13,3,1,12,6,11,2,9,15,5,14,4,10,7,8)], [(0,14,2,10,4,7,12,8,15,1,13,9,5,11,6,3)], [(0,15,4,13,12,10,8,1,5,3,9,2,11,7,14,6)]

(38) HWP*$(16; 8^4, 16^{11})$,
[(0,6,12,15,5,14,7,1,),(2,9,4,3,13,10,8,11)], [(0,9,15,12,6,11,4,7),(1,10,2,5,13,8,14,3)], [(0,12,3,2,7,14,8,4),(1,11,10,5,15,6,13,9)], [(0,13,11,1,15,9,12,5),(2,14,6,10,3,7,4,8)], [(0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15)], [(0,2,4,6,8,10,12,14,1,3,5,7,9,11,15,13)], [(0,3,8,6,15,10,1,14,5,4,9,7,13,2,12,11)], [(0,4,2,1,8,3,6,5,11,9,14,10,13,15,7,12)], [(0,5,2,6,1,9,3,10,7,11,13,4,15,14,12,8)], [(0,7,5,1,4,10,14,13,3,12,2,8,15,11,6,9)], [(0,8,1,7,15,3,11,14,4,13,5,12,9,6,2,10)], [(0,10,15,4,12,1,6,3,14,11,8,5,9,13,7,2)], [(0,11,3,15,2,13,1,12,4,14,9,5,8,7,10,6)], [(0,14,2,15,1,13,6,4,11,5,10,9,8,12,7,3)], [(0,15,8,13,12,10,4,1,5,3,9,2,11,7,6,14)]

(39) HWP*$(16; 8^6, 16^9)$,
[(0,4,2,1,8,3,6,5),(7,11,10,13,9,12,15,14)], [(0,5,2,6,1,9,15,12),(3,10,7,14,8,11,13,4)], [(0,6,11,1,10,8,4,7),(2,14,12,5,13,15,9,3)], [(0,9,14,10,3,1,11,4),(2,5,15,7,12,6,13,8)], [(0,12,3,13,11,9,4,8),(1,15,5,14,6,10,2,7)], [(0,13,10,5,11,2,9,1),(3,7,4,15,6,12,8,14)], [(0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15)], [(0,2,4,6,8,10,12,14,1,3,5,7,9,11,15,13)], [(0,3,8,6,15,10,1,14,5,4,9,7,13,2,12,11)], [(0,7,5,1,4,10,14,13,3,12,2,8,15,11,6,9)], [(0,8,1,7,15,3,11,14,4,13,5,12,9,6,2,10)], [(0,10,15,4,12,1,6,3,14,11,8,5,9,13,7,2)], [(0,11,3,15,2,13,1,12,4,14,9,5,8,7,10,6)], [(0,14,2,15,1,13,6,4,11,5,10,9,8,12,7,3)], [(0,15,8,13,12,10,4,1,5,3,9,2,11,7,6,14)]

(40) HWP*$(16; 8^8, 16^7)$,
[(0,4,2,1,7,3,6,5),(8,11,9,12,15,14,10,13)], [(0,5,2,6,1,8,3,10),(4,7,11,13,15,12,9,14)], [(0,6,2,5,10,3,7,12),(1,11,14,8,4,13,9,15)], [(0,8,1,9,3,2,10,7),(4,15,5,14,12,6,13,11)], [(0,9,8,12,7,14,6,4),(1,10,2,15,3,13,5,11)], [(0,12,8,14,2,9,4,3),(1,13,6,11,10,5,15,7)], [(0,13,4,11,5,12,3,1),(2,14,7,15,9,6,10,8)], [(0,14,3,11,2,7,4,8),(1,15,6,12,5,13,10,9)], [(0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15)], [(0,2,4,6,8,10,12,14,1,3,5,7,9,11,15,13)], [(0,3,8,6,15,10,1,14,5,4,9,7,13,2,12,11)], [(0,7,5,1,4,10,14,13,3,12,2,8,15,11,6,9)], [(0,10,15,4,12,1,6,3,14,11,8,5,9,13,7,2)], [(0,11,3,15,2,13,1,12,4,14,9,5,8,7,10,6)], [(0,15,8,13,12,10,4,1,5,3,9,2,11,7,6,14)]

(41) HWP*$(16; 8^{10}, 16^5)$,
[(0,4,2,1,6,3,7,5),(8,11,9,12,15,14,10,13)], [(0,5,1,4,3,2,6,9),(7,11,8,14,13,10,15,12)], [(0,6,1,7,2,5,9,3),(4,10,14,12,8,15,11,13)], [(0,7,1,8,2,9,4,12),(3,6,13,15,5,14,11,10)] [(0,8,1,9,6,2,10,7),(3,12,5,13,11,14,4,15)], [(0,9,1,10,2,7,14,8),(3,11,5,15,4,13,6,12)], [(0,10,8,3,13,9,15,1),(2,14,7,4,11,6,5,12)], [(0,12,1,15,9,13,5,10),(2,8,4,7,3,14,6,11)], [(0,13,7,15,6,10,5,2),(1,11,4,8,12,9,14,3)], [(0,14,2,15,7,12,6,4),(1,13,3,10,9,8,5,11)], [(0,11,3,15,2,13,1,12,4,14,9,5,8,7,10,6)], [(0,15,8,13,12,10,4,1,5,3,9,2,11,7,6,14)], [(0,3,8,6,15,10,1,14,5,4,9,7,13,2,12,11)], [(0,2,4,6,8,10,12,14,1,3,5,7,9,11,15,13)], [(0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15)]

(42) HWP*$(16; 8^{12}, 16^3)$,
[(0,4,1,5,2,6,3,7),(8,11,9,12,10,13,15,14)], [(0,5,1,4,2,7,3,6),(8,12,9,13,11,14,10,15)], [(0,6,1,7,2,5,3,9),(4,8,13,10,14,12,15,11)], [(0,7,1,6,2,8,3,10),(4,11,5,14,13,9,15,12)],

[(0,8,1,9,2,10,3,12),(4,7,5,15,6,14,11,13)], [(0,9,1,8,2,11,3,14),(4,10,5,12,6,13,7,15)],
[(0,10,2,1,11,6,4,3),(5,13,8,15,9,14,7,12)], [(0,11,1,10,4,13,12,2),(3,15,7,6,5,9,8,14)],
[(0,12,8,4,14,2,15,1),(3,13,5,11,7,10,6,9)], [(0,13,6,10,7,11,8,5),(1,15,2,14,9,4,12,3)],
[(0,14,6,11,10,9,5,8),(1,12,7,4,15,3,2,13)], [(0,15,5,10,8,7,14,4),(1,13,3,11,2,9,6,12)],
[(0,3,8,6,15,10,1,14,5,4,9,7,13,2,12,11)], [(0,2,4,6,8,10,12,14,1,3,5,7,9,11,15,13)],
[(0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15)]

(43) $\mathrm{HWP}^*(16; 8^{14}, 16^1)$,
[(0,2,1,3,5,4,6,8),(7,9,11,10,12,14,13,15)], [(0,3,1,4,2,5,7,6),(8,10,9,12,15,13,11,14)],
[(0,4,1,5,2,6,3,7),(8,11,9,13,10,15,14,12)], [(0,5,1,6,2,4,3,9),(7,10,13,8,14,11,15,12)],
[(0,6,1,7,2,8,3,10),(4,9,14,5,15,11,13,12)], [(0,7,1,8,2,9,3,11),(4,12,5,13,6,15,10,14)],
[(0,8,1,9,2,7,3,13),(4,15,5,12,11,6,14,10)], [(0,9,1,10,2,11,3,12),(4,7,14,6,13,5,8,15)],
[(0,10,1,11,2,12,3,14),(4,8,13,7,5,9,15,6)], [(0,11,1,12,2,13,9,5),(3,15,8,6,10,7,4,14)],
[(0,12,9,6,11,7,13,1),(2,15,3,8,4,10,5,14)], [(0,13,3,2,14,7,11,4),(1,15,9,8,5,10,6,12)],
[(0,14,1,13,4,11,5,3),(2,10,8,12,6,9,7,15)], [(0,15,1,14,9,4,13,2),(3,6,5,11,8,7,12,10)],
[(0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15)]

(44) $\mathrm{HWP}^*(16; 4^1, 16^{14})$,
[(0,1,2,3),(4,5,6,7),(8,9,10,11),(12,13,14,15)], [(0,2,4,3,7,1,8,5,9,6,10,12,15,14,11,13)],
[(0,3,9,12,1,5,13,8,2,6,14,7,11,15,10,4)], [(0,4,12,2,1,11,7,5,8,14,9,13,10,3,6,15)],
[(0,5,12,4,1,15,3,11,2,14,6,8,7,10,13,9)], [(0,6,11,5,10,7,12,9,2,15,4,8,3,13,1,14)],
[(0,7,2,8,1,4,6,3,10,5,15,9,14,13,11,12)], [(0,8,4,15,13,2,12,14,1,3,5,7,9,11,10,6)],
[(0,9,15,8,11,14,12,10,2,5,3,1,6,13,4,7)], [(0,10,1,7,3,2,9,4,13,5,14,8,15,6,12,11)],
[(0,11,1,9,3,4,2,13,15,7,14,10,8,12,6,5)], [(0,12,3,8,6,1,13,7,15,11,9,5,4,14,2,10)],
[(0,13,3,14,5,11,4,9,8,10,15,1,12,7,6,2)], [(0,14,3,15,2,11,6,4,10,9,7,8,13,12,5,1)],
[(0,15,5,2,7,13,6,9,1,10,14,4,11,3,12,8)]

(45) $\mathrm{HWP}^*(16; 4^2, 16^{13})$,
[(0,1,2,3),(4,5,6,7),(8,9,10,11),(12,13,14,15)], [(0,2,4,6),(8,10,12,14),(1,3,5,7),(9,11,13,15)],
[(0,3,9,12,1,5,13,8,2,6,14,7,11,15,10,4)], [(0,4,12,2,1,11,7,5,8,14,9,13,10,3,6,15)],
[(0,5,12,4,1,15,3,11,2,14,6,8,7,10,13,9)], [(0,6,11,5,10,7,12,9,2,15,4,8,3,13,1,14)],
[(0,7,2,8,1,4,3,10,5,9,6,12,15,14,13,11)], [(0,8,4,2,7,3,12,5,1,9,14,11,6,10,15,13)],
[(0,9,15,8,11,14,12,10,2,5,3,1,6,13,4,7)], [(0,10,1,7,6,2,9,3,8,13,5,14,4,15,11,12)],
[(0,11,1,8,5,2,12,3,15,6,9,4,13,7,14,10)], [(0,12,6,1,10,8,15,2,11,4,9,7,13,3,14,5)],
[(0,13,2,10,6,3,4,14,1,12,11,9,5,15,7,8)], [(0,14,3,2,13,6,5,4,11,10,9,8,12,7,15,1)],
[(0,15,5,11,3,7,9,1,13,12,8,6,4,10,14,2)]

(46) $\mathrm{HWP}^*(16; 4^3, 16^{12})$,
[(0,1,2,3),(4,5,6,7),(8,9,10,11),(12,13,14,15)], [(0,2,4,6),(8,10,12,14),(1,3,5,7),(9,11,13,15)],
[(0,3,9,12),(1,5,13,8),(2,6,14,7),(10,15,11,4)], [(0,4,12,2,1,11,7,5,8,14,9,13,10,3,6,15)],
[(0,5,12,4,1,15,3,11,2,14,6,8,7,10,13,9)], [(0,6,11,5,10,7,12,9,2,15,4,8,3,13,1,14)],
[(0,7,3,2,8,4,9,1,10,5,11,6,12,15,14,13)], [(0,8,2,7,6,1,4,3,10,9,14,5,15,13,12,11)],
[(0,9,15,8,11,14,12,10,2,5,3,1,6,13,4,7)], [(0,10,1,7,8,5,14,2,13,11,12,3,15,6,9,4)],
[(0,11,1,8,6,2,9,3,12,7,15,10,14,4,13,5)], [(0,12,1,9,5,2,10,4,14,11,15,7,13,6,3,8)],
[(0,13,3,4,11,9,7,14,10,6,5,1,12,8,15,2)], [(0,14,3,7,11,10,8,13,2,12,5,9,6,4,15,1)],
[(0,15,5,4,2,11,3,14,1,13,7,9,8,12,6,10)]

(47) $\mathrm{HWP}^*(16; 4^4, 16^{11})$,
[(0,1,2,3),(4,5,6,7),(8,9,10,11),(12,13,14,15)], [(0,2,4,6),(8,10,12,14),(1,3,5,7),(9,11,13,15)],
[(0,3,9,12),(1,5,13,8),(2,6,14,7),(10,15,11,4)], [(0,4,15,2),(1,11,7,5),(3,14,9,13),(10,8,12,6)],
[(0,5,12,4,1,15,3,11,2,14,6,8,7,10,13,9)], [(0,6,11,5,10,7,12,9,2,15,4,8,3,13,1,14)],
[(0,7,3,2,1,4,9,5,8,6,12,11,15,10,14,13)], [(0,8,2,7,6,1,9,3,4,11,12,15,13,5,14,10)],
[(0,9,15,8,11,14,12,10,2,5,3,1,6,13,4,7)], [(0,10,1,7,8,4,2,13,6,9,14,11,3,12,5,15)],
[(0,11,1,8,5,2,9,7,13,12,3,10,6,15,14,4)], [(0,12,1,10,3,6,2,8,15,7,14,5,9,4,13,11)],
[(0,13,7,9,8,14,1,12,2,11,10,4,3,15,6,5)], [(0,14,3,7,15,1,13,2,10,5,11,9,6,4,12,8)],
[(0,15,5,4,14,2,12,7,11,6,3,8,13,10,9,1)]

(48) $\mathrm{HWP}^*(16; 4^5, 16^{10})$,
[(0,1,2,3),(4,5,6,7),(8,9,10,11),(12,13,14,15)], [(0,2,4,6),(8,10,12,14),(1,3,5,7),(9,11,13,15)],
[(0,3,9,12),(1,5,13,8),(2,6,14,7),(10,15,11,4)], [(0,4,15,2),(1,11,7,5),(3,14,9,13),(10,8,12,6)],
[(0,5,12,4),(1,15,3,11),(8,14,6,2),(7,10,13,9)], [(0,6,11,5,10,7,12,9,2,15,4,8,3,13,1,14)],
[(0,7,3,1,4,2,5,8,6,9,14,11,12,15,13,10)], [(0,8,2,1,6,3,4,7,9,5,15,10,14,13,12,11)],
[(0,9,1,7,6,4,3,2,10,5,11,15,14,12,8,13)], [(0,10,1,8,4,9,3,6,13,11,14,2,12,7,15,5)],
[(0,11,2,7,8,5,3,12,1,13,4,14,10,9,6,15)], [(0,12,2,9,4,11,3,10,6,8,15,7,13,5,14,1)],

[(0,13,2,11,6,1,10,3,15,8,7,14,4,12,5,9)], [(0,14,5,4,1,9,15,6,12,3,8,11,10,2,13,7)],
[(0,15,1,12,10,4,13,6,5,2,14,3,7,11,9,8)]

(49) HWP*$(16; 4^6, 16^9)$,
[(0,1,2,3),(4,5,6,7),(8,9,10,11),(12,13,14,15)], [(0,2,4,6),(8,10,12,14),(1,3,5,7),(9,11,13,15)],
[(0,3,9,12),(1,5,13,8),(2,6,14,7),(10,15,11,4)], [(0,4,15,2),(1,11,7,5),(3,14,9,13),(10,8,12,6)],
[(0,5,12,4),(1,15,3,11),(8,14,6,2),(7,10,13,9)], [(0,6,11,5),(1,7,12,9),(2,15,4,8),(3,13,10,14)],
[(0,7,3,1,4,2,5,8,6,9,14,11,15,13,12,10)], [(0,8,3,2,1,6,4,7,11,12,5,10,9,15,14,13)],
[(0,9,2,7,6,1,8,4,13,5,11,14,10,3,12,15)], [(0,10,1,9,3,4,11,2,12,7,8,13,6,15,5,14)],
[(0,11,3,6,5,2,9,4,14,12,8,15,10,7,13,1)], [(0,12,1,10,2,11,6,13,4,3,15,7,14,5,9,8)],
[(0,13,2,14,4,1,12,11,9,6,3,10,5,15,8,7)], [(0,14,1,13,7,15,6,12,2,10,4,9,5,3,8,11)],
[(0,15,1,14,2,13,11,10,6,8,5,4,12,3,7,9)]

(50) HWP*$(16; 4^7, 16^8)$,
[(0,1,2,3),(4,5,6,7),(8,9,10,11),(12,13,14,15)], [(0,2,4,6),(8,10,12,14),(1,3,5,7),(9,11,13,15)],
[(0,3,9,12),(1,5,13,8),(2,6,14,7),(10,15,11,4)], [(0,4,15,2),(1,11,7,5),(3,14,9,13),(10,8,12,6)],
[(0,5,12,4),(1,15,3,11),(8,14,6,2),(7,10,13,9)], [(0,6,11,5),(1,7,12,9),(2,15,4,8),(3,13,10,14)],
[(0,7,15,10),(1,12,2,14),(3,6,13,11),(4,9,8,5)], [(0,8,3,1,4,2,5,9,6,12,10,7,11,15,14,13)],
[(0,9,2,1,6,3,4,7,8,13,12,11,14,10,5,15)], [(0,11,2,9,3,7,13,1,10,4,12,8,6,15,5,14)],
[(0,12,1,9,4,11,10,2,13,7,14,5,3,15,6,8)], [(0,10,1,8,4,3,2,7,6,9,14,12,15,13,5,11)],
[(0,13,4,14,2,11,9,15,8,7,3,12,5,10,6,1)], [(0,14,11,6,4,1,13,2,12,3,10,9,5,8,15,7)],
[(0,15,1,14,4,13,6,5,2,10,3,8,11,12,7,9)]

(51) HWP*$(16; 4^8, 16^7)$,
[(0,1,2,3),(4,5,6,7),(8,9,10,11),(12,13,14,15)], [(0,2,4,6),(8,10,12,14),(1,3,5,7),(9,11,13,15)],
[(0,3,9,12),(1,5,13,8),(2,6,14,7),(10,15,11,4)], [(0,4,15,2),(1,11,7,5),(3,14,9,13),(10,8,12,6)],
[(0,5,12,4),(1,15,3,11),(8,14,6,2),(7,10,13,9)], [(0,6,11,5),(1,7,12,9),(2,15,4,8),(3,13,10,14)],
[(0,7,15,10),(1,12,2,14),(3,6,13,11),(4,9,8,5)], [(0,8,15,7),(1,9,2,13),(3,10,4,12),(14,11,6,5)],
[(0,9,3,1,4,2,5,8,6,12,10,7,11,15,14,13)], [(0,10,1,6,3,2,7,8,4,11,12,15,13,5,9,14)],
[(0,11,2,1,8,3,4,13,7,14,12,5,10,6,9,15)], [(0,12,1,10,2,9,4,14,5,3,7,13,6,15,8,11)],
[(0,13,2,12,8,7,3,15,6,4,1,14,10,5,11,9)], [(0,14,2,11,10,9,5,15,1,13,4,3,12,7,6,8)],
[(0,15,5,2,10,3,8,13,12,11,14,4,7,9,6,1)]

(52) HWP*$(16; 4^9, 16^6)$,
[(0,1,2,3),(4,5,6,7),(8,9,10,11),(12,13,14,15)], [(0,2,4,6),(8,10,12,14),(1,3,5,7),(9,11,13,15)],
[(0,3,9,12),(1,5,13,8),(2,6,14,7),(10,15,11,4)], [(0,4,15,2),(1,11,7,5),(3,14,9,13),(10,8,12,6)],
[(0,5,12,4),(1,15,3,11),(8,14,6,2),(7,10,13,9)], [(0,6,11,5),(1,7,12,9),(2,15,4,8),(3,13,10,14)],
[(0,7,15,10),(1,12,2,14),(3,6,13,11),(4,9,8,5)], [(0,8,15,7),(1,9,2,13),(3,10,4,12),(14,11,6,5)],
[(0,9,15,8),(1,6,12,10),(2,5,11,14),(7,13,4,3)], [(0,10,2,1,4,7,3,8,6,9,5,15,14,13,12,11)],
[(0,11,2,7,6,1,8,3,4,13,5,10,9,14,12,15)], [(0,12,1,10,3,2,9,6,4,14,5,8,7,11,15,13)],
[(0,13,6,3,15,1,14,10,7,8,4,2,11,12,5,9)], [(0,14,4,11,10,6,15,5,2,12,8,13,7,9,3,1)],
[(0,15,6,8,11,9,4,1,13,2,10,5,3,12,7,14)]

(53) HWP*$(16; 4^{10}, 16^5)$,
[(0,1,2,3),(4,5,6,7),(8,9,10,11),(12,13,14,15)], [(0,2,4,6),(8,10,12,14),(1,3,5,7),(9,11,13,15)],
[(0,3,9,12),(1,5,13,8),(2,6,14,7),(10,15,11,4)], [(0,4,15,2),(1,11,7,5),(3,14,9,13),(10,8,12,6)],
[(0,5,12,4),(1,15,3,11),(8,14,6,2),(7,10,13,9)], [(0,6,11,5),(1,7,12,9),(2,15,4,8),(3,13,10,14)],
[(0,7,15,10),(1,12,2,14),(3,6,13,11),(4,9,8,5)], [(0,8,15,7),(1,9,2,13),(3,10,4,12),(14,11,6,5)],
[(0,9,15,8),(1,6,12,10),(2,5,11,14),(7,13,4,3)], [(0,10,5,9),(2,12,7,11),(3,15,1,8),(4,14,13,6)],
[(0,11,9,3,1,4,2,7,8,13,12,5,10,6,15,14)], [(0,12,1,10,2,9,6,3,4,7,14,5,8,11,15,13)],
[(0,13,2,11,10,7,6,9,14,12,15,5,3,8,4,1)], [(0,14,4,11,12,8,6,1,13,7,3,2,10,9,5,15)],
[(0,15,6,8,7,9,4,13,5,2,1,14,10,3,12,11)]

(54) HWP*$(16; 4^{11}, 16^4)$,
[(0,1,2,3),(4,5,6,7),(8,9,10,11),(12,13,14,15)], [(0,2,1,5),(3,6,4,8),(7,9,12,11),(10,14,13,15)],
[(0,3,5,2),(1,6,8,7),(4,12,9,13),(10,15,11,14)], [(0,4,3,7),(1,8,2,15),(5,9,11,13),(6,10,12,14)],
[(0,5,4,9),(1,12,2,8),(3,10,13,11),(6,14,7,15)], [(0,6,12,10),(1,3,11,5),(2,14,9,7),(4,13,8,15)],
[(0,7,12,4),(1,15,3,9),(2,13,6,11),(5,14,8,10)], [(0,8,14,11),(1,9,4,10),(2,5,12,6),(3,15,7,13)],
[(0,9,15,8),(1,7,5,11),(2,12,3,14),(4,6,13,10)], [(0,10,8,12),(1,14,3,13),(2,4,15,9),(5,7,11,6)],
[(0,15,2,6),(1,11,4,14),(3,12,7,10),(5,13,9,8)], [(0,11,9,3,1,4,2,7,8,13,12,5,10,6,15,14)],
[(0,12,1,10,2,9,6,3,4,7,14,5,8,11,15,13)], [(0,13,2,11,10,7,6,9,14,12,15,5,3,8,4,1)],
[(0,14,4,11,12,8,6,1,13,7,3,2,10,9,5,15)]

(55) HWP*$(16; 4^{12}, 16^3)$,
[(0,1,2,3),(4,5,6,7),(8,9,10,11),(12,13,14,15)], [(0,2,1,5),(3,6,4,8),(7,9,11,12),(10,14,13,15)],

[(0,3,2,4),(1,6,5,7),(8,10,12,14),(9,15,11,13)], [(0,4,3,7),(1,8,2,5),(6,12,10,15),(9,13,11,14)],
[(0,5,2,6,),(1,3,10,13),(4,9,12,11),(7,15,8,14)], [(0,6,1,9),(2,10,5,11),(3,13,4,14),(7,12,8,15)],
[(0,7,2,12),(1,11,6,10),(3,14,4,13),(5,15,9,8)], [(0,8,6,2),(1,13,10,9),(3,12,4,15),(5,14,11,7)],
[(0,9,2,8),(1,12,3,15),(4,6,14,10),(5,13,7,11)], [(0,10,3,11),(1,15,2,14),(4,12,9,5),(6,8,7,13)],
[(0,14,2,15),(1,7,10,8),(3,9,4,11),(5,12,6,13)], [(0,15,4,10),(1,14,6,11),(2,13,8,12),(3,5,9,7)],
[(0,11,9,3,1,4,2,7,8,13,12,5,10,6,15,14)], [(0,12,1,10,2,9,6,3,4,7,14,5,8,11,15,13)],
[(0,13,2,11,10,7,6,9,14,12,15,5,3,8,4,1)]

(56) HWP*$(16; 4^{13}, 16^2)$,
[(0,1,2,3),(4,5,6,7),(8,9,10,11),(12,13,14,15)], [(0,2,1,5),(3,6,4,8),(7,9,11,12),(10,14,13,15)],
[(0,3,2,4),(1,6,5,7),(8,10,9,13),(11,14,12,15)], [(0,4,1,7),(2,5,3,8),(6,12,10,15),(9,14,11,13)],
[(0,5,1,8),(2,6,9,12),(3,10,13,11),(4,14,7,15)], [(0,6,1,9),(2,8,4,11),(3,12,14,10),(5,15,7,13)],
[(0,7,2,10),(1,12,9,8),(3,15,5,14),(4,13,6,11)], [(0,8,6,2),(1,14,9,15),(3,5,13,7),(4,12,11,10)],
[(0,9,4,15),(1,13,2,14),(3,7,10,12),(5,11,6,8)], [(0,10,5,12),(1,15,3,11),(2,13,4,9),(6,14,8,7)],
[(0,13,1,11),(2,15,8,14),(3,9,5,4),(6,10,7,12)], [(0,14,4,6),(1,3,13,10),(2,12,8,15),(5,9,7,11)],
[(0,15,9,1),(2,11,7,5),(3,14,6,13),(4,10,8,12)], [(0,11,9,3,1,4,2,7,8,13,12,5,10,6,15,14)],
[(0,12,1,10,2,9,6,3,4,7,14,5,8,11,15,13)]

(57) HWP*$(16; 4^{14}, 16^1)$,
[(0,1,2,3),(4,5,6,7),(8,9,10,11),(12,13,14,15)], [(0,2,1,5),(3,4,6,8),(7,9,11,12),(10,14,13,15)],
[(0,3,2,4),(1,6,5,7),(8,10,9,13),(11,14,12,15)], [(0,4,1,7),(2,5,3,6),(8,12,9,14),(10,15,13,11)],
[(0,5,1,8),(2,6,3,7),(4,14,9,15),(10,12,11,13)], [(0,6,1,9),(2,8,4,10),(3,12,14,11),(5,13,7,15)],
[(0,7,3,10),(1,11,2,9),(4,12,6,13),(5,15,8,14)], [(0,8,1,12),(2,10,5,14),(3,9,7,13),(4,11,15,6)],
[(0,9,12,2),(1,3,8,15),(4,13,5,11),(6,14,10,7)], [(0,10,13,1),(2,15,9,8),(3,14,6,12),(4,7,11,5)],
[(0,12,8,6),(1,10,3,11),(2,13,9,5),(4,15,7,14)], [(0,13,6,11),(1,15,2,14),(3,5,9,4),(7,12,10,8)],
[(0,14,3,15),(1,13,2,12),(4,9,6,10),(5,8,11,7)], [(0,15,3,13),(1,14,7,10),(2,11,6,9),(4,8,5,12)],
[(0,11,9,3,1,4,2,7,8,13,12,5,10,6,15,14)]

(58) HWP*$(15; 3^1, 15^{13})$,
[(0,1,2),(3,4,5),(6,7,8),(9,10,11),(12,13,14)], [(0,2,4,8,10,6,12,14,1,3,5,7,9,11,13)],
[(0,3,9,1,8,4,6,14,7,5,10,13,2,12,11)], [(0,4,7,1,9,3,2,14,13,6,5,8,11,12,10)],
[(0,5,13,1,4,10,2,9,14,3,6,11,7,12,8)], [(0,6,1,5,2,3,7,4,9,8,13,11,14,10,12)],
[(0,7,2,1,6,3,8,5,9,13,12,4,11,10,14)], [(0,8,1,7,3,10,4,2,13,9,12,5,14,11,6)],
[(0,9,2,5,1,10,3,11,4,14,6,13,8,12,7)], [(0,10,1,11,2,6,4,3,14,8,7,13,5,12,9)],
[(0,11,1,12,2,7,6,10,8,14,9,5,4,13,3)], [(0,12,1,14,2,10,5,6,9,7,11,8,3,13,4)],
[(0,13,10,7,14,5,11,3,12,6,2,8,9,4,1)], [(0,14,4,12,3,1,13,7,10,9,6,8,2,11,5)]

(59) HWP*$(15; 3^3, 15^{11})$,
[(0,1,2),(3,4,5),(6,7,8),(9,10,11),(12,13,14)], [(0,2,4),(6,8,10),(12,14,1),(3,5,7),(9,11,13)],
[(0,3,9),(1,8,2),(4,14,7),(5,10,13),(6,12,11)], [(0,4,7,1,9,3,2,14,13,6,5,8,11,12,10)],
[(0,5,13,1,4,10,2,9,14,3,6,11,7,12,8)], [(0,6,1,3,7,2,5,4,8,9,13,11,14,10,12)],
[(0,7,5,1,6,2,3,8,12,9,4,13,10,14,11)], [(0,8,1,5,2,6,3,13,4,11,10,9,12,7,14)],
[(0,9,1,7,6,4,12,2,10,3,11,5,14,8,13)], [(0,10,1,11,2,7,9,5,12,6,13,8,14,4,3)],
[(0,11,1,10,4,2,8,3,14,6,9,7,13,12,5)], [(0,12,4,9,6,14,2,13,7,10,8,5,11,3,1)],
[(0,13,3,12,1,14,5,9,2,11,8,4,6,10,7)], [(0,14,9,8,7,11,4,1,13,2,12,3,10,5,6)]

(60) HWP*$(15; 3^5, 15^9)$,
[(0,1,2),(3,4,5),(6,7,8),(9,10,11),(12,13,14)], [(0,2,4),(6,8,10),(12,14,1),(3,5,7),(9,11,13)],
[(0,3,9),(1,8,2),(4,14,7),(5,10,13),(6,12,11)], [(0,4,7),(1,9,3),(2,14,13),(5,11,8),(10,12,6)],
[(0,5,13),(1,4,10),(2,9,14),(3,6,11),(7,12,8)], [(0,6,1,3,2,5,4,8,9,7,13,10,14,11,12)],
[(0,7,1,5,2,3,8,4,6,13,11,14,9,12,10)], [(0,8,1,6,2,7,5,9,13,12,3,14,10,4,11)],
[(0,9,1,7,2,6,3,11,10,5,12,4,13,8,14)], [(0,10,2,8,3,7,11,1,13,4,12,9,5,14,6)],
[(0,11,2,10,3,12,1,14,4,9,8,13,7,6,5)], [(0,12,2,11,4,3,13,1,10,7,9,6,14,5,8)],
[(0,13,6,9,4,2,12,7,14,3,10,8,11,5,1)], [(0,14,8,12,5,6,4,1,11,7,10,9,2,13,3)]

(61) HWP*$(15; 3^7, 15^7)$,
[(0,1,2),(3,4,5),(6,7,8),(9,10,11),(12,13,14)], [(0,2,4),(6,8,10),(12,14,1),(3,5,7),(9,11,13)],
[(0,3,9),(1,8,2),(4,14,7),(5,10,13),(6,12,11)], [(0,4,7),(1,9,3),(2,14,13),(5,11,8),(10,12,6)],
[(0,5,13),(1,4,10),(2,9,14),(3,6,11),(7,12,8)], [(0,6,1),(2,13,10),(3,8,14),(4,11,5),(7,9,12)],
[(0,7,11),(2,6,5),(1,14,9),(4,12,10),(3,13,8)], [(0,8,1,3,2,5,6,4,9,13,7,10,14,11,12)],
[(0,9,2,3,7,1,5,8,12,4,6,13,11,14,10)], [(0,10,3,11,1,6,2,7,5,14,4,13,12,9,8)],
[(0,11,2,8,4,1,10,7,13,3,12,5,9,6,14)], [(0,12,2,11,4,3,10,8,9,7,14,5,1,13,6)],
[(0,13,1,7,2,12,3,14,6,9,4,8,11,10,5)], [(0,14,8,13,4,2,10,9,5,12,1,11,7,6,3)]

(62) HWP*(15; $3^9, 15^5$),
[(0,1,2),(3,4,5),(6,7,8),(9,10,11),(12,13,14)], [(0,2,4),(6,8,10),(12,14,1),(3,5,7),(9,11,13)],
[(0,3,9),(1,8,2),(4,14,7),(5,10,13),(6,12,11)], [(0,4,7),(1,9,3),(2,14,13),(5,11,8),(10,12,6)],
[(0,5,13),(1,4,10),(2,9,14),(3,6,11),(7,12,8)], [(0,6,1),(2,13,10),(3,8,14),(4,11,5),(7,9,12)],
[(0,7,11),(2,6,5),(1,14,9),(4,12,10),(3,13,8)], [(0,8,12),(1,13,4),(2,10,3),(7,14,11),(5,6,9)],
[(0,9,8),(2,5,12),(11,4,13),(3,14,6),(1,10,7)], [(0,10,5,1,3,7,2,8,11,12,4,9,13,6,14)],
[(0,11,1,5,8,9,2,7,10,14,4,6,13,12,3)], [(0,12,1,7,13,3,10,8,4,2,11,14,5,9,6)],
[(0,13,7,5,14,8,1,6,2,12,9,4,3,11,10)], [(0,14,10,9,7,6,4,8,13,1,11,2,3,12,5)]

(63) HWP*(15; $3^{11}, 15^3$),
[(0,1,2),(3,4,5),(6,7,8),(9,10,11),(12,13,14)], [(0,2,4),(6,8,10),(12,14,1),(3,5,7),(9,11,13)],
[(0,3,9),(1,8,2),(4,14,7),(5,10,13),(6,12,11)], [(0,4,7),(1,9,3),(2,14,13),(5,11,8),(10,12,6)],
[(0,5,13),(1,4,10),(2,9,14),(3,6,11),(7,12,8)], [(0,6,1),(2,13,10),(3,8,14),(4,11,5),(7,9,12)],
[(0,7,11),(2,6,5),(1,14,9),(4,12,10),(3,13,8)], [(0,8,12),(1,13,4),(2,10,3),(7,14,11),(5,6,9)],
[(0,9,8),(2,5,12),(11,4,13),(3,14,6),(1,10,7)],[(0,10,14),(12,4,3),(2,8,11),(7,5,9),(1,6,13)],
[(0,11,10),(1,5,8),(2,12,9),(3,7,13),(4,6,14)], [(0,12,1,3,10,9,6,4,8,13,7,2,11,14,5)],
[(0,13,12,5,14,10,8,9,4,2,3,11,1,7,6)], [(0,14,8,4,9,13,6,2,7,10,5,1,11,12,3)]

(64) HWP*(15; $5^1, 15^{13}$),
[(0,1,2,3,4),(5,6,7,8,9),(10,11,12,13,14)], [(0,2,8,6,4,10,14,12,1,3,5,7,9,11,13)],
[(0,3,9,1,8,2,4,14,7,5,10,13,6,12,11)], [(0,4,7,1,9,3,2,14,13,5,11,8,10,12,6)],
[(0,5,13,1,4,2,10,9,14,3,6,11,7,12,8)], [(0,6,1,5,2,7,3,8,4,9,13,12,14,11,10)],
[(0,7,2,1,6,3,10,4,5,8,13,11,14,9,12)], [(0,8,1,7,4,3,11,2,12,9,6,13,10,5,14)],
[(0,9,2,5,1,10,3,13,4,11,6,14,8,12,7)], [(0,10,1,11,3,7,6,2,13,8,14,4,12,5,9)],
[(0,11,1,12,2,6,9,4,13,7,10,8,3,14,5)], [(0,12,10,6,5,4,8,11,9,7,13,3,1,14,2)],
[(0,13,9,8,7,14,6,10,2,11,5,3,12,4,1)], [(0,14,1,13,2,9,10,7,11,4,6,8,5,12,3)]

(65) HWP*(15; $5^3, 15^{11}$),
[(0,1,2,3,4),(5,6,7,8,9),(10,11,12,13,14)], [(0,2,8,6,5),(10,14,12,1,3),(7,4,9,11,13)],
[(0,3,9,1,7),(2,4,14,8,5),(10,13,6,12,11)], [(0,4,7,1,9,3,2,14,13,5,11,8,10,12,6)],
[(0,5,13,1,4,2,10,9,14,3,6,11,7,12,8)], [(0,6,1,5,3,7,2,9,4,10,8,13,12,14,11)],
[(0,7,3,1,6,2,5,4,11,14,9,13,8,12,10)], [(0,8,1,10,2,6,3,5,7,14,4,13,11,9,12)],
[(0,9,2,1,8,3,11,4,5,12,7,13,10,6,14)], [(0,10,1,11,2,7,5,8,4,12,3,14,6,13,9)],
[(0,11,1,12,2,13,4,8,14,5,9,7,6,10,3)], [(0,12,5,10,4,1,14,7,9,6,8,11,3,13,2)],
[(0,13,3,8,2,12,4,6,9,10,7,11,5,14,1)], [(0,14,2,11,6,4,3,12,9,8,7,10,5,1,13)]

(66) HWP*(15; $5^5, 15^9$),
[(0,1,2,3,4),(5,6,7,8,9),(10,11,12,13,14)], [(0,2,8,6,5),(10,14,12,1,3),(7,4,9,11,13)],
[(0,3,9,1,7),(2,4,14,8,5),(10,13,6,12,11)], [(0,4,7,1,9),(3,2,14,13,5),(11,8,10,12,6)],
[(0,5,13,2,1),(4,10,9,14,6),(3,11,7,12,8)], [(0,6,1,4,2,5,7,3,8,11,14,9,13,12,10)],
[(0,7,2,6,3,1,5,4,12,14,11,9,10,8,13)], [(0,8,1,6,2,7,9,12,3,13,10,4,11,5,14)],
[(0,9,2,10,1,8,4,3,6,14,5,12,7,13,11)], [(0,10,2,9,3,5,1,12,4,13,8,14,7,11,6)],
[(0,11,1,10,3,14,2,12,5,9,7,6,13,4,8)], [(0,12,9,4,5,10,6,8,7,14,1,11,2,13,3)],
[(0,13,1,14,3,7,10,5,11,4,6,9,8,12,2)], [(0,14,4,1,13,9,6,10,7,5,8,2,11,3,12)]

(67) HWP*(15; $5^7, 15^7$),
[(0,1,2,3,4),(5,6,7,8,9),(10,11,12,13,14)], [(0,2,8,6,5),(10,14,12,1,3),(7,4,9,11,13)],
[(0,3,9,1,7),(2,4,14,8,5),(10,13,6,12,11)], [(0,4,7,1,9),(3,2,14,13,5),(11,8,10,12,6)],
[(0,5,13,2,1),(4,10,9,14,6),(3,11,7,12,8)], [(0,6,13,11,2),(1,10,3,7,14),(4,8,12,5,9)],
[(0,7,6,2,11),(1,14,3,8,4),(13,10,5,12,9)], [(0,8,1,4,2,5,7,3,13,9,12,14,11,6,10)],
[(0,9,2,6,1,5,4,3,12,10,7,13,8,11,14)], [(0,10,1,6,3,5,8,14,2,9,7,11,4,13,12)],
[(0,11,1,8,7,5,10,2,13,4,12,3,14,9,6)], [(0,12,2,7,10,8,13,1,11,5,14,4,6,9,3)],
[(0,13,3,6,14,5,1,12,7,2,10,4,11,9,8)], [(0,14,7,9,10,6,8,2,12,4,5,11,3,1,13)],

(68) HWP*(15; $5^9, 15^5$),
[(0,1,2,3,4),(5,6,7,8,9),(10,11,12,13,14)], [(0,2,8,6,5),(10,14,12,1,3),(7,4,9,11,13)],
[(0,3,9,1,7),(2,4,14,8,5),(10,13,6,12,11)], [(0,4,7,1,9),(3,2,14,13,5),(11,8,10,12,6)],
[(0,5,13,2,1),(4,10,9,14,6),(3,11,7,12,8)], [(0,6,13,11,2),(1,10,3,7,14),(4,8,12,5,9)],
[(0,7,6,2,11),(1,14,3,8,4),(13,10,5,12,9)], [(0,8,13,3,12),(1,11,4,6,10),(2,5,14,7,9)],
[(0,9,10,6,8),(1,13,12,3,5),(2,7,11,14,4)], [(0,10,2,6,1,4,3,13,8,7,5,11,9,12,14)],
[(0,11,1,5,4,12,7,10,8,14,2,13,9,3,6)], [(0,12,4,13,1,8,2,9,6,14,11,5,10,7,3)],
[(0,13,4,11,3,14,5,8,1,6,9,7,2,12,10)], [(0,14,9,8,11,6,3,1,12,2,10,4,5,7,13)]

(69) HWP*(15; $5^{11}, 15^3$),
[(0,1,2,3,4),(5,6,7,8,9),(10,11,12,13,14)], [(0,2,8,6,5),(10,14,12,1,3),(7,4,9,11,13)],

[(0,3,9,1,7),(2,4,14,8,5),(10,13,6,12,11)], [(0,4,7,1,9),(3,2,14,13,5),(11,8,10,12,6)],
[(0,5,13,2,1),(4,10,9,14,6),(3,11,7,12,8)], [(0,6,2,7,3),(1,12,4,5,10),(8,13,11,14,9)],
[(0,7,9,4,13),(1,8,12,3,5),(2,10,6,14,11)], [(0,8,4,2,11),(1,14,7,6,3),(5,12,9,13,10)],
[(0,9,6,13,12),(1,10,7,11,4),(2,5,14,3,8], [(0,12,5,9,2),(1,13,3,7,14),(4,8,11,6,10)],
[(0,14,4,6,8),(1,11,5,7,13),(2,9,10,3,12)], [(0,10,2,6,1,4,3,13,8,7,5,11,9,12,14)],
[(0,11,1,5,4,12,7,10,8,14,2,13,9,3,6)], [(0,13,4,11,3,14,5,8,1,6,9,7,2,12,10)]

(70) HWP$^*(15; 5^{13}, 15^1)$,
[(0,1,2,3,4),(5,6,7,8,9),(10,11,12,13,14)], [(0,2,8,6,5),(10,14,12,1,3),(7,4,9,11,13)],
[(0,3,9,1,7),(2,4,14,8,5),(10,13,6,12,11)], [(0,4,7,1,9),(3,2,14,13,5),(11,8,10,12,6)],
[(0,5,13,2,1),(4,10,9,14,6),(3,11,7,12,8)], [(0,6,13,11,2),(1,10,3,7,14),(4,8,12,5,9)],
[(0,7,2,11,14),(1,6,9,12,3),(4,13,10,5,8)], [(0,8,13,3,12),(1,11,4,6,10),(2,5,14,7,9)],
[(0,9,6,2,10),(1,8,7,11,5),(3,13,12,14,4)], [(0,10,4,11,3),(1,14,5,7,6),(2,12,9,13,8)],
[(0,12,4,1,13),(2,9,7,5,10),(3,8,11,6,14)], [(0,13,4,5,11),(1,12,2,6,8),(3,14,9,10,7)],
[(0,14,11,9,8),(1,4,2,7,13),(3,5,12,10,6)], [(0,11,1,5,4,12,7,10,8,14,2,13,9,3,6)]

(71) HWP$^*(15; 3^1, 5^{13})$,
[(0,1,2),(3,4,5),(6,7,8),(9,10,11),(12,13,14)], [(0,2,1,4,7),(3,14,5,9,8),(6,12,11,13,10)],
[(0,3,9,1,8),(2,11,5,6,13),(4,12,14,7,10)], [(0,4,2,9,14),(1,6,10,7,12),(3,13,8,5,11)],
[(0,5,14,13,1),(2,6,4,10,12),(3,7,11,8,9)], [(0,6,1,13,11),(2,10,3,12,7),(4,8,14,9,5)],
[(0,7,1,14,10),(2,5,8,13,6),(3,11,4,9,12)], [(0,8,1,5,12),(2,3,10,14,11),(4,6,9,7,13)],
[(0,9,13,12,5),(1,11,10,8,4),(2,7,14,6,3)], [(0,10,5,1,3),(9,2,8,11,12),(4,13,7,6,14)],
[(0,11,14,2,13),(1,12,6,8,10),(3,5,7,9,4)], [(0,12,8,7,4),(1,10,2,14,3),(5,13,9,11,6)],
[(0,13,5,10,9),(1,7,3,6,11),(2,12,4,14,8)], [(0,14,1,9,6),(2,4,11,7,5),(3,8,12,10,13)]

(72) HWP$^*(15; 3^3, 5^{11})$,
[(0,1,2),(3,4,5),(6,7,8),(9,10,11),(12,13,14)], [(0,2,4),(6,8,10),(12,14,1),(3,5,7),(9,11,13)],
[(0,3,9),(1,8,2),(4,14,7),(5,10,13),(6,12,11)], [(0,4,2,9,14),(1,6,10,7,12),(3,13,8,5,11)],
[(0,5,14,13,1),(2,6,4,10,12),(3,7,11,8,9)], [(0,6,1,13,11),(2,10,3,12,7),(4,8,14,9,5)],
[(0,7,1,14,10),(2,5,8,13,6),(3,11,4,9,12)], [(0,8,1,5,12),(2,3,10,14,11),(4,6,9,7,13)],
[(0,9,13,12,5),(1,11,10,8,4),(2,7,14,6,3)], [(0,10,5,1,3),(9,2,8,11,12),(4,13,7,6,14)],
[(0,11,14,2,13),(1,7,10,4,3),(5,9,8,12,6)], [(0,12,8,3,6),(1,4,11,7,9),(2,14,5,13,10)],
[(0,13,3,14,8),(1,10,9,6,11),(2,12,4,7,5)], [(0,14,3,8,7),(1,9,4,12,10),(2,11,5,6,13)]

(73) HWP$^*(15; 3^5, 5^9)$,
[(0,1,2),(3,4,5),(6,7,8),(9,10,11),(12,13,14)], [(0,2,4),(6,8,10),(12,14,1),(3,5,7),(9,11,13)],
[(0,3,9),(1,8,2),(4,14,7),(5,10,13),(6,12,11)], [(0,4,7),(1,9,3),(2,14,13),(5,11,8),(10,12,6)],
[(0,5,13),(1,4,10),(2,9,14),(3,6,11),(7,12,8)], [(0,6,1,13,11),(2,10,3,12,7),(4,8,14,9,5)],
[(0,7,1,14,10),(2,5,8,13,6),(3,11,4,9,12)], [(0,8,1,5,12),(2,3,10,14,11),(4,6,9,7,13)],
[(0,9,13,12,5),(1,11,10,8,4),(2,7,14,6,3)], [(0,10,5,1,3),(9,2,8,11,12),(4,13,7,6,14)],
[(0,11,7,5,14),(1,10,4,2,12),(3,8,9,6,13)], [(0,12,2,13,1),(3,7,10,9,8),(4,11,14,5,6)],
[(0,13,10,2,6),(1,7,11,5,9),(3,14,8,12,4)], [(0,14,3,13,8),(1,6,5,2,11),(4,12,10,7,9)]

(74) HWP$^*(15; 3^7, 5^7)$,
[(0,1,2),(3,4,5),(6,7,8),(9,10,11),(12,13,14)], [(0,2,4),(6,8,10),(12,14,1),(3,5,7),(9,11,13)],
[(0,3,9),(1,8,2),(4,14,7),(5,10,13),(6,12,11)], [(0,4,7),(1,9,3),(2,14,13),(5,11,8),(10,12,6)],
[(0,5,13),(1,4,10),(2,9,14),(3,6,11),(7,12,8)], [(0,6,1),(2,13,10),(3,8,14),(4,11,5),(7,9,12)],
[(0,7,11),(2,6,5),(1,14,9),(4,12,10),(3,13,8)], [(0,8,1,5,12),(2,3,10,14,11),(4,6,9,7,13)],
[(0,9,13,12,5),(1,11,10,8,4),(2,7,14,6,3)], [(0,10,5,1,3),(9,2,8,11,12),(4,13,7,6,14)],
[(0,11,1,13,6),(2,5,14,8,12),(3,7,10,9,4)], [(0,12,3,14,10),(1,7,5,8,13),(2,11,4,9,6)],
[(0,13,3,11,14),(1,10,7,2,12),(4,8,9,5,6)], [(0,14,5,9,8),(1,6,13,11,7),(2,10,3,12,4)]

(75) HWP$^*(15; 3^9, 5^5)$,
[(0,1,3),(2,6,7),(4,5,8),(9,13,12),(10,11,14)], [(0,2,10),(1,6,8),(3,13,7),(4,12,14),(5,11,9)],
[(0,7,13),(1,8,6),(2,9,12),(3,10,14),(4,11,5)], [(0,9,8),(1,2,11),(3,14,12),(4,13,10),(5,7,6)],
[(0,10,5),(1,4,3),(2,13,8),(6,14,9),(7,11,12)], [(0,11,6),(1,12,13),(2,8,14),(3,4,9),(5,10,7)],
[(0,12,4),(1,14,5),(2,3,6),(7,10,8),(9,11,13)], [(0,13,14),(1,5,12),(2,7,4),(3,8,11),(6,9,10)],
[(0,14,11),(1,13,4),(2,12,10),(3,5,6),(7,8,9)], [(0,3,9,1,7),(2,4,14,8,5),(10,13,6,12,11)],
[(0,4,7,1,9),(3,2,14,13,5),(11,8,10,12,6)], [(0,5,13,2,1),(4,10,9,14,6),(3,11,7,12,8)],
[(0,6,13,11,2),(1,10,3,7,14),(4,8,12,5,9)], [(0,8,13,3,12),(1,11,4,6,10),(2,5,14,7,9)]

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