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REFLEXIBLE COVERS OF PRISMS

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ABSTRACT. The Tomotope provided the first well understood example of an abstract 4-polytope whose connection (monodromy) group was not a string C-group, and which also did not have a unique minimal regular cover. Conversely, we know that if the connection group of a polytope is a string C-group (if the polytope is *C-connected*), then the polytope will have a unique minimal regular cover. Since the discovery of the Tomotope, an active area of investigation has been determining which abstract d-polytopes are C-connected and the ways various constructions for abstract polytopes result in polytopes that do or do not possess unique minimal regular covers. In the current work we show that the prism over every abstract polyhedron is C-connected, or equivalently, that it has a unique minimal regular cover. We also describe a conjecture positing a general condition on the C-connectedness of prisms over polytopes that is independent of rank.

1. INTRODUCTION

Abstract polytopes have been used extensively since the 1980s to investigate and generalize the combinatorial and symmetry properties of convex polytopes. While much of the early work focused on the most symmetric class, the regular abstract polytopes, since the early 1990s there has been increasing activity investigating abstract polytopes that, while still highly symmetric, are not regular. Two major closely related threads have emerged in the study of less symmetric abstract polytopes: the use of covers of abstract polytopes by abstract regular polytopes, pioneered by Hartley [9, 10], and the analysis of the connection (or monodromy) group of a polytope (see, especially, [12, 15]. In this paper we apply methods developed by Cunningham, Pellicer and Williams [16, 3] to investigate properties of the connection groups of prisms over abstract polytopes. The main result of the paper is Theorem 4.10, in which we show that every prism constructed over a polyhedron has a unique minimal regular cover.

2. POLYTOPES, MANIPLEXES AND PRE-MANIPLEXES

Here we will review some of the basic definitions and results required for the present work, as well as introduce our notational conventions. For additional details on regular abstract polytopes the standard reference is [13]; for more information about connection groups we refer the reader to [15] (where they are called *monodromy groups*); further details about maniplexes can be found in [19] and [4, 6, 17] (though the definitions in this area are still evolving).

2.1. **Groups.** If $\Gamma = \langle \gamma_0, \gamma_1, \ldots, \gamma_{d-1} \rangle$ is a group generated by involutions γ_i such that $|\gamma_i \gamma_j| = 2$ for all $|i-j| > 1$, then we say Γ is a *string group generated by involutions* or *sggi*. They are called string groups since this is exactly how a Coxeter group whose diagram is a string (or simple path) is generated (see below). If k_i is the order of $\gamma_{i-1}\gamma_i$ for $1 \leq i \leq d-1$, then we say that the *type* of Γ is $\{k_1, \ldots, k_{d-1}\}$ (to be clear, this denotes a sequence rather than a set, but we use curly braces

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for historical reasons). An sggi Γ is a *string C-group* if it also satisfies the *intersection condition*, namely, that

$$\langle \gamma_i | i \in I \rangle \cap \langle \gamma_j | j \in J \rangle = \langle \gamma_k | k \in I \cap J \rangle$$

for all $I, J \subseteq \{0, 1, \ldots, d-1\}$. The Coxeter group $[k_1, \ldots, k_{d-1}]$ is the group $\langle \gamma_0, \ldots, \gamma_{d-1} \rangle$ with defining relations

$$\gamma_i^2 = 1 \text{ for } 0 \le i \le d-1,$$

$$(\gamma_{i-1}\gamma_i)^{k_i} = 1 \text{ for } 1 \le i \le d-1,$$

$$(\gamma_i \gamma_j)^2 = 1 \text{ for } 0 \le i < j-1 \le d-2.$$

2.2. **Polytopes.** An *abstract d-polytope* $\mathcal{P}$ is a ranked partially ordered set whose elements are called *faces* satisfying the following four properties:

P1. There is a unique maximal face of rank d, and a unique minimal face of rank -1.
P2. All maximal chains have the same length, and contain a face of each rank.
P3. (Diamond condition) Given faces $F \le G$, with $\text{rank}(F) = i-1$ and $\text{rank}(G) = i+1$, there are exactly two faces H of rank i satisfying $F \le H \le G$.
P4. The poset is strongly flag-connected (see below).

Maximal chains in P are known as *flags*; we denote the set of flags of $\mathcal{P}$ by $\mathcal{F}(\mathcal{P})$. Two flags are said to be *i-adjacent* if they differ by exactly one face of rank i (or *i-face*). A poset satisfying P1 and P2 is said to be *flag-connected* if for any two flags Φ and Ψ of $\mathcal{P}$, there exists a sequence of flags $\Phi = \Phi_0, \Phi_1, \ldots, \Phi_k = \Psi$ such that each Φ_j is adjacent to Φ_{j+1}, and it is *strongly flag-connected* if such a sequence can always be found where each Φ_j contains $\Phi \cap \Psi$. We denote the unique flag differing from Φ at rank i by Φ^i. A function $f : \mathcal{P} \to \mathcal{Q}$ is a *rank and adjacency preserving map* or a *rap-map* if $\text{rank}(F) = \text{rank}((F)f)$ for all faces $F \in \mathcal{P}$ and if $(\Phi^i)f = (\Phi f)^i$ for all $i \in \{0, 1, \ldots, d-1\}$ and $\Phi \in \mathcal{F}(\mathcal{P})$. The *automorphism group* $\text{Aut}(\mathcal{P})$ is the set of rank and adjacency preserving bijective maps of $\mathcal{P}$ to itself; elements of $\text{Aut}(\mathcal{P})$ are typically written using Greek lower-case letters, and act on the right. A polytope $\mathcal{P}$ is *regular* if $\text{Aut}(\mathcal{P})$ acts transitively on $\mathcal{F}(\mathcal{P})$.

For any (base) flag Φ, the automorphism group of a regular polytope can be generated by the involutions ρ_i that send Φ to Φ^i for each i. Furthermore, if $\Gamma = \langle \rho_0, \ldots, \rho_{d-1} \rangle$ is the automorphism group of a regular polytope $\mathcal{P}$, then the group $\Gamma^+ = \langle \rho_0\rho_1, \rho_1\rho_2, \ldots, \rho_{d-2}\rho_{d-1} \rangle$ is called the *rotation subgroup* of Γ. The index of Γ^+ in Γ is at most 2. When the index is 2, we say that $\mathcal{P}$ is an orientable regular polytope.

A surjective rap-map $\pi : \mathcal{R} \to \mathcal{P}$ is a *covering* of polytopes; such a covering is indicated by $\mathcal{R} \searrow \mathcal{P}$. We note that if $\mathcal{P}$ is known to be flag connected then the rap-map is always surjective. The automorphism group of any regular abstract polytope is a string C-group [13, Section 2E]. A *minimal regular cover* $\text{MRC}(\mathcal{P})$ of a polytope $\mathcal{P}$ is a regular abstract polytope such that whenever there is a regular abstract polytope $\mathcal{R}$ such that $\text{MRC}(\mathcal{P}) \searrow \mathcal{R} \searrow \mathcal{P}$, it must be the case that $\mathcal{R} \cong \text{MRC}(\mathcal{P})$. Not every polytope has a unique minimal regular cover; see [14, Theorem 5.9].

We will also be interested in the *connection* (or *monodromy*) group $\text{Conn}(\mathcal{P})$ of the polytope, which is the sggi $\langle r_0, r_1, \ldots, r_{d-1} \rangle$ where $r_i : \mathcal{F}(\mathcal{P}) \to \mathcal{F}(\mathcal{P})$ sends each flag Φ of $\mathcal{P}$ to its i-adjacent neighbor. In our treatment, elements of $\text{Conn}(\mathcal{P})$ act on the left. By Proposition 3.16 of [15], if $\text{Conn}(\mathcal{P})$ is a string C-group, then there is, up to isomorphism, a unique minimal regular cover $\text{MRC}(\mathcal{P})$ of $\mathcal{P}$ whose automorphism group is an sggi isomorphic to $\text{Conn}(\mathcal{P})$.

An abstract 3-polytope is also known as a *polyhedron.* The connection group of any polyhedron is a string C-group (see, e.g., [15]).

2.3. **Graphs.** The terminology of graph theory in applications such as this is sufficiently inconsistent that we feel compelled to take a moment and clarify how we use the standard vocabulary. A *graph* $\mathcal{G} := (V, E)$ is an ordered pair where V is a set of *vertices* or *nodes*, and the *edges* E is a list of multisets of elements of V such that if $e \in E$, then $1 \leq |e| \leq 2$. If $|e| = 1$, we say that e is a *semi-edge*. We could represent a loop as a multiset with the same vertex repeated twice; however, since we will not consider graphs with loops in this manuscript, all edges will be actual subsets of V.

The *degree* of a vertex $v \in V$ is the cardinality of $\text{star}(v) := \{e \in E : v \in e\}$. An *edge coloring* of a graph with *colors* from a set S is a function $\phi : E \to S$, and it is *proper* if $\phi(e_1) \neq \phi(e_2)$ for all distinct pairs of edges e_1 and e_2 with $e_1 \cap e_2 \neq \emptyset$; if $\mathcal{G}$ is properly edge-colored with $|S| = d$, then $\mathcal{G}$ is *properly d-edge-colored*. The graph is *k-regular* if the degree of every vertex in $\mathcal{G}$ is k. The graph $\mathcal{G}$ is *simple* if E is a set (that is, there are no multiple edges between nodes) and if $|e| = 2$ for all $e \in E$ (that is, there are no semi-edges). A graph is *connected* if for all pairs $v_1, v_2 \in V$, there exists a sequence of edges $e_1, e_2, \ldots, e_n$ such that $v_1 \in e_1, v_2 \in e_n$ and $e_i \cap e_{i+1} \neq \emptyset$ for all $i \in \{1, 2, \ldots, n-1\}$. Naturally, we may visualize a graph by drawing the vertices as dots and connecting them by curves when they belong to an edge. For a semi-edge, only one endpoint of the curve is designated as a vertex.

We may associate to every d-polytope $\mathcal{P}$ its *flag graph* $\mathcal{FG}(\mathcal{P})$, whose nodes are $\mathcal{F}(\mathcal{P})$, with edges colored i connecting flags if they are i-adjacent. Such a graph is a d-regular, connected, properly d-edge-colored simple graph. The automorphism group of $\mathcal{P}$ has an induced natural faithful action on $\mathcal{FG}(\mathcal{P})$ via its action on the flags, since the automorphisms preserve flag adjacency.

2.4. **Maniplexes.** Let $\mathcal{G}$ be a connected d-regular graph, with proper edge coloring by the labels $\{0, 1, \ldots, d-1\}$. Let the *connection group of* $\mathcal{G}$, $\text{Conn}(\mathcal{G}) = \langle g_0, g_1, \ldots, g_{d-1}\rangle$, be the permutation group on the vertices of $\mathcal{G}$ where for all $v, w \in \mathcal{G}$, $g_i v = w$ iff v and w are in an edge of $\mathcal{G}$ with label i. We note that g_i fixes v if v is in a semi-edge with label i. Vertices $v, w \in \mathcal{G}$ are *i-adjacent* iff $g_i v = w$ (thus v is adjacent to itself if v is incident to a semi-edge of color i). If $g_i g_j = g_j g_i$ whenever $|i - j| > 1$, then we say $\mathcal{G}$ is a *pre-maniplex* of rank d. If we further require that $\mathcal{G}$ be a simple graph then $\mathcal{G}$ is a *maniplex* of rank d. (In some works, such as [11], pre-maniplexes are not required to be connected.). Note that $\text{Conn}(\mathcal{G})$ is an sggi when $\mathcal{G}$ is a maniplex.

Observe that every flag graph of an abstract polytope is a maniplex (though the implication does not go the other direction). It is natural then to call the nodes of a pre-maniplex *flags* without causing confusion. Given a flag $\Phi \in \mathcal{G}$, with $\mathcal{G}$ a pre-maniplex, we denote by Φ^i the flag that is i-adjacent to Φ, i.e., $g_i\Phi = \Phi^i$. We extend this notation by defining $\Phi^{i_1\cdots i_k} = (\Phi^{i_1\cdots i_{k-1}})^{i_k}$. Notice that $\Phi^{i_1\cdots i_k} = g_{i_k}\cdots g_{i_1}\Phi$.

Note that every sggi $C = \langle c_0, c_1, \ldots, c_{d-1}\rangle$ that is a permutation group determines a pre-maniplex $\mathcal{G}$ by letting the flags of $\mathcal{G}$ be the moved points of C with edges determined by the action of the generators of C and labeled with the index of the corresponding generator, and vice versa. (Note that individual generators c_i may have fixed points, but in order to get a connected structure, we need every flag to be moved by at least one generator.) Arbitrary sggis determine maniplexes by their left action on themselves, with the flags being the elements of C and edges labeled with the index of the corresponding generator of C.

Rap-maps are a way of distinguishing those polytope homomorphisms that have a well behaved action on the associated flag graph, and maniplexes and pre-maniplexes are the family of graphs that best describe the broader family of edge-labeled graphs to which flag graphs belong. We therefore find it convenient when adapting results about rap-maps to maniplexes and pre-maniplexes to have similar, but distinct, terminology at our disposal. Hence, an *elap-map* $\pi : \mathcal{G} \to \mathcal{H}$ of pre-maniplexes preserves edge labels and adjacency; equivalently, it is an edge-color preserving graph homomorphism. It is worth noting here that we mean that π is color preserving in the strict sense that if edge e has label i, then so does $e\pi$. A surjective elap-map is a *covering* of pre-maniplexes, and if there exists a surjective elap-map from $\mathcal{G}$ to $\mathcal{H}$, we denote that by $\mathcal{G} \searrow \mathcal{H}$. We note that if $\mathcal{G}$ and $\mathcal{H}$ are connected, then an elap-map from $\mathcal{G}$ to $\mathcal{H}$ is always a covering.

Lemma 2.1. *Suppose $\eta : \mathcal{M} \to \mathcal{N}$ is a elap-map of pre-maniplexes with rank d. For any flag Φ of $\mathcal{M}$ and $i_1, i_2, \ldots, i_k \in \{0, \ldots, d-1\}$, we have*

$$((\Phi)\eta)^{i_1 i_2 \cdots i_k} = (\Phi^{i_1 i_2 \cdots i_k})\eta.$$

Proof. Since η is edge label and adjacency preserving, we have $(\Psi^j)\eta = (\Psi\eta)^j$ for any flag Ψ and $0 \leq j \leq d-1$. The result follows from induction. □

The set of bijective elap-maps from a pre-maniplex $\mathcal{G}$ to itself is the *automorphism group* $\text{Aut}(\mathcal{G})$ of $\mathcal{G}$, and is the set of edge-color-preserving graph automorphisms. If the automorphism group $\text{Aut}(\mathcal{G})$ acts transitively on the flags of $\mathcal{G}$, then $\mathcal{G}$ is *reflexible*. (Thus, the flag graph of a regular polytope is a reflexible maniplex.) In fact, the action of $\text{Aut}(\mathcal{G})$ on flags is free, and thus sharply transitive (also called regular) when $\mathcal{G}$ is a reflexible maniplex. When $\mathcal{G}$ is any reflexible maniplex, the automorphism group is an sggi. In particular, if we fix a base flag Φ, then for each $i \in \{0, \ldots, d-1\}$ there is a unique automorphism ρ_i that sends Φ to Φ^i. We define the *type* of $\mathcal{G}$ to be the type of $\text{Aut}(\mathcal{G})$. More generally, if $\mathcal{G}$ is a reflexible pre-maniplex, then $\text{Aut}(\mathcal{G})$ is a quotient of an sggi; some generators may be trivial or some generators may coincide.

We will frequently make use of the following simple but fundamental fact: if $\mathcal{M}$ is a reflexible maniplex, then $\text{Conn}(\mathcal{M})$ is isomorphic to $\text{Aut}(\mathcal{M})$ in the natural way that sends generators to generators; see [3, Thm. 2.1].

The *smallest reflexible cover* $\text{SRC}(\mathcal{M})$ of a maniplex $\mathcal{M}$ is the Cayley graph of the group $\text{Conn}(\mathcal{M})$ on its standard set of generators, which is itself a reflexible maniplex. It is straightforward to adapt Proposition 3.16 from [15] to show that every other reflexible cover of $\mathcal{M}$ is also a cover of $\text{SRC}(\mathcal{M})$, and as a corollary that $\text{SRC}(\mathcal{FG}(\mathcal{P}))$ is the flag graph of the unique minimal regular cover of $\mathcal{P}$ when $\text{Conn}(\mathcal{P})$ is a string C-group. Hence, we say that a pre-maniplex is *C-connected* if its connection group is a string C-group.

By Lemma 2.1, elap-maps behave in most settings exactly like rap-maps of abstract polytopes, particularly in settings where it is the connectivity of the flag graph that is central to the argument. Consequently, it is straightforward to adapt many results from the literature to this setting, with slight rewordings. Here is one important example.

Lemma 2.2 (Proposition 3.11 from [15])**.** *Let $\mathcal{M}$, $\mathcal{N}$ be pre-maniplexes, let $\kappa : \mathcal{M} \to \mathcal{N}$ a covering of pre-maniplexes. Then there exists an epimorphism $\overline{\kappa} : \text{Conn}(\mathcal{M}) \to \text{Conn}(\mathcal{N})$. Suppose also κ maps a flag Λ' of $\mathcal{M}$ to a flag Λ of $\mathcal{N}$. Then*

$$\text{Stab}_{\text{Conn}(\mathcal{M})}\Lambda'\overline{\kappa} \subseteq \text{Stab}_{\text{Conn}(\mathcal{N})}\Lambda.$$

It is worth nothing the this group epimorphism is induced by the natural mapping of standard generators to standard generators.

Given a polytope $\mathcal{P}$ and a subgroup H of $\text{Aut}(\mathcal{P})$, the quotient $\mathcal{P}/H$ of $\mathcal{P}$ by H is the partially ordered set whose elements are the orbits of the faces of $\mathcal{P}$ under the action of H with the induced partial order. The important things to note about such quotients is that while a quotient by a group of automorphisms will frequently induce a covering of polytopes, this won't always be the case since the quotient object need not be a polytope (see [15, Sec. 2]). However, such a quotient does induce a corresponding natural quotient of the flag graph of $\mathcal{P}$ by the action of H on $\mathcal{F}(\mathcal{P})$, and the resulting quotient of the edge-labeled flag graph will be a maniplex or pre-maniplex. An example of the latter phenomenon occurs if adjacent flags in $\mathcal{P}$ are identified in $\mathcal{P}/H$. In particular, semi-edges arise from graph automorphisms which swap the two flags on a proper edge of the flag graph. We note that semi-edges are needed to preserve vertex degrees and the proper edge-coloring.

Now let us adapt Lemma 5.2 from [16].

Lemma 2.3. *Let $\mathcal{P}$ be an abstract polytope, $H \leq Aut(\mathcal{P})$. Let $\mathcal{Q} = \mathcal{P}/H$ and let*

$$L = \{l \in \text{Conn}(\mathcal{P}) : \forall \Phi \in \mathcal{F}(\mathcal{P}), \exists h \in H \ s.t.\ l\Phi = \Phi h\}.$$

Then $L \lhd \text{Conn}(\mathcal{P})$ and $\text{Conn}(\mathcal{P})/L \cong \text{Conn}(\mathcal{Q})$.

Proof. Let L be as above. Notice that the h in the definition of L is allowed to depend on your choice of flag Φ. Let $l \in L$ and $m \in \text{Conn}(\mathcal{P}), \Phi \in \mathcal{F}(\mathcal{P})$. We claim that for each $\Phi \in \mathcal{F}(\mathcal{P})$ there exists $h \in H$ such that $m^{-1}lm\Phi = \Phi h$. Note that by the definition of L, there exists $h \in H$ such that $lm\Phi = (m\Phi)h$, so $m^{-1}lm\Phi = m^{-1}m\Phi h = \Phi h$, as desired.

Let $\kappa : \mathcal{P} \to \mathcal{Q}$ be the natural quotient map induced by H. By Lemma 2.2 the map κ induces a surjective map $\overline{\kappa} : \text{Conn}(\mathcal{P}) \to \text{Conn}(\mathcal{Q})$. We claim that $\ker(\overline{\kappa}) = L$. Let $\Phi \in \mathcal{F}(\mathcal{P})$. First, let $k \in \ker(\overline{\kappa})$, then $k\Phi H = \Phi H$, and so $k \in L$, thus $\ker(\overline{\kappa}) \leq L$. Next, let $l \in L$, then there exists $h \in H$ such that $l\Phi = \Phi h$, and so $l\Phi H = (\Phi h)H = \Phi H$. Hence, $l \in \ker(\overline{\kappa})$, and $L \leq \ker(\overline{\kappa})$. Hence $L \lhd \text{Conn}(\mathcal{P})$ and $\text{Conn}(\mathcal{P})/L \cong \text{Conn}(\mathcal{Q})$. □

Lemma 2.4. *Let $\mathcal{R}$ be an orientable regular polyhedron, and let $\gamma \in \Gamma^+(\mathcal{R})$ fix an incident vertex-facet pair. Then γ is the identity.*

Proof. We first note that $\Gamma^+(\mathcal{R})$ preserves orientation and acts transitively on each orientation class of $\mathcal{R}$. Let v, f, respectively, be our incident vertex-facet pair, and let Φ be a flag of $\mathcal{R}$ containing them, and, without loss of generality, let α_i be the generators of $\Gamma(\mathcal{R})$ associated with this choice of base flag. Then we may express γ as an element of $\Gamma^+(\mathcal{R}) = \langle \alpha_0\alpha_1, \alpha_1\alpha_2 \rangle$. By assumption γ fixes v and f, and thus $\gamma \in \langle \alpha_0, \alpha_1 \rangle \cap \langle \alpha_1, \alpha_2 \rangle$. Then, since $\Gamma(\mathcal{R})$ is a string C-group, $\gamma \in \langle \alpha_1 \rangle$. We note that the identity is the only element of $\langle \alpha_1 \rangle$ that preserves orientation, completing the argument. □

2.5. **Mixing.** Suppose that $\mathcal{M}$ and $\mathcal{N}$ are d-maniplexes (or indeed, pre-maniplexes), with base flags Φ_0 and Ψ_0, respectively. Consider a new graph whose nodes are pairs (Φ, Ψ) with Φ a flag of $\mathcal{M}$ and Ψ a flag of $\mathcal{N}$, and define $(\Phi, \Psi)^i = (\Phi^i, \Psi^i)$. Then the *mix of* $\mathcal{M}$ *with* $\mathcal{N}$, denoted $\mathcal{M} \diamondsuit \mathcal{N}$, is the (pre-)maniplex we obtain by taking the connected component of this graph that contains (Φ_0, Ψ_0).

If $\mathcal{N}$ (or $\mathcal{M}$) is reflexible, then the graph we obtain has a single connected component, and so in that case it is not necessary to specify base flags of $\mathcal{M}$ and $\mathcal{N}$ (see [2, Corollary 3.13]). Furthermore, in this case, $\mathcal{M} \diamondsuit \mathcal{N}$ covers $\mathcal{M}$ and $\mathcal{N}$ and satisfies the following universal property: any pre-maniplex that covers both $\mathcal{M}$ and $\mathcal{N}$ also covers $\mathcal{M} \diamondsuit \mathcal{N}$. Note that the mix of two reflexible pre-maniplexes is itself reflexible.

Proposition 2.5. *Suppose that $\mathcal{M}$ and $\mathcal{R}$ are pre-maniplexes, with $\mathcal{R}$ reflexible. Then*

$$\text{SRC}(\mathcal{M} \diamondsuit \mathcal{R}) \cong \text{SRC}(\mathcal{M}) \diamondsuit \mathcal{R}.$$

Proof. Let $\mathcal{L} = \text{SRC}(\mathcal{M} \diamondsuit \mathcal{R})$. Then $\mathcal{L}$ covers $\mathcal{M} \diamondsuit \mathcal{R}$ and thus it covers $\mathcal{M}$ and $\mathcal{R}$. Futhermore, since $\mathcal{L}$ is reflexible, it must cover $\text{SRC}(\mathcal{M})$, and so it covers $\text{SRC}(\mathcal{M}) \diamondsuit \mathcal{R}$.

In the other direction, if $\mathcal{N} = \text{SRC}(\mathcal{M}) \diamondsuit \mathcal{R}$, then $\mathcal{N}$ covers $\mathcal{M}$ and $\mathcal{R}$, and so it covers $\mathcal{M} \diamondsuit \mathcal{R}$. Since $\mathcal{N}$ is the mix of two reflexible pre-maniplexes, it is reflexible, and so it covers $\text{SRC}(\mathcal{M} \diamondsuit \mathcal{R})$, showing that $\mathcal{L} \cong \mathcal{N}$. □

3. Stratified Operations and Connection Groups

3.1. **Stratification and Cover Preservation.** Constructing a prism over a base maniplex is an example of a broader class of operations called *stratified operations* introduced in [3]. The connection group of the result of a stratified operation can be described nicely using the connection group of the input. In this section, we will see how the connection group of a prism can be described in terms of the connection group of the base.

Let M_d denote the family of maniplexes of rank d, and $W_d = [\infty, \ldots, \infty]$ the universal string Coxeter group of rank d. The group $W_d = \langle w_0 \ldots, w_{d-1} \rangle$ acts on the flags of any d-maniplex $\mathcal{M}$, where the action of each w_i can be identified with the corresponding generator r_i of the connection group of $\mathcal{M}$. Following [3] we consider a maniplex operation $F : M_n \to M_m$ to be *stratified* if there is a set A (called the *strata*, and each element of A is a *stratum*) such that

(a) If Ω is the set of flags of an n-maniplex $\mathcal{M}$, then the set of flags of $F(\mathcal{M})$ is a subset of $A \times \Omega$ such that the canonical projections into A and Ω are surjective.
(b) Let $S = \{w_0, \ldots, w_{m-1}\}$ be the set of generators of W_m; then W_m has a well-defined action on A, where we denote by σ_i the permutation of A induced by $w_i \in S$.
(c) There is a function $\phi : A \times S \to W_n$ such that, for every maniplex $\mathcal{M}$ and flag Φ, the action of w_i on $A \times \Omega$ is described by

$$w_i(a, \Phi) = (\sigma_i a, \phi(a, w_i)\Phi).$$

Furthermore, if the set of flags of $F(\mathcal{M})$ is all of $A \times \Omega$, then we say that F is *fully stratified.* Fully stratified operations are *cover-preserving* [3, Proposition 3.8]; that is to say, if F is fully stratified, and $\mathcal{M}$ covers $\mathcal{L}$, then $F(\mathcal{M})$ covers $F(\mathcal{L})$. Additionally, when examining connection groups, fully stratified operations commute with the smallest reflexible cover operation [3, Theorem 3.9]. Formally, it follows from [3, Remark 2.3] that, if F is fully stratified and $\mathcal{M}$ is any maniplex, then $\mathrm{Conn}\,(F(\mathcal{M})) \cong \mathrm{Conn}\,(\mathrm{SRC}\,(F(\mathcal{M}))) \cong \mathrm{Conn}\,(F(\mathrm{SRC}\,(\mathcal{M})))$.

Proposition 3.1. *If F is a stratified operation and $\mathcal{M}$ is a maniplex, then* $\mathrm{Aut}\,(\mathcal{M}) \leq \mathrm{Aut}\,(F(\mathcal{M}))$.

Proof. Given $\alpha \in \mathrm{Aut}\,(\mathcal{M})$, we define $\overline{\alpha} \in \mathrm{Aut}\,(F(\mathcal{M}))$ by $(a, \Phi)\overline{\alpha} = (a, \Phi\alpha)$. This is easily seen to commute with the action of W_m, and thus it does define an automorphism. □

Remark 3.2. The definition of stratified operation, fully stratified operation, and so on, all apply equally well to pre-maniplexes. Indeed, the essential feature of maniplexes in [3] is just the action of W_d on maniplexes of rank d, and generalizing to pre-maniplexes is straightforward.

3.2. **Orientability.** A pre-maniplex is said to be *orientable* if it is a bipartite graph; otherwise it is *non-orientable.* We note that orientable (or *directly*) regular polytopes are orientable when considered as pre-maniplexes. Since we consider a flag to be adjacent to itself whenever it is incident to a semi-edge, an orientable pre-maniplex cannot have any semi-edges. The smallest orientable pre-maniplex of rank d consists of two flags, connected to each other with d edges with labels $0, \ldots, d-1$. We will denote this pre-maniplex by $\mathcal{G}_d$.

Every non-orientable pre-maniplex $\mathcal{M}$ has a unique orientable double cover. We can obtain this by mixing $\mathcal{M}$ with $\mathcal{G}_d$ (with d the rank of $\mathcal{M}$); see [18, p. 541], which describes the analogous construction for maps. Note that if $\mathcal{M}$ is orientable, then $\mathcal{M} \diamondsuit \mathcal{G}_d \cong \mathcal{M}$. Thus, we will define $\mathrm{OC}\,(\mathcal{M})$ as $\mathcal{M} \diamondsuit \mathcal{G}_d$, and $\mathrm{OC}\,(\mathcal{M})$ is the *minimal orientable cover of* $\mathcal{M}$.

We will use the following simple result.

Proposition 3.3. *Let* $\mathrm{OC}\,(\mathcal{M})$ *be the orientable cover of a maniplex. Then OC is cover-preserving, and* $\mathrm{SRC}\,(\mathrm{OC}\,(\mathrm{SRC}\,(\mathcal{M}))) = \mathrm{SRC}\,(\mathrm{OC}\,(\mathcal{M}))$.

Proof. Suppose that $\mathcal{L}$ and $\mathcal{M}$ are maniplexes of rank d such that $\mathcal{L}$ covers $\mathcal{M}$. Note that $\mathrm{OC}\,(\mathcal{L}) = \mathcal{L} \diamondsuit \mathcal{G}_d$, which covers both $\mathcal{L}$ and $\mathcal{G}_d$. Since it covers $\mathcal{L}$, it covers $\mathcal{M}$, and since it covers $\mathcal{M}$ and $\mathcal{G}_d$, it covers their mix, which is $\mathrm{OC}\,(\mathcal{M})$. So $\mathrm{OC}\,(\mathcal{L})$ covers $\mathrm{OC}\,(\mathcal{M})$, and thus OC is cover-preserving.

Next we will show that $\mathrm{SRC}\,(\mathrm{OC}\,(\mathrm{SRC}\,(\mathcal{M}))) = \mathrm{SRC}\,(\mathrm{OC}\,(\mathcal{M}))$. First, note that $\mathrm{OC}\,(\mathrm{SRC}\,(\mathcal{M}))$ is reflexible, and so the left side can be simplified to $\mathrm{OC}\,(\mathrm{SRC}\,(\mathcal{M}))$. Then by the definition of OC, we want to show that

$$\mathrm{SRC}\,(\mathcal{M}) \;\diamondsuit\; \mathcal{G}_d = \mathrm{SRC}\,(\mathcal{M} \diamondsuit \mathcal{G}_d)\,.$$

That follows from Proposition 2.5. □

A reflexible maniplex is orientable if and only if the defining relators of its automorphism group all have even length.

3.3. **The Prism Operation.** In [3, Theorem 6.1] it is shown that the products, including the prisms, are fully stratified. Furthermore, in [7, Section 8.2], the connection group of a prism is explicitly described. Strictly speaking, this connection group was defined for a prism over a polytope, but the same definition can be used to define a prism over a general pre-maniplex. (See also Section 4 in [11].) This description will also show that the prism operation is fully stratified, where the exact permutations σ_i and the function ϕ can be seen from their description. In [7, Section 8.2] the authors represent each flag in the prism as a triple $\Psi = (\Lambda, b, \Phi)$ where Λ is either one of the two flags of the rank 1 polytope, b is one of the elements e_i with $i \in \{1, 2, \ldots, d+1\}$, and Φ is a flag in the base*. To match our notation for stratified operations, we can think of each pair $[\Lambda, b]$ as representing a stratum. Thus we will have $2(d+1)$ strata, and can construct $s_i(a, \Phi) = s_i([\Lambda, b], \Phi)$, in the notation of stratified operations, as $s_i(\Lambda, b, \Phi)$ in the notation of [7]. Let $\langle s_0, \ldots, s_d\rangle$ be the connection group of a prism, $\langle r_0, \ldots, r_{d-1}\rangle$ be the connection group of the base. The generators of the connection group of the prism are as follows:

$$s_0([\Lambda, b], \Phi) = \begin{cases} ([\Lambda, b], r_0\Phi), & \text{if } b \neq e_1, \\ ([\Lambda^0, b], \Phi), & \text{if } b = e_1. \end{cases} \tag{1}$$

For $i > 0$ the connections are as follows:

$$s_i([\Lambda, b], \Phi) = \begin{cases} ([\Lambda, e_{i+1}], \Phi) \text{ if } b = e_i \\ ([\Lambda, e_i], \Phi) \text{ if } b = e_{i+1} \\ ([\Lambda, b], r_{i-1}\Phi) \text{ if } b \in \{e_1, \ldots, e_{i-1}\} \\ ([\Lambda, b], r_i\Phi) \text{ if } b \in \{e_{i+2}, \ldots, e_{d+1}\}. \end{cases} \tag{2}$$

Thus the connection group of a prism over a pre-maniplex can be seen as an imprimitive group with $2d + 2$ blocks, and it can be embedded in a wreath product. In particular, the connection group acts on the strata according to the pre-maniplex shown in Figure 1. (Compare to Figure 5 in [11].) To simplify our notation, fix our choice for the variable Λ to be the flag λ, and we will label the strata, which correspond to pairs $[\Lambda, b]$, using the numbers 1 through $2d + 2$.

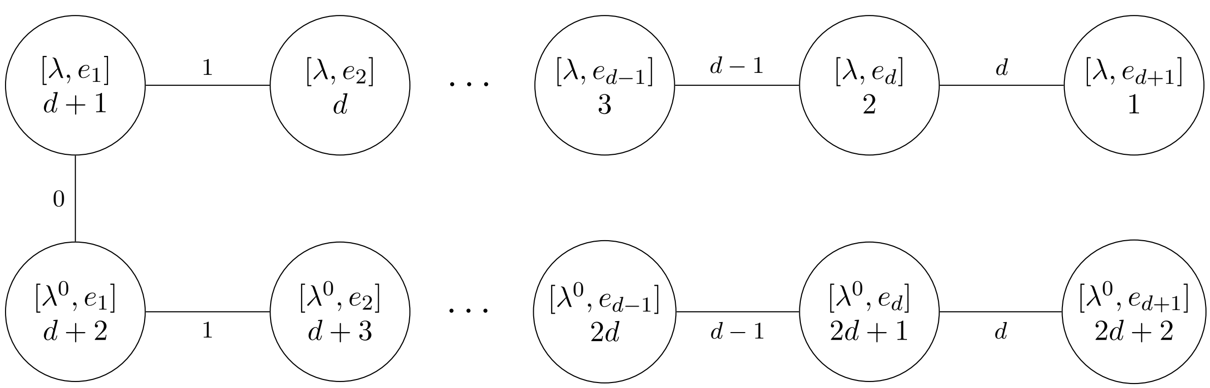

FIGURE 1. The pre-maniplex determined by the action of a prism on strata. The semi-edges, which make the graph regular, are not shown for simplicity. Indices for the strata are listed below the stratum in the figure, e.g., $[\lambda, e_2]$ has label d.

*For ease of reading, we have changed the order of the triple, making the flag in the base come after the other information that determines the stratum. In [7] each flag is actually written as a triple $\Psi = (\Phi, \Lambda, b)$.

Using the notation of [16], we can now represent the connection group of a prism over a 3-maniplex $\mathcal{B}$ as a subgroup of $S_8 \ltimes (\mathrm{Conn}\,(\mathcal{B}))^8 = S_8 \wr \mathrm{Conn}\,(\mathcal{B})$. (Note that we are choosing to reverse the usual ordering of the groups in the wreath product to respect the side of action usually associated with the connection group.) To do this we consider a base flag Φ of $\mathcal{P}$, and we may then associate the positions in our representation with the sequence of flags $\Phi, \Phi^3, \Phi^{32}, \Phi^{321}, \Phi^{3210}, \Phi^{32101}, \Phi^{321012}, \Phi^{3210123}$. So if $\mathrm{Conn}\,(\mathcal{B}) = \langle r_0, r_1, r_2\rangle$ with identity element e, then $\mathrm{Conn}\,(\mathcal{P}) = \langle s_0, \ldots, s_3\rangle$ will have generators

$$\begin{aligned} s_0 &:= ((4,5), [r_0, r_0, r_0, e, e, r_0, r_0, r_0]),\\ s_1 &:= ((3,4)(5,6), [r_1, r_1, e, e, e, e, r_1, r_1]),\\ s_2 &:= ((2,3)(6,7), [r_2, e, e, r_1, r_1, e, e, r_2]),\\ s_3 &:= ((1,2)(7,8), [e, e, r_2, r_2, r_2, r_2, e, e]). \end{aligned}$$

If we quotient $\mathrm{Conn}\,(\mathcal{P})$ by the normal subgroup of elements which fix each block setwise, we get a group B which gives the action of $\mathrm{Conn}\,(\mathcal{P})$ on these 8 blocks. Let $B = \langle b_0, b_1, b_2, b_3\rangle$ where the generators are the images of the generators of $\mathrm{Conn}\,(\mathcal{P})$ under the quotient map. Then we have the following permutation representation for B:

$$b_0 = (4,5),\ b_1 = (3,4)(5,6),\ b_2 = (2,3)(6,7),\ b_3 = (1,2)(7,8).$$

Due to the symmetry in the terms in $(\mathrm{Conn}\,(\mathcal{B}))^8$, we will usually denote elements in this representation without the redundant terms and with a doubled right brace, e.g.,

$$s_2 := ((2,3)(6,7), [r_2, e, e, r_1]\!]).$$

An immediate observation is that

$$\begin{aligned} (s_0s_1)^4 &= ((), [(r_0r_1)^4, (r_0r_1)^4, e, e]\!]),\\ (s_1s_2)^3 &= ((), [(r_1r_2)^3, e, e, e]\!]),\\ (s_2s_3)^3 &= ((), [e, e, e, (r_1r_2)^3]\!]). \end{aligned} \tag{3}$$

4. Properties of the Connection Groups of Prisms of Rank 4

In this section we provide some facts about the connection groups of prisms with a special focus on prisms over 3-maniplexes.

Proposition 4.1. *Let $\langle s_0, \ldots, s_3\rangle$ be the connection group of a rank 4 prism $\mathcal{P}$. Let K be the normal subgroup of $\langle s_1, s_2, s_3\rangle$ that fixes the eight blocks setwise. Then K is the normal closure of $\langle (s_1s_2)^3, (s_2s_3)^3\rangle$ in $\langle s_1, s_2, s_3\rangle$.*

Proof. Equation (3) shows that $(s_1s_2)^3$ and $(s_2s_3)^3$ fix all blocks, and so K contains their normal closure. Now, let $\pi : \langle s_1, s_2, s_3\rangle \to [3,3]$ be projection in the first coordinate. In other words, $\pi(s_i)$ gives the action of s_i on blocks, so that $K = \ker \pi$. Now, let $W = \langle w_1, w_2, w_3\rangle$ be the universal Coxeter group $[\infty, \infty]$. Then there is a group epimorphism from W to $\langle s_1, s_2, s_3\rangle$ that sends each w_i to s_i. Furthermore, W covers $[3,3]$, and the kernel of this cover is the normal closure of $(s_1s_2)^3$ and $(s_2s_3)^3$ in W. Then K is the image of that kernel in $\langle s_1, s_2, s_3\rangle$, and the result follows. □

Corollary 4.2. *Let $\langle s_0, \ldots, s_3\rangle$ be the connection group of a rank 4 prism $\mathcal{P}$, and let $\langle r_0, r_1, r_2\rangle$ be the connection group of its base. Let K be the normal subgroup of $\langle s_1, s_2, s_3\rangle$ that fixes the eight blocks setwise. Then*

$$K = \{((), [(r_1r_2)^{3i_1}, (r_1r_2)^{3i_2}, (r_1r_2)^{3i_3}, (r_1r_2)^{3i_4}]\!]) : i_1, i_2, i_3, i_4 \in \mathbb{Z}\}.$$

Proof. Let S be the set (in fact, subgroup) on the right. First, note that conjugation by s_1, s_2, and s_3 all fix S, and so S is a normal subgroup of $\langle s_1, s_2, s_3\rangle$ that contains $(s_1s_2)^3$ and $(s_2s_3)^3$. Thus S contains K. Now, we have

$$\begin{aligned}(s_1s_2)^3 &= ((), [\![(r_1r_2)^3, e, e, e]\!])\\ ((s_1s_2)^3)^{s_3} &= ((), [\![e, (r_1r_2)^3, e, e]\!])\\ ((s_2s_3)^3)^{s_1} &= ((), [\![e, e, (r_1r_2)^3, e]\!])\\ (s_2s_3)^3 &= ((), [\![e, e, e, (r_1r_2)^3]\!]).\end{aligned}$$

Since K is generated by $(s_1s_2)^3$ and $(s_2s_3)^3$ and their conjugates, this shows that K contains S, proving that they are equal. □

For what follows, we will be using the pre-maniplex $\mathcal{Q}$ determined by the connection group with generators

$$\begin{aligned}\rho_0 &:= (1,3)(2,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16),\\ \rho_1 &:= (1,5)(2,6)(3,7)(4,8)(9,10)(11,12)(13,14)(15,16),\\ \rho_2 &:= (1,2)(3,4)(5,9)(6,10)(7,11)(8,12)(13,14)(15,16),\\ \rho_3 &:= (1,2)(3,4)(5,6)(7,8)(9,13)(10,14)(11,15)(12,16);\end{aligned}$$

which turns out to be precisely the prism over the pre-maniplex $\mathcal{G}_3$ with connection group

$$\mathrm{Conn}\,(\mathcal{G}_3) = \langle (1,2), (1,2), (1,2)\rangle;$$

in this interpretation, the two facets isomorphic to $\mathcal{G}_3$ correspond to the pairs of vertices $\{13, 14\}$ and $\{15, 16\}$. It is easily verified computationally that $\mathrm{Conn}\,(\mathcal{Q})$ is a string C-group. Additionally, as will be seen in Proposition 4.7, the smallest reflexible cover of $\mathrm{Prism}(\mathcal{G}_3)$ is the 4-cube. We note that while $\mathrm{Conn}\,(\mathcal{Q}) \cong [4, 3, 3]$, this does not represent the usual action of $[4, 3, 3]$ on the 16 vertices of the 4-cube. In that action there exists a vertex stabilized by three of the four generators of the automorphism group of the 4-cube.

Theorem 4.3. *Prisms over orientable regular polyhedra are C-connected. In other words, if $\mathcal{B}$ is an orientable regular polyhedron, and $\mathcal{P} = \mathrm{Prism}\,(\mathcal{B})$, then $\mathrm{Conn}\,(\mathcal{P})$ is a string C-group, and $\mathrm{SRC}\,(\mathcal{P})$ is a polytope.*

Proof. Let $\Gamma = \langle \rho_0, \rho_1, \rho_2\rangle = \mathrm{Aut}\,(\mathcal{B})$ in terms of the usual generators for a regular polyhedron, and let Γ^+ be the rotation subgroup of Γ. By Proposition 3.1, we can naturally embed Γ^+ into $\mathrm{Aut}\,(\mathcal{P})$; let H denote the image of this embedding. Let L be defined as in Lemma 2.3 for this group H.

We first note that, as shown in [7, Theorem A], $\mathcal{P}$ is itself a polytope. Since $\mathcal{B}$ is an orientable polyhedron, its flag graph is bipartite, and the action of Γ^+ on the flags preserves the partition and acts transitively on the flags in each part, hence $\mathcal{B}/\Gamma^+ \cong \mathcal{G}_3$ as pre-maniplexes. Let us extend the coloring of $\mathcal{B}$ to a coloring of $\mathcal{P}$ by coloring each flag (a, Φ) with the color of Φ. Then it isn't hard to see that $\mathcal{P}/H \cong \mathcal{Q}$ since the action of H will identify flags in the same stratum of the same color, and thus $\mathrm{Conn}\,(\mathcal{Q}) \cong \mathrm{Conn}\,(\mathcal{P})\,/L$ by Lemma 2.3. Let $\overline{\kappa}$ be the quotient map from Lemma 2.3.

Let $\{r_0, r_1, r_2, r_3\}$ be the canonical generators of $C = \mathrm{Conn}\,(\mathcal{P})$. Observe that since $\mathcal{P}$ is a polytope, that the groups $C_0 = \langle r_1, r_2, r_3\rangle$ and $C_3 = \langle r_0, r_1, r_2\rangle$ are string C-groups by Proposition 5.1 of [16]. Thus by [13, Proposition 2E16], to show that C is a string C-group it suffices to show that $C_0 \cap C_3 = \langle r_1, r_2\rangle$.

Let $\gamma \in C_0 \cap C_3$, and let $s_i := r_i\overline{\kappa}$. Then $\gamma\overline{\kappa} \in (C_0 \cap C_3)\overline{\kappa}$. Since $\mathrm{Conn}\,(\mathcal{Q})$ is a string C-group, it follows that $(C_0 \cap C_3)\overline{\kappa} \subseteq C_0\overline{\kappa} \cap C_3\overline{\kappa} = \langle s_1, s_2\rangle = \langle r_1, r_2\rangle\overline{\kappa}$. Thus $\gamma \in \langle r_1, r_2\rangle L$, i.e., $\gamma = sl$ for some $s \in \langle r_1, r_2\rangle$ and $l \in L$. Since $\gamma \in C_0 \cap C_3$ fixes every vertex and facet of $\mathcal{P}$, so does $l = s^{-1}\gamma$. We consider now the action of l on the flags of $\mathcal{P}$. Let Φ be a flag of $\mathcal{P}$, and let v, f be the incident vertex and facet of Φ.

If Φ is a flag in a prismatic facet of $\mathcal{P}$, then $l\Phi = \Phi h$ for some $h \in H$ by the definition of L. We claim h is the identity element. Since l fixes v and f, h must fix the corresponding vertex and 2-face of $\mathcal{B}$, but since $\mathcal{B}$ is a polytope, Lemma 2.4 implies that the only element of h that does this is the identity element, hence $l\Phi = \Phi$.†

Now, consider the action of l on the eight blocks (as in Figure 1). The argument above shows that it acts trivially on blocks 2 through 7. Furthermore, since $l \in C_3$, inspection of the action on blocks shows that l fixes blocks 1 and 8 setwise. So l fixes all of the blocks. Since $l \in C_0$ as well, Corollary 4.2 implies that l acts on block 1 (and 8) like an automorphism $(\rho_1\rho_2)^{3k}$ for some k. Since l also fixes blocks 2-7, it follows that l is a power of

$$(s_1s_2)^3 = ((), [(r_1r_2)^3, e, e, e]).$$

Thus, $l \in \langle r_1, r_2\rangle$. From this we conclude that γ may be represented as a product of elements in $\langle r_1, r_2\rangle$. Therefore $\mathrm{Conn}(\mathcal{P})$ is a string C-group and $\mathrm{SRC}(\mathcal{P})$ is a polytope.

□

Remark 4.4. The fact that $\mathcal{B}$ was a polyhedron, and not just an orientable and reflexible 3-maniplex is important. In fact there are prisms over orientable and reflexible 3-maniplexes that do not have string C connection groups. For instance consider the reflexible maniplex $\mathcal{M}$ with automorphism group

$$\langle \rho_0, \rho_1, \rho_2 \mid \rho_0^2 = \rho_1^2 = \rho_2^2 = (\rho_0\rho_1)^8 = (\rho_1\rho_2)^6 = (\rho_0\rho_2)^2 = (\rho_0\rho_1)^4(\rho_1\rho_2)^3 = 1\rangle.$$

This maniplex is not a polytope; it fails the diamond condition, as for each incident vertex and facet, there are four edges incident with both. However, it is orientable as all the relators have even length in the generators. In this example both ρ_1 and $(\rho_0\rho_1)^4$ will fix an incident vertex and facet pair. However $(\rho_0\rho_1)^4$ is in $\Gamma^+(\mathcal{M})$ and thus Lemma 2.4 will not hold.

Remark 4.5. If the base were non-orientable, the pre-maniplex $\mathcal{P}/H$ is determined by the connection group with generators $\rho_0 := (1,2), \rho_1 := (1,3)(2,4), \rho_2 := (3,5)(4,6), \rho_3 := (5,7)(6,8)$.

Lemma 4.6. *Let $\mathcal{M}$ be a reflexible 3-maniplex, and let $\mathcal{P} = Prism(\mathcal{M})$ be the prism with $\mathcal{M}$ as a base. Then the smallest reflexible cover of $\mathcal{P}$ covers the regular 4-cube, and* $\mathrm{SRC}(\mathcal{P}) \diamondsuit \mathrm{Cube}(4) = \mathrm{SRC}(\mathcal{P})$.

Proof. Let $C = \mathrm{Conn}(\mathcal{P})$. In Section 3.3 we saw that C can be represented as a subgroup of the imprimitive wreath product, where C acts with 8 blocks of imprimitivity on the flags, and that when we quotient C by the normal subgroup of elements which fix each block setwise, we get a group B which gives the action of C on these 8 blocks with the following generators.

$$b_0 = (4,5),\ b_1 = (3,4)(5,6),\ b_2 = (2,3)(6,7),\ b_3 = (1,2)(7,8);$$

As this is an sggi of type $\{4,3,3\}$ and has order 384, this can be shown to be equivalent to the automorphism group of the 4-cube acting faithfully on its 8 facets.

Since the connection group of the regular 4-cube is isomorphic to the automorphism group, we can conclude that $C = \mathrm{Conn}(\mathcal{P}) \searrow \mathrm{Conn}(\mathrm{Cube}(4))$ and thus $\mathrm{SRC}(\mathcal{P}) \diamondsuit \mathrm{Cube}(4) = \mathrm{SRC}(\mathcal{P})$ (by Lemma 5.3 of [15]). □

Proposition 4.7. *If $\mathcal{G}_d$ is the orientable reflexible rank d pre-maniplex with two flags, then* $\mathrm{Prism}(\mathcal{G}_d)$ *is orientable, and the smallest reflexible cover of* $\mathrm{Prism}(\mathcal{G}_d)$ *is the cube of rank $d+1$.*

Proof. We first note that the $\mathrm{Conn}(\mathcal{G}_d) = \langle (1,2), \ldots, (1,2)\rangle$. From Equations (1) and (2), since $\mathcal{G}_d$ has only two flags, it is straightforward to check that $|s_0s_1| = 4$, and that $|s_is_{i+1}| = 3$ for $i > 0$.

†To be even more pedantic: h is an element of the automorphism group Γ, which acts freely on $\mathcal{B}$. The only non-trivial element of Γ that fixes an incident vertex and 2-face is $\rho_1 \notin H$.

For instance, let ϕ be either of the two flags in the base pre-maniplex $\mathcal{G}_d$ and λ be either of the two flags in the rank 1 polytope. Then we get one 8-cycle of flags when alternately applying s_0 and s_1:

$$(([\lambda, e_2], \phi), ([\lambda, e_2], \phi^0), ([\lambda, e_1], \phi^0), ([\lambda^0, e_1], \phi^0), ([\lambda^0, e_2], \phi^0), ([\lambda^0, e_2], \phi), ([\lambda^0, e_1], \phi), ([\lambda, e_1], \phi)).$$

Thus $s_0 s_1$ has order 4 when acting on any of these 8 flags. Additionally $s_0 s_1$ acts like identity on any flags not in this 8-cycle. The calculation for $|s_i s_{i+1}| = 3$ for $i > 0$ is similar.

Thus $\mathrm{Conn}(\mathrm{Prism}(\mathcal{G}_d))$ satisfies all the relations of the Coxeter group $[4, 3, \ldots, 3]$. Additionally $|s_0 s_1 \ldots s_d| = 2d + 2$ and thus $\mathrm{Conn}(\mathrm{Prism}(\mathcal{G}_d))$ does not satisfy the relations of the connection group of the hemi-cube (which satisfies $|s_0 s_1 \ldots s_d| = d + 1$ instead). In particular, using Figure 1, we see that the action of $s_0 s_1 \cdots s_d$ on the strata is given by:

$$s_0 s_1 \cdots s_d = (1, 2, \ldots, d + 1, 2d + 2, 2d + 1, \ldots, d + 2).$$

Therefore $\mathrm{Prism}(\mathcal{G}_d)$ is orientable and the smallest reflexible cover of $\mathrm{Prism}(\mathcal{G}_d)$ is the cube of rank $d + 1$.

□

We will now work to show that the conclusion of Theorem 4.3 works even if the base is non-orientable.

Lemma 4.8. *Let $\mathcal{M}$ be a reflexible d-maniplex. Then* $\mathrm{Prism}(\mathrm{OC}(\mathcal{M})) \cong \mathrm{OC}(\mathrm{Prism}(\mathcal{M}))$.

Proof. First, note that if $\mathcal{M}$ is orientable, then $\mathrm{OC}(\mathcal{M}) \cong \mathcal{M}$. Furthermore, $\mathrm{Prism}(\mathcal{M})$ is orientable. Thus the result is clear in this case. So let us suppose that $\mathcal{M}$ is non-orientable.

Let us consider the flags of $\mathrm{Prism}(\mathrm{OC}(\mathcal{M}))$. First, the flags of $\mathrm{OC}(\mathcal{M})$ can be seen as pairs (Φ, x) with Φ a flag of $\mathcal{M}$ and $x \in \{-1, 1\}$. Then $\mathrm{Prism}(\mathrm{OC}(\mathcal{M}))$ consists of pairs $(a, (\Phi, x))$, with $a \in \{1, 2, \ldots, 2d + 2\}$ representing the stratum in the definition of Prism as a stratified operation, as shown in Figure 1. Similarly, we find that the flags of $\mathrm{OC}(\mathrm{Prism}(\mathcal{M}))$ can be represented as $((a, \Phi), x)$. Note that since $\mathcal{M}$ is non-orientable, so is $\mathrm{Prism}(\mathcal{M})$, so every pair of a and Φ appears with every x. Furthermore, since Prism is fully stratified, every Φ occurs with every a.

We claim that the function f that sends each $(a, (\Phi, x))$ to $((a, \Phi), (-1)^{a+1} x)$ is an elap-map. If we can show that, then it is clear that it is one-to-one since both sides consist of all triples of a, Φ, and x.

First, note that in the definition of the Prism operation, each s_i either fixes the stratum a or fixes the flag Φ. Let us consider an arbitrary flag $(a, (\Phi, x))$. Suppose that a is odd; the case with a even is similar. If s_i fixes a, then

$$(a, (\Phi, x))^i = (a, (\Phi, x)^j) = (a, (\Phi^j, -x))$$

for some j depending only on a and i. Similarly,

$$((a, \Phi), x)^i = ((a, \Phi)^i, -x) = ((a, \Phi^j), -x).$$

Note that f sends $(a, (\Phi^j, -x))$ to $((a, \Phi^j), -x)$, as it is supposed to. Similarly, if s_i sends a to a^i while fixing Φ, then

$$(a, (\Phi, x))^i = (a^i, (\Phi, x))$$

while

$$((a, \Phi), x)^i = ((a, \Phi)^i, -x) = ((a^i, \Phi), -x).$$

We note that whenever $a^i \neq a$, then a^i has opposite parity from a. Again, f sends $(a^i, (\Phi, x))$ to $((a^i, \Phi), -x)$ as it should. Thus f preserves adjacency and edge-labels, completing the proof. □

Lemma 4.9. *Let $\mathcal{M}$ be a non-orientable reflexible 3-maniplex. Then*

$$\mathrm{SRC}(\mathrm{Prism}(\mathcal{M})) = \mathrm{SRC}(\mathrm{Prism}(\mathrm{OC}(\mathcal{M}))).$$

Proof. Let $\mathcal{P}$ be the prism with base $\mathcal{M}$, and let $\mathcal{C}$ be the 4-cube. Since $\mathrm{SRC}(\mathrm{Prism}(\mathrm{OC}(\mathcal{M})))$ covers $\mathrm{SRC}(\mathcal{P})$, it suffices to show that $\mathrm{SRC}(\mathcal{P})$ covers $\mathrm{SRC}(\mathrm{Prism}(\mathrm{OC}(\mathcal{M})))$.

Since $\mathcal{C}$ is orientable, $\mathrm{SRC}(\mathcal{P}) \diamondsuit \mathcal{C}$ is also orientable. Thus $\mathrm{SRC}(\mathcal{P}) \diamondsuit \mathcal{C}$ covers $\mathrm{OC}(\mathrm{SRC}(\mathcal{P}))$. Then due to the fact that $\mathrm{SRC}(\mathcal{P}) \diamondsuit \mathcal{C}$ is reflexible, it also covers $\mathrm{SRC}(\mathrm{OC}(\mathrm{SRC}(\mathcal{P})))$. By Proposition 3.3, $\mathrm{SRC}(\mathrm{OC}(\mathcal{P})) = \mathrm{SRC}(\mathrm{OC}(\mathrm{SRC}(\mathcal{P})))$. Therefore, $\mathrm{SRC}(\mathcal{P}) \diamondsuit \mathcal{C}$ covers $\mathrm{SRC}(\mathrm{OC}(\mathcal{P}))$. Applying Lemma 4.6 and Lemma 4.8 we can conclude that $\mathrm{SRC}(\mathcal{P})$ covers

$$\mathrm{SRC}(\mathrm{Prism}(\mathrm{OC}(\mathcal{M}))).$$

□

Theorem 4.10. *Prisms over polyhedra are C-connected. In other words, if $\mathcal{B}$ is a polyhedron, and $\mathcal{P} = \mathrm{Prism}(\mathcal{B})$, then $\mathrm{Conn}(\mathcal{P})$ is a string C-group, and $\mathrm{SRC}(\mathcal{P})$ is a polytope.*

Proof. First, Theorem 4.3 proves the case where $\mathcal{B}$ is orientable and reflexible, and combined with Lemma 4.9, this implies that the result holds for all reflexible polyhedra. Moreover, by [15, Prop. 6.1] $\mathrm{Conn}(\mathcal{B})$ is a string C-group for all polyhedra $\mathcal{B}$; in other words, $\mathrm{Conn}(\mathcal{B}) = \mathrm{Conn}(\mathcal{R})$ where $\mathcal{R} = \mathrm{SRC}(\mathcal{B})$. Since the prism operation is connection preserving, $\mathrm{Conn}(\mathcal{P}) = \mathrm{Conn}(\mathrm{Prism}(\mathcal{R}))$, completing the proof. □

5. Open Problems

As mentioned in the introduction, there are numerous examples of 4-polytopes $\mathcal{P}$ for which $\mathrm{Conn}(\mathrm{Prism}(\mathcal{P}))$ is not a string C-group. In fact, the role of orientability in the classification of which polytopes of rank 4 and higher is unclear since the universal polytope $\mathcal{P}$ of type $\{3,3,2\}$ is orientable, as is $\{3,6,4\} * 144$[§], and yet $\mathrm{Prism}(\mathcal{P})$ has a smallest reflexible cover that is not polytopal in both cases.

Question 5.1. *For which polytopes $\mathcal{P}$ of rank $d \geq 4$ is the smallest reflexible cover of $\mathrm{Prism}(\mathcal{P})$ a regular polytope?*

Lemmas 4.6, 4.8, and 4.9 all generalize to higher ranks (using Proposition 4.7). In fact, Lemma 4.6 suggests another possibility; namely that being able to form a C-connected prism over a polytope is related to the structure of the mix of the polytope with the cube. Testing of a wide range of examples in various ranks in GAP [5] with the package RAMP [1] suggests the following conjecture.

Conjecture 5.2. *Let $\mathcal{P}$ be a d-polytope. Then $\mathrm{Prism}(\mathcal{P})$ is C-connected iff $\mathcal{P} \diamondsuit \mathrm{Cube}(d)$ is C-connected.*

Note that a number of possible weakenings of the conjecture above fail. For example, changing both instances of 'C-connected' to 'polytopal' fails; since while all prisms over polytopes are polytopal, there are polytopes (such as the 4-cross-polytope) that have a non-polytopal mix with the 4-cube.

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[§] Here we are using the canonical name in [8].

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