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THE INTERLACING PROPERTIES OF GENERALIZED NARAYANA POLYNOMIALS

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Abstract. In this paper, we obtain two new interlacing properties on the zeros of a type of generalized Narayana polynomials arising in the study of the infinite log-concavity of the Boros–Moll polynomials. Our tools include a criterion established by Liu and Wang and two new recurrence relations for these Narayana polynomials. The new recurrences are verified with the help of `Maple` package `APCI` given by Hou. Our results also imply the real-rootedness of these Narayana polynomials.

1. Introduction

Let RZ be the set of real polynomials with only real zeros. Given two polynomials $f(x), g(x) \in \mathrm{RZ}$, suppose $u_1, u_2, \ldots$ and $v_1, v_2, \ldots$ are the zeros of $f(x)$ and $g(x)$, respectively, in nonincreasing order. We say that $g(x)$ interlaces $f(x)$, denoted $g(x) \preccurlyeq f(x)$, if either $\deg f(x) = \deg g(x) = n$ and

$$v_n \leq u_n \leq \cdots \leq v_2 \leq u_2 \leq v_1 \leq u_1,$$

or $\deg f(x) = \deg g(x) + 1 = n$ and

$$u_n \leq v_{n-1} \leq u_{n-1} \leq \cdots \leq v_2 \leq u_2 \leq v_1 \leq u_1.$$

Following the notation of Liu and Wang [12], we also let $a \preccurlyeq bx + c$ for any nonnegative a, b, c, and let $0 \preccurlyeq f$ and $f \preccurlyeq 0$ for any $f \in \mathrm{RZ}$. Let PF be the set of polynomials in RZ with nonnegative coefficients, including any nonnegative constant for convenience. The sequence of the coefficients of each polynomial in PF is called a *Pólya frequency sequence* in the theory of total positivity, see Karlin [11] and Brenti [3]. Given a polynomial sequence $\{f_n(x)\}_{n\geq 0}$, if each $f_n(x) \in \mathrm{PF}$ and

$$f_0(x) \preccurlyeq f_1(x) \preccurlyeq \cdots \preccurlyeq f_{n-1}(x) \preccurlyeq f_n(x) \preccurlyeq \cdots,$$

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then $\{f_n(x)\}_{n\geq 0}$ is said to be a generalized Sturm sequence.

There are many generalized Sturm sequences. For example, the sequence of the classical Narayana polynomials $\{N_n(x)\}_{n\geq 0}$, where

$$N_n(x) = \sum_{k=0}^{n} \frac{1}{n+1}\binom{n+1}{k}\binom{n+1}{k+1}x^k. \tag{1.1}$$

These polynomials were extensively studied by the combinatorial community, see for instance [1, 10, 4, 13]. Chen, Yang and Zhang [4] found the following generalized Narayana polynomials,

$$N_{n,m}(x) = \sum_{k=0}^{n} \left(\binom{n}{k}\binom{m}{k} - \binom{n}{k+1}\binom{m}{k-1}\right) x^k, \tag{1.2}$$

when they studied the infinite log-concavity of the Boros–Moll polynomials, which were first introduced by Boros and Moll [2] while studying a quartic integral. When $n = m + 1$, it is clear that (1.2) reduces to (1.1), namely, $N_{m+1,m}(x) = N_m(x)$. With the aid of a criterion established by Liu and Wang [12, Theorem 2.3], Chen et al. [4, Theorem 1.4] had proved the real-rootedness of $N_{n,m}(x)$ by showing two interlacing relations on the zeros of $N_{n,m}(x)$.

Theorem 1.1 ([4, Theorem 3.2])**.** *For any integers $0 \leq n \leq m$, the generalized Narayana polynomial $N_{n,m}(x)$ has only real zeros, and moreover, $N_{n,m}(x) \preccurlyeq N_{n+1,m+1}(x)$.*

Theorem 1.2 ([4, Theorem 3.4])**.** *For $n \geq m \geq 0$, the polynomial $N_{n,m}(x)$ has only real zeros. If $n \geq m+2$, then $N_{n,m}(x)$ has one and only one positive zero and m negative zeros, and moreover, for $m \geq 1$, the negative zeros of $N_{n,m}(x)$ and $N_{n+1,m+1}(x)$ have interlacing relations. That is,*

$$r_{n+1,m+1}^{(m+1)} < r_{n,m}^{(m)} < r_{n+1,m+1}^{(m)} < \cdots < r_{n,m}^{(2)} < r_{n+1,m+1}^{(2)} < r_{n,m}^{(1)} < r_{n+1,m+1}^{(1)} < 0,$$

where $\{r_{n,m}^{(i)}\}_{i=1}^{m}$ are negative zeros of $N_{n,m}(x)$ for all $m \geq 1$ and $n \geq m+2$.

In this paper, we obtain two new interlacing properties on the zeros of $N_{n,m}(x)$. One for $n \leq m+1$, the other for $n \geq m+2$. The main results of this paper are as follows.

Theorem 1.3. *Let $N_{n,m}(x)$ be given by (1.2). Then for any fixed $m \geq 0$, the sequence $\{N_{n,m}(x)\}_{n=0}^{m+1}$ is a generalized Sturm sequence. More precisely, for $1 \leq n \leq m+1$,*

$$N_{n-1,m}(x) \preccurlyeq N_{n,m}(x).$$

When $n \geq m+2$, the leading term of $N_{n,m}(x)$ is $-\binom{n}{m+2}x^{m+1}$ by (1.2). So $N_{n,m}(x) \notin \mathrm{PF}$ for $n \geq m+2$. But, we still have the following result on the interlacing property of the zeros of $N_{n,m}(x)$.

Theorem 1.4. *For any integers $n \geq 3$ and $1 \leq m \leq n-2$, we have*

$$N_{n,m-1}(x) \preccurlyeq N_{n,m}(x).$$

That is,

$$r_{n,m+1}^{(m+1)} < r_{n,m}^{(m)} < r_{n,m+1}^{(m)} < \cdots < r_{n,m}^{(1)} < r_{n,m+1}^{(1)} < 0 < r_{n,m}^{+} < r_{n,m+1}^{+}, \tag{1.3}$$

where $\{r_{n,m}^{(i)}\}_{i=1}^{m}$ and $r_{n,m}^{+}$ are negative zeros and positive zero of $N_{n,m}(x)$, respectively, for all $m \geq 0$ and $n \geq m+2$.

Our method is similar to that of Chen et al. [4]. However, we establish two new recurrence relations of $N_{n,m}(x)$ and hence deduce two new results, Theorems 1.3 and 1.4.

The remainder of this paper is organized as follows. In Section 2, we give two new recurrence relations of $N_{n,m}(x)$, and show symbolic proofs based on the extended Zeilberger algorithm. In Section 3, we complete the proofs of our main results, Theorem 1.3 and Theorem 1.4, by using the new recurrence relations obtained in Section 2.

2. Recurrence relations

In this section, we show two new recurrence relations of the generalized Narayana polynomials $N_{n,m}(x)$. Based on these crucial recurrence relations, we can prove the main results of this paper, the interlacing properties of the zeros of $N_{n,m}(x)$.

The main results of this section are as follows.

Theorem 2.1. *For any integer $m \geq 0$, we have*

$$\begin{aligned}(n+1)(m+2-n)N_{n+1,m}(x) = {} & [(m+2-n)(m-n)x + 2n(m-n) \\ & + m + n + 2]N_{n,m}(x) \\ & + (x-1)(m-n)nN_{n-1,m}(x),\end{aligned} \tag{2.1}$$

for $n \geq 1$, with initial values $N_{0,m}(x) = 1$ and $N_{1,m}(x) = mx + 1$.

Theorem 2.2. *For any integer $n \geq 0$, we have*

$$\begin{aligned}(m+3)(ax-2)N_{n,m+1}(x) = {} & (b_0x^2 + b_1x - 4m - 6)N_{n,m}(x) \\ & + m(x-1)(cx-2)N_{n,m-1}(x)\end{aligned} \tag{2.2}$$

for $m \geq 1$, where

$$\begin{cases} a = (n-m)(n-m-1), \\ b_0 = (n-m)(n-m-1)(n-m-2), \\ b_1 = (m-n)(2m^2 - 2mn + 7m - 3n + 7) - 2n, \\ c = (n-m-1)(n-m-2), \end{cases} \tag{2.3}$$

with initial values $N_{n,0}(x) = -\binom{n}{2}x + 1$ and $N_{n,1}(x) = -\binom{n}{3}x^2 - \frac{n(n-3)}{2}x + 1$.

The elementary proofs of Theorems 2.1 and 2.2 by hand are somewhat tedious. So we shall show alternative proofs by using a symbolic method, the extended Zeilberger algorithm [5], to derive these recurrences from the expression of $N_{n,m}(x)$. The extended Zeilberger algorithm was developed by Chen, Hou, and Mu [5] on the basis of the Zeilberger algorithm [15, 14].

Let us first have a brief overview of this algorithm with the notation and terminology of [5]. Given ℓ hypergeometric terms, namely, $f_1(k, p_1, p_2, \ldots, p_\nu), \ldots, f_\ell(k, p_1, p_2, \ldots, p_\nu)$ of k with parameters $p_1, p_2, \ldots, p_\nu$ such that both

$$\frac{f_i(k, p_1, p_2, \ldots, p_\nu)}{f_j(k, p_1, p_2, \ldots, p_\nu)} \quad \text{and} \quad \frac{f_i(k+1, p_1, p_2, \ldots, p_\nu)}{f_i(k, p_1, p_2, \ldots, p_\nu)}$$

are all rational functions of k and $p_1, p_2, \ldots, p_\nu$ for any $1 \le i, j \le \ell$. The extended Zeilberger algorithm is devised to find a hypergeometric term $g(k, p_1, p_2, \ldots, p_\nu)$ and polynomial coefficients $a_i(p_1, p_2, \ldots, p_\nu)$ for $1 \le i \le \ell$ which are independent of k such that

$$a_1 f_1(k) + a_2 f_2(k) + \cdots + a_\ell f_\ell(k) = g(k+1) - g(k), \tag{2.4}$$

where a_i stands for $a_i(p_1, p_2, \ldots, p_\nu)$, $f_i(k)$ stands for $f_i(k, p_1, p_2, \ldots, p_\nu)$, and $g(k)$ stands for $g(k, p_1, p_2, \ldots, p_\nu)$ for brevity. For $1 \le i \le \ell$, let $F_i = \sum_k f_i(k)$. Summing the telescoping relation (2.4) over k usually leads to a homogeneous relation

$$a_1 F_1 + a_2 F_2 + \cdots + a_\ell F_\ell = 0. \tag{2.5}$$

The extended Zeilberger algorithm has been implemented as the function `Ext_Zeil` in the Maple package `APCI` by Hou [9], which consists of three files "apci.help", "apci.lib", and "apci.ind". The calling sequence of this function is of the form `Ext_Zeil(`$[f_1, f_2, \ldots, f_\ell]$`, `$k$`)`. If the algorithm is applicable, it outputs $[C, Ca_2/a_1, Ca_3/a_1, \ldots, Ca_\ell/a_1]$, where C is a k-free nonzero constant.

Now we are ready to show the proofs of Theorems 2.1 and 2.2, respectively.

Proof of Theorem 2.1. To use the package `APCI`, we first import it in Maple as follows.

```
[> with(APCI);
```

[AbelZ, Ext_Zeil, Gosper, MZeil, Zeil, hyper_simp, hyperterm, poch, qExt_Zeil, qGosper, qZeil, qbino, qhyper_simp, qhyperterm, qpoch]

Observe that (2.1) is of the form (2.5) with

$$f_1 = N(n+1, m, k)x^k, \quad f_2 = N(n, m, k)x^k, \quad f_3 = N(n-1, m, k)x^k,$$

where

$$N(n, m, k) = \binom{n}{k}\binom{m}{k} - \binom{n}{k+1}\binom{m}{k-1}. \tag{2.6}$$

In order to prove (2.1), we continue the following set of f_i:

[> $f_1 := \left(\binom{n+1}{k}\binom{m}{k} - \binom{n+1}{k+1}\binom{m}{k-1}\right) x^k$:

[> $f_2 := \left(\binom{n}{k}\binom{m}{k} - \binom{n}{k+1}\binom{m}{k-1}\right) x^k$:

[> $f_3 := \left(\binom{n-1}{k}\binom{m}{k} - \binom{n-1}{k+1}\binom{m}{k-1}\right) x^k$:

Then we run the command of the main function

[> Ext_Zeil($[f_1, f_2, f_3], k$);

$$\left[-\frac{k\_free_3(n+1)(m+2-n)}{(x-1)(m-n)n},\right.$$

$$\left.\frac{(m^2x-2mnx+n^2x+2mn+2mx-2n^2-2nx+m+n+2)k\_free_3}{(x-1)(m-n)n}, k\_free_3\right].$$

The above output implies that there exists some nonzero constant C free of k such that

$$C \cdot \frac{a_2}{a_1} \cdot N_{n+1,m}(x) + C \cdot \frac{a_3}{a_1} \cdot N_{n,m}(x) + C \cdot N_{n-1,m}(x) = 0,$$

where

$$\begin{cases} a_1 = (x-1)(m-n)n, \\ a_2 = -(n+1)(m+2-n), \\ a_3 = m^2x - 2mnx + n^2x + 2mn + 2mx - 2n^2 - 2nx + m + n + 2. \end{cases}$$

This relation is equivalent to (2.1). It is clear that $N_{0,m}(x) = 1$ and $N_{1,m}(x) = mx + 1$ by (1.2). This completes the proof. □

Proof of Theorem 2.2. In a similar manner as in the proof of Theorem 2.1, one can reach (2.2) by setting

$$f_1 = N(n, m+1, k)x^k, \quad f_2 = N(n, m, k)x^k, \quad f_3 = N(n, m-1, k)x^k,$$

where $N(n, m, k)$ is as defined in (2.6). The details are omitted here. □

3. Proofs of the main results

This section is devoted to the proofs of the main results of this paper, Theorem 1.3 and Theorem 1.4, the interlacing properties of the zeros of $N_{n,m}(x)$. Our proofs also imply the real-rootedness of $N_{n,m}(x)$ for $m \geq 0$ and $n \geq 1$.

In order to complete the proofs, we need the following sufficient condition, established by Liu and Wang [12], for determining whether two polynomials have interlaced zeros.

Theorem 3.1 ([12, Theorem 2.1])**.** *Let $F(x), f(x), g(x)$ be real polynomials satisfying the following conditions*

(*i*) *There exist real polynomials $\phi(x)$ and $\psi(x)$ such that*

$$F(x) = \phi(x)f(x) + \psi(x)g(x), \tag{3.1}$$

and $\deg F(x) = \deg f(x)$ *or* $\deg F(x) = \deg f(x) + 1$.

(*ii*) $f(x), g(x)$ *are polynomials with only real zeros, and* $g(x) \preccurlyeq f(x)$.

(*iii*) *The leading coefficients of* $F(x)$ *and* $g(x)$ *have the same sign.*

Suppose that $\psi(r) \leq 0$ *for each zero* r *of* $f(x)$. *Then* $F(x)$ *has only real zeros and* $f(x) \preccurlyeq F(x)$.

Remark 3.2. *It should be mentioned that the result of Theorem 3.1 still holds if* $\phi(x)$ *and* $\psi(x)$ *are rational functions of* x, *provided* $F(x), f(x)$, *and* $g(x)$ *are polynomials in* x *and the original conditions are satisfied. This can be obtained by a similar argument of the proof of Theorem 3.1 given by Liu and Wang* [12].

Now we are ready to give a proof of Theorem 1.3.

Proof of Theorem 1.3. Note that the coefficients of x^k in $N_{n,m}(x)$, namely

$$\begin{aligned} N(n,m,k) &= \binom{n}{k}\binom{m}{k} - \binom{n}{k+1}\binom{m}{k-1} \\ &= \binom{n+1}{k+1}\binom{m+1}{k}\frac{(m-n)k+m+1}{(n+1)(m+1)}, \end{aligned} \tag{3.2}$$

are nonnegative for $0 \leq n \leq m$. Thus each $N_{n,m}(x)$ is a polynomial with nonnegative coefficients for $0 \leq n \leq m$. By (1.2), it is clear that $N_{m+1,m}(x) = N_{m,m}(x)$, the classical Narayana polynomial, which had been proved to be real-rooted [12]. So $N_{m,m}(x), N_{m+1,m}(x) \in \mathrm{PF}$ and $N_{m,m}(x) \preccurlyeq N_{m+1,m}(x)$. It remains to prove that $\{N_{n,m}(x)\}_{n=0}^{m}$ is a generalized Sturm sequence.

By (1.2) and (3.2), each polynomial $N_{n,m}(x)$ has only nonnegative coefficients. By Theorem 2.1

$$N_{0,m}(x) = 1, \quad N_{1,m}(x) = mx + 1,$$

and hence $N_{0,m}(x), N_{1,m}(x) \in \mathrm{PF}$, and

$$N_{0,m}(x) \preccurlyeq N_{1,m}(x), \quad m \geq 0.$$

For $1 \leq n \leq m-1$, assume $N_{n-1,m}(x) \preccurlyeq N_{n,m}(x)$, we aim to prove that $N_{n,m}(x) \preccurlyeq N_{n+1,m}(x)$. For this purpose, let us consider the recurrence relation (2.1). Since $n \leq m-1$, the recurrence (2.1) can be rewritten as

$$\begin{aligned} N_{n+1,m}(x) = &\frac{(m+2-n)(m-n)x + 2n(m-n) + m + n + 2}{(n+1)(m+2-n)} N_{n,m}(x) \\ &+ \frac{(x-1)(m-n)n}{(n+1)(m+2-n)} N_{n-1,m}(x), \qquad n \geq 1. \end{aligned} \tag{3.3}$$

Observe that (3.3) is in the form of (3.1) where

$$F(x) = N_{n+1,m}(x), \quad f(x) = N_{n,m}(x), \quad g(x) = N_{n-1,m}(x),$$

and

$$\phi(x) = \frac{(m+2-n)(m-n)x + 2n(m-n) + m + n + 2}{(n+1)(m+2-n)},$$

$$\psi(x) = \frac{(x-1)(m-n)n}{(n+1)(m+2-n)}.$$

Clearly, each $N_{n,m}(x)$ is a polynomial with degree n for $0 \le n \le m$. It is easy to verify that the conditions in (i),(ii), and (iii) of Theorem 3.1 are satisfied for $1 \le n \le m-1$. Moreover, $\psi(x) \le 0$ for any $x \le 0$, hence $\psi(r) \le 0$ for any zero r of $f(x)$ since $f(x)$ is a polynomial with nonnegative coefficients. Then by Theorem 3.1, we obtain that $f(x) \preccurlyeq F(x)$, namely $N_{n,m}(x) \preccurlyeq N_{n+1,m}(x)$ for $1 \le n \le m-1$.

It is clear that $N_{n,m}(x) \in \mathrm{PF}$ for $0 \le n \le m+1$ with $m \ge 0$, and hence $\{N_{n,m}(x)\}_{n=0}^{m+1}$ is a generalized Sturm sequence for any fixed $m \ge 0$. That is, $N_{n-1,m}(x) \preccurlyeq N_{n,m}(x)$ for $1 \le n \le m+1$. This completes the proof. □

We proceed to prove Theorem 1.4.

Proof of Theorem 1.4. Let $n \ge m+2$ and $m \ge 0$. Clearly, each $N_{n,m}(x)$ is a polynomial with degree $m+1$. Moreover, each $N_{n,m}(x)$ has one and only one positive zero which has been proved in [4, Theorem 3.4] by applying Descartes's Rule (see [6]).

By the Chu–Vandermonde convolution (see [7] or [8, §5.1]), we have

$$N_{n,m}(1) = \binom{n+m}{n} - \binom{n+m}{n-2} = \frac{m+2-n}{m+2}\binom{n+m+1}{n} \le 0, \quad n \ge m+2.$$

Evidently, $N_{n,m}(1) = 0$ only for $n = m+2$. Denote by $r_{n,m}^{+}$ the positive zero of $N_{n,m}(x)$. It follows that

$$0 < r_{n,m}^{+} \le 1, \qquad \text{for } m \ge 0,\ n \ge m+2, \tag{3.4}$$

where the equality holds only for $n = m+2$.

We continue the proof of Theorem 1.4. Let $n \ge m+2$. By Theorem 2.2, we have

$$N_{n,0}(x) = -\binom{n}{2}x + 1, \qquad N_{n,1}(x) = -\binom{n}{3}x^2 - \frac{n(n-3)}{2}x + 1.$$

Clearly, $N_{n,0}(x)$ has only one positive zero $r_{n,0}^{+} = 2/n(n-1)$, and $N_{n,1}(x)$ has two real zeros, say, $r_{n,1}^{(1)}$ and $r_{n,1}^{+}$, where $r_{n,1}^{(1)} < 0 < r_{n,1}^{+}$. In view of

$$N_{n,1}(r_{n,0}^{+}) = \frac{4(n+1)}{3n(n-1)} > 0, \quad n \ge 3,$$

it follows that $r_{n,1}^{(1)} < r_{n,0}^{+} < r_{n,1}^{+}$. That is,

$$N_{n,0}(x) \preccurlyeq N_{n,1}(x), \quad n \ge 3.$$

Moreover, $N_{n,1}(x)$ has one positive zero $r_{n,1}^{+}$, and one negative zero $r_{n,1}^{(1)}$, for $n \ge 3$.

For $n \geq 4$ and $1 \leq m \leq n-3$, suppose $N_{n,m-1}(x) \preccurlyeq N_{n,m}(x)$. It suffices to show that $N_{n,m}(x) \preccurlyeq N_{n,m+1}(x)$. To this end, we will use Theorem 2.2 and Theorem 3.1 together with Remark 3.2. By Theorem 2.2, when $x \neq 2/(n-m)(n-m-1)$, the recurrence relation (2.2) can be rewritten as

(3.5)

$$N_{n,m+1}(x) = \frac{b_0x^2 + b_1x - 4m - 6}{(m+3)(ax-2)} N_{n,m}(x) + \frac{m(x-1)(cx-2)}{(m+3)(ax-2)} N_{n,m-1}(x).$$

Observe that the recurrence (3.5) is of the form of (3.1) where

$$F(x) = N_{n,m+1}(x), \quad f(x) = N_{n,m}(x), \quad g(x) = N_{n,m-1}(x),$$

and

$$\phi(x) = \frac{b_0x^2 + b_1x - 4m - 6}{(m+3)(ax-2)}, \qquad \psi(x) = \frac{m(x-1)(cx-2)}{(m+3)(ax-2)},$$

where a, b_0, b_1, c are given by (2.3). It is easy to verify that the conditions in (i),(ii), and (iii) of Theorem 3.1 and Remark 3.2 are satisfied. So it remains to prove that $\psi(r) \leq 0$ for each zero r of $f(x)$, that is to prove

$$\psi(r) = \frac{m(r-1)((n-m-1)(n-m-2)r-2)}{(m+3)((n-m)(n-m-1)r-2)} \leq 0 \tag{3.6}$$

for $1 \leq m \leq n-3$ and any r whenever $N_{n,m}(r) = 0$.

Since $N_{n,m}(x)$ has only real zeros, it follows from (3.4) that

$$r_{n,m}^{(m)} < r_{n,m}^{(m-1)} < \cdots < r_{n,m}^{(2)} < r_{n,m}^{(1)} < 0 < r_{n,m}^{+} < 1$$

where $\{r_{n,m}^{(k)}\}_{k=1}^{m}$ and $r_{n,m}^{+}$ stand for the negative zeros and the positive zero of $N_{n,m}(x)$, respectively.

By a result [16, Theorem 3.1] given by Zhao, we have for $m \geq 1$ and $n \geq m+3$,

$$r_{n,m}^{+} < \frac{2}{(n-m)(n-m-1)}.$$

So for $m \geq 1$ and $n \geq m+3$, we get

$$(n-m)(n-m-1)r - 2 < 0, \qquad (n-m-1)(n-m-2)r - 2 < 0$$

for all zeros r of $N_{n,m}(x)$. This means that the inequality in (3.6) holds strictly for each zero r of $N_{n,m}(x)$. Hence by Theorem 3.1 and Remark 3.2, the polynomial $F(x)$, that is $N_{n,m+1}(x)$, has only real zeros and $N_{n,m}(x) \preccurlyeq N_{n,m+1}(x)$ for $1 \leq m \leq n-3$.

By the fact that each polynomial $N_{n,m}(x)$ has one and only one positive zero, it follows that $N_{n,m}(x)$ has one positive zero and m negative zeros for any $n \geq m+2$ and (1.3) holds true. This completes the proof. □

The following result is an immediate consequence of Theorem 1.4.

Corollary 3.3. *The negative zeros of $N_{n,m}(x)$ interlace the negative zeros of $N_{n,m+1}(x)$ for $n \geq m+3$ and $m \geq 0$.*

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