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ON SOME NEW FAMILIES OF k-MERSENNE AND GENERALIZED k-GAUSSIAN MERSENNE NUMBERS AND THEIR POLYNOMIALS

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Abstract. In this paper, we define the generalized k-Mersenne numbers, give a formula for generalized Mersenne polynomials and study their properties. Moreover, we define Gaussian Mersenne numbers and obtain some identities such as a Binet type formula, Cassini type identity, D'Ocagne type identity, and generating functions. The generalized Gaussian Mersenne numbers are described and their relation with the classical Mersenne numbers are explained. We also introduce a generalization of Gaussian Mersenne polynomials and establish some properties of these polynomials.

1. Introduction

Mersenne sequence $\{M_n\}_{n\geq 0}$ [3] is given by the recurrence relation,

$$M_{n+2} = 3M_{n+1} - 2M_n, \qquad n \geq 0, \ with \ M_0 = 0, \ M_1 = 1, \tag{1.1}$$

and the terms of this sequence are known as Mersenne numbers.

The characteristic equation corresponding to the above recurrence relation is

$$\lambda^2 - 3\lambda + 2 = 0. \tag{1.2}$$

The Binet type formula for the Mersenne numbers is given by

$$M_n = \lambda_1^n - \lambda_2^n, \tag{1.3}$$

where $\lambda_1 = 2$ and $\lambda_2 = 1$ are the roots of the characteristic eqn. (1.2).

Recall that, in number theory Mersenne numbers $\{M_n\}$ are sequences of integers of the form $2^n - 1$ for non-negative integer n, which can also be obtained from the Binet type formula. In this paper, we study and generalize one of the recursive sequences of integers and we give the corresponding polynomials, some well-known identities for this type of sequence. Some well-known recursive sequences are Fibonacci, Lucas, Horadam, Pell,

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Perrin, Fibonacci-Lucas, Jacobsthal, etc. that are studied over the years and still are a great area of interest for generalization and their applications in other disciplines like cryptography, coding theory, matrix theory, etc. Horadam [7, 8] introduced the concept of Gaussian Fibonacci numbers and complex Fibonacci numbers. Further, Jordan [9] considered Gaussian Fibonacci and Lucas numbers and extended classical relations. Moreover, studies on the different Gaussian sequences like Gaussian Fibonacci, Gaussian Lucas, Gaussian Pell, Gaussian Pell-Lucas, and their polynomials can be found in the papers [10, 2, 6, 15]. Also, some work in the direction of generalization of the recursive sequences like Lucas, Pell, Horadam, etc. has been done in [11, 12, 16]. Prasad et. al. [13] discussed the generalization of higher-order Fibonacci sequences and showed their application in cryptography as a key matrix. Constructions of identities related to Mersenne numbers and generalized Mersenne numbers and study of their properties have been studied in papers [1, 3, 5, 4, 14] using generating functions and matrix methods.

Theorem 1.1 (Cassini Type Identity [14]). *For $n \geq 1$,*

$$M_n^2 - M_{n+1}M_{n-1} = 2^{n-1}. \tag{1.4}$$

Theorem 1.2 (Generating function [5]). *Generating function for Mersenne numbers M_n is given by,*

$$M(x) = \sum_{i=0}^{\infty} M_i x^i = \frac{x}{(1-3x+2x^2)}. \tag{1.5}$$

This paper is organized as follows. In Section 2, we define generalized k-Mersenne numbers and their polynomials and establish relations between classical Mersenne numbers and generalized k-Mersenne numbers. In Section 3, we introduce Gaussian Mersenne numbers and obtain some identities such as a Binet type Formula, Cassini type identity, D'Ocagne type identity and generating functions related to these numbers. At last, we define k-generalized Gaussian Mersenne numbers and discussed their properties, and obtain some identities involving Mersenne numbers.

2. Main Work

2.1. Generalized k-Mersenne numbers.

Definition 2.1. *Let $k \in \mathbb{N}, n \in \mathbb{N} \cup \{0\}$ and let $s, r \in \mathbb{N} \cup \{0\}$ be the unique natural numbers such that such that $n = sk + r$, where $0 \leq r < k$. Then the generalized k-Mersenne numbers $M_n^{(k)}$ are defined as*

$$M_n^{(k)} = (\lambda_1^s - \lambda_2^s)^{k-r}(\lambda_1^{s+1} - \lambda_2^{s+1})^r, \ n = sk + r \tag{2.1}$$

where λ_1 and λ_2 are the roots of equation (1.2).

From eqn. (1.3) and Definition 2.1, the generalized k-Mersenne numbers and Mersenne numbers are related as

$$M_n^{(k)} = M_s^{k-r} M_{s+1}^r, \; n = sk + r. \tag{2.2}$$

If $k = 1$ then $r = 0$ and hence $n = s$. So, from eqn. (2.2) we have $M_n^{(1)} = M_n$.

From the above derivations, we have noted the following identities showing relations between generalized k-Mersenne numbers and Mersenne numbers for $k = 2, 3$.

$$\begin{aligned}
M_{2s}^{(2)} &= M_s^2. \\
M_{2s+1}^{(2)} &= M_s M_{s+1}. \\
M_{2s+1}^{(2)} &= 3M_{2s}^{(2)} - 2M_{2s-1}^{(2)}. \\
M_{3s}^{(3)} &= M_s^3. \\
M_{3s+1}^{(3)} &= M_s^2 M_{s+1}. \\
M_{3s+1}^{(3)} &= 3M_{3s}^{(3)} - 2M_{3s-1}^{(3)}. \\
M_{3s+2}^{(3)} &= M_s M_{s+1}^2.
\end{aligned}$$

Some generalized k-Mersenne numbers $M_n^{(k)}$ are shown in the following table.

$M_n^{(k)}$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
$M_0^{(k)}$	0	0	0	0	0
$M_1^{(k)}$	1	0	0	0	0
$M_2^{(k)}$	3	1	0	0	0
$M_3^{(k)}$	7	3	1	0	0
$M_4^{(k)}$	15	9	3	1	0
$M_5^{(k)}$	31	21	9	3	1

TABLE 1. List of some generalized k-Mersenne numbers ($M_n^{(k)}$).

Proposition 2.2. *Let $k, s \in \mathbb{N}$ then $M_{sk}^{(k)} = M_s^k$.*

Proof. For $n = sk$, $r = 0$, so from eqn. (2.2) we have

$$M_{sk}^{(k)} = M_s^{k-0} M_{s+1}^0 = M_s^k.$$

□

Theorem 2.3. *For $n, s \in \mathbb{N}$, $M_{sn+1}^{(s)} = 3M_{sn}^{(s)} - 2M_{sn-1}^{(s)}$.*

Proof. From eqn. (1.1) and eqn. (2.2), we have

$$\begin{aligned} 3M^{(s)}_{sn} - 2M^{(s)}_{sn-1} &= 3M^s_n - 2M_{n-1}M^{s-1}_n \\ &= M^{s-1}_n(3M_n - 2M_{n-1}) \\ &= M^{s-1}_n M_{n+1} \\ &= M^{(s)}_{sn+1}. \end{aligned}$$

□

Theorem 2.4. *For $k, s \in \mathbb{N}$ we have, $M^k_{s+1} - M^k_s = M^{(k)}_{sk+k} - M^{(k)}_{sk}$.*

Proof. From eqn. (2.2), we have

$$\begin{aligned} M^{(k)}_{sk+k} - M^{(k)}_{sk} &= [M^{k-k}_s M^k_{s+1}] - [M^{k-0}_s M^0_{s+1}] \\ &= M^k_{s+1} - M^k_s. \end{aligned}$$

□

Theorem 2.5. *For $n, m \geq 0$ such that $n + m > 1$,*

$$M^{(2)}_{2(n+m-1)} - M_{n+m}M_{n+m-2} = 2^{n+m-2}.$$

Proof. By eqn. (1.4) and Proposition 2.2, we get

$$M^{(2)}_{2(n+m-1)} - M_{n+m}M_{n+m-2} = M^2_{(n+m-1)} - M_{n+m}M_{n+m-2} = 2^{n+m-2}.$$

□

Theorem 2.6. *Let $n, k \geq 2$ then Cassini type identity for $M^{(k)}_n$ is,*

$$M^{(k)}_{nk+a}M^{(k)}_{nk+a-2} - (M^{(k)}_{nk+a-1})^2 = \begin{cases} -2^{n-1}M^{2k-2}_n, & a = 1 \\ 0, & a \neq 1. \end{cases}$$

Proof. If a$\neq$ 1, then from eqn. (2.2)

$$\begin{aligned} & M^{(k)}_{nk+a}M^{(k)}_{nk+a-2} - (M^{(k)}_{nk+a-1})^2 \\ =&(M^{k-a}_n M^a_{n+1})(M^{k-a+2}_n M^{a-2}_{n+1}) - (M^{k-a+1}_n M^{a-1}_{n+1})^2 \\ =&M^{2k-2a+2}_n[M^{2a-2}_{n+1} - (M_{n+1})^{2a-2}]. \\ =&0, \end{aligned}$$

and if $a = 1$,

$$\begin{aligned} M^{(k)}_{nk+1}M^{(k)}_{nk-1} - (M^{(k)}_{nk})^2 &= (M^{k-1}_n M_{n+1})(M_{n-1}M^{k-1}_n) - (M^k_n)^2 \\ &= M^{2k-2}_n[M_{n+1}M_{n-1} - M^2_n] \\ &= -2^{n-1}M^{2k-2}_n. \qquad \text{(using eqn. (1.4))} \end{aligned}$$

□

2.2. **Mersenne Polynomials and Generalized k-Mersenne polynomials.**

Definition 2.7. *The* Mersenne Polynomials $\{M_n(x)\}$ *are defined by the recurrence relation,*

$$M_{n+2}(x) = 3xM_{n+1}(x) - 2M_n(x) \qquad n \geq 0, \tag{2.3}$$

with $M_0(x) = 0,\ M_1(x) = 1.$

The characteristic equation corresponding to the recurrence relation (2.3) is

$$\lambda^2 - 3x\lambda + 2 = 0. \tag{2.4}$$

Theorem 2.8. *For $n \in \mathbb{N}$, we can write the Binet type formula for the Mersenne polynomials as*

$$M_n(x) = \frac{\lambda_1^n(x) - \lambda_2^n(x)}{\lambda_1(x) - \lambda_2(x)}, \tag{2.5}$$

where $\lambda_1(x) = \dfrac{3x + \sqrt{9x^2 - 8}}{2}$ *and* $\lambda_2(x) = \dfrac{3x - \sqrt{9x^2 - 8}}{2}$ *are the roots of the characteristic equation (2.4).*

Proof. By the theory of difference equations, the n^{th} term of Mersenne polynomials can be written as,

$$M_n(x) = a\lambda_1^n(x) + b\lambda_2^n(x). \tag{2.6}$$

From eqn. (2.3) we have, $M_0(x) = a + b$ and $M_1(x) = a\lambda_1(x) + b\lambda_2(x)$.

On solving $M_0(x)$ and $M_1(x)$ we get,

$$a = \frac{1}{\lambda_1(x) - \lambda_2(x)} \text{ and } b = \frac{-1}{\lambda_1(x) - \lambda_2(x)}.$$

Now, using these values of a and b in eqn. (2.6), we get

$$M_n(x) = \frac{\lambda_1^n(x) - \lambda_2^n(x)}{\lambda_1(x) - \lambda_2(x)}.$$

□

Theorem 2.9. *For $n \geq 1$,*

$$M_n^2(x) - M_{n+1}(x)M_{n-1}(x) = 2^{n-1}. \tag{2.7}$$

Proof. We proceed by using mathematical induction on n.

For $n = 1$,

$$M_1^2(x) - M_2(x)M_0(x) = 1 - 0(3x) = 2^0.$$

Thus the result is true for $n = 1$.

Assume that result is true for $n = k$, *i.e.*

$$M_k^2(x) - M_{k+1}(x)M_{k-1}(x) = 2^{k-1}. \tag{2.8}$$

Now for $n = k + 1$, using eqn. (2.3) and eqn. (2.8), we have

$$
\begin{aligned}
&M_{k+1}^2(x) - M_{k+2}(x)M_k(x)\\
=&M_k^2(x) - \left[(3M_{k+1}(x) - 2M_k(x))\left(\frac{1}{3}M_{k+1}(x) + \frac{2}{3}M_{k-1}(x)\right)\right]\\
=&-2M_{k+1}(x)M_{k-1}(x) + \frac{2}{3}M_{k+1}(x)M_k(x) + \frac{4}{3}M_{k-1}(x)M_k(x)\\
=&2M_k^2(x) + (2)2^{k-1} - \frac{2}{3}M_{k+1}(x)M_k(x) + \frac{4}{3}M_{k-1}(x)M_k(x)\\
=&2^k.
\end{aligned}
$$

□

Definition 2.10 (Generalized k-Mersenne polynomials $\{M_n^{(k)}(x)\}$)**.** *Let $k \in \mathbb{N}, n \in \mathbb{N} \cup \{0\}$ and let $s, r \in \mathbb{N} \cup \{0\}$ be the unique natural numbers such that $n = sk + r$, and $0 \leq r < k$. Then the generalized k-Mersenne polynomials $M_n^{(k)}(x)$ are defined by*

$$
M_n^{(k)}(x) = \left(\frac{\lambda_1^s(x) - \lambda_2^s(x)}{\lambda_1(x) - \lambda_2(x)}\right)^{k-r}\left(\frac{\lambda_1^{s+1}(x) - \lambda_2^{s+1}(x)}{\lambda_1(x) - \lambda_2(x)}\right)^r, \tag{2.9}
$$

where $\lambda_1(x)$ and $\lambda_2(x)$ are the roots of the characteristic equation (2.4).

From eqn. (2.5) and Definition 2.10, we have the relation between generalized k-Mersenne polynomials and Mersenne polynomials as

$$
M_n^{(k)}(x) = M_s^{k-r}(x)M_{s+1}^r(x),\ n = sk + r. \tag{2.10}
$$

If $k = 1$ then $r = 0$ and $n = s$. From eqn. (2.10), we have $M_n^{(1)}(x) = M_n(x)$.

Some values of generalized k-Mersenne polynomials $M_n^{(k)}(x)$ are shown in following table.

$M_n^{(k)}(x)$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
$M_0^{(k)}(x)$	0	0	0	0	0
$M_1^{(k)}(x)$	1	0	0	0	0
$M_2^{(k)}(x)$	$3x$	1	0	0	0
$M_3^{(k)}(x)$	$9x^2-2$	$3x$	1	0	0
$M_4^{(k)}(x)$	$27x^3-12x$	$9x^2$	$3x$	1	0
$M_5^{(k)}(x)$	$81x^4-54x^2+4$	$27x^3-6x$	$9x^2$	$3x$	1

TABLE 2. Some generalized k-Mersenne polynomials $M_n^{(k)}(x)$.

From Table 2 and eqn. (2.10), we have the following relations between the generalized k-Mersenne polynomials and Mersenne polynomials for $k = 2, 3$,

$$\begin{aligned}
M_{2s}^{(2)}(x) &= M_s^2(x). \\
M_{2s+1}^{(2)}(x) &= M_s(x)M_{s+1}(x). \\
M_{2s+1}^{(2)}(x) &= 3M_{2s}^{(2)}(x) - 2M_{2s-1}^{(2)}(x). \\
M_{3s}^{(3)}(x) &= M_s^3(x). \\
M_{3s+1}^{(3)}(x) &= M_s^2(x)M_{s+1}(x). \\
M_{3s+1}^{(3)}(x) &= 3M_{3s}^{(3)}(x) - 2M_{3s-1}^{(3)}(x). \\
M_{3s+2}^{(3)}(x) &= M_s(x)M_{s+1}^2(x).
\end{aligned}$$

Proposition 2.11. *For $k, s \in \mathbb{N}$, we have $M_{ks}^{(k)}(x) = M_s^k(x)$.*

Proof. The proof is similar to Proposition 2.2. □

Theorem 2.12. *For $n, s \in \mathbb{N}$, we have*

$$M_{sn+1}^{(s)}(x) = 3xM_{sn}^{(s)}(x) - 2M_{sn-1}^{(s)}(x).$$

Proof. From eqn. (2.10) and eqn. (2.3), we have

$$\begin{aligned}
3xM_{sn}^{(s)}(x) - 2M_{sn-1}^{(s)}(x) &= 3xM_n^s(x) - 2M_{n-1}(x)M_n^{s-1}(x) \\
&= M_n^{s-1}(x)(3xM_n(x) - 2M_{n-1}(x)) \\
&= M_n^{s-1}(x)M_{n+1}(x) \\
&= M_{sn+1}^{(s)}(x).
\end{aligned}$$

□

Theorem 2.13. *For $k, s \in \mathbb{N}$ we have,*

$$M_{s+1}^k(x) - M_s^k(x) = M_{sk+k}^{(k)}(x) - M_{sk}^{(k)}(x).$$

Proof. From eqn. (2.10), we have

$$\begin{aligned}
M_{sk+k}^{(k)}(x) - M_{sk}^{(k)}(x) &= [M_s^{k-k}(x)M_{s+1}^k(x)] - [M_s^{k-0}(x)M_{s+1}^0(x)] \\
&= M_{s+1}^k(x) - M_s^k(x).
\end{aligned}$$

□

Theorem 2.14. *Let $n, k \geq 2$, we can write the Cassini type identity for $M_n^{(k)}(x)$ as*

$$M_{nk+a}^{(k)}(x)M_{nk+a-2}^{(k)}(x) - (M_{nk+a-1}^{(k)})^2(x) = \begin{cases} -2^{n-1}M_n^{2k-2}(x), & a = 1 \\ 0, & a \neq 1. \end{cases}$$

Proof. Let $a \neq 1$, so from eqn. (2.2) we have

$$\begin{aligned} & M^{(k)}_{nk+a}(x)M^{(k)}_{nk+a-2}(x) - \left(M^{(k)}_{nk+a-1}(x)\right)^2 \\ =& \left(M_n^{k-a}(x)M^{a}_{n+1}(x)\right)\left(M_n^{k-a+2}(x)M^{a-2}_{n+1}(x)\right) - \left(M_n^{k-a+1}(x)M^{a-1}_{n+1}(x)\right)^2 \\ =& M_n^{2k-2a+2}(x)\left[M^{2a-2}_{n+1}(x) - M^{2a-2}_{n+1}(x)\right] \\ =& 0 \end{aligned}$$

and if $a = 1$,

$$\begin{aligned} & M^{(k)}_{nk+1}(x)M^{(k)}_{nk-1}(x) - (M^{(k)}_{nk})^2(x) \\ =& \left(M_n^{k-1}(x)M_{n+1}(x)\right)\left(M_{n-1}M_n^{k-1}(x)\right) - M_n^{2k}(x) \\ =& M_n^{2k-2}(x)\left[M_{n+1}(x)M_{n-1}(x) - M_n^2(x)\right] \\ =& -2^{n-1}M_n^{2k-2}(x). \qquad \text{(using Theorem 2.9)} \end{aligned}$$

□

3. Generalized Gaussian Mersenne numbers and their polynomials

3.1. Gaussian Mersenne Numbers.

Definition 3.1. *The* Gaussian Mersenne sequence $\{GM_k\}_{k\geq 0}$ *is defined by the recurrence relation,*

$$GM_{k+2} = 3GM_{k+1} - 2GM_k, \qquad k \geq 0 \tag{3.1}$$

with $GM_0 = -i/2,\ GM_1 = 1$.

The first few Gaussian Mersenne numbers are $-i/2, 1, 3+i, 7+3i, 15+7i, ...$ The relation between Gaussian Mersenne numbers and classical Mersenne numbers is

$$GM_{k+2} = M_{k+2} + iM_{k+1},$$

where GM_k is the k^{th}-Gaussian Mersenne number.

Theorem 3.2. *For $n \in \mathbb{N}$, the Binet type formula for the Gaussian Mersenne numbers is*

$$GM_n = (\lambda_1^n - \lambda_2^n) + i(\lambda_1^{n-1} - \lambda_2^{n-1}), \tag{3.2}$$

where λ_1 and λ_2 are the roots of the characteristic equation (1.2).

Proof. The n^{th} term of Gaussian Mersenne numbers for the difference equation (3.1) is,

$$GM_n = a\lambda_1^n + b\lambda_2^n. \tag{3.3}$$

To eliminate constants a and b, we use initial conditions given in eqn. (3.1). Since we have $GM_0 = a + b$ and $GM_1 = a\lambda_1 + b\lambda_2$, we find $a = 1 + i/2$ and $b = -1 - i$.

From eqn. (3.3) we get

$$GM_n = (\lambda_1^n - \lambda_2^n) + i(\lambda_1^{n-1} - \lambda_2^{n-1}).$$

Furthermore, by using values of λ_1 and λ_2 in eqn. (3.2), we get another form of the Binet type formula which is

$$GM_n = (2^n - 1) + i(2^{n-1} - 1). \tag{3.4}$$

□

Theorem 3.3 (Catalan Type Identity)**.** *For $n, m \geq 1$, we have*

$$GM_{n+m}GM_{n-m} - GM_n^2 = \left[(2^n - 2^{n+m-1}) + (2^{n-m-1} - 2^{n-m})\right] + i3(2^n - 2^{n+m-1} - 2^{n-m-1}). \tag{3.5}$$

Proof. Using the Binet type formula(3.4), we have

$$\begin{aligned} &GM_{n+m}GM_{n-m} - GM_n^2 \\ &= \left[(2^{n+m} - 1) + i(2^{n+m-1} - 1)\right]\left[(2^{n-m} - 1) + i(2^{n-m-1} - 1)\right] \\ &\quad - \left[(2^n - 1) + i(2^{n-1} - 1)\right]^2 \\ &= \left[(2^n - 2^{n+m-1}) + (2^{n-m-1} - 2^{n-m})\right] \\ &\quad + i3(2^n - 2^{n+m-1} - 2^{n-m-1}). \end{aligned}$$

□

Note: If $m = 1$ in the Catalan type identity (3.5), we get the Cassini type identity for the Gaussian Mersenne numbers and hence the following result.

Theorem 3.4 (Cassini Type Identity)**.** *For $n \geq 1$,*

$$GM_{n+1}GM_{n-1} - GM_n^2 = (2^{n-2} - 2^{n-1}) - i3(2^{n-2}). \tag{3.6}$$

Theorem 3.5 (D'Ocagne Type Identity)**.** *For $n, m \geq 1$,*

$$GM_{m+1}GM_n - GM_mGM_{n+1} = (2^{n-1} - 2^{m-1}) + i3(2^{n-1} - 2^{m-1}). \tag{3.7}$$

Proof. From eqn. (3.4), we have

$$\begin{aligned} &GM_{m+1}GM_n - GM_mGM_{n+1} \\ &= \left[(2^{m+1} - 1) + i(2^m - 1)(2^n - 1) + i(2^{n-1} - 1)\right] \\ &\quad - \left[\ (2^m - 1) + i(2^{m-1} - 1)(2^{n+1} - 1) + i(2^n - 1)\right] \\ &=(2^{n-1} - 2^{m-1}) + i3(2^{n-1} - 2^{m-1}). \end{aligned}$$

□

Theorem 3.6. *The generating function for the Gaussian Mersenne numbers is*

$$GM(z) = \frac{z + i\left(\frac{3}{2}z - \frac{1}{2}\right)}{(1 - 3z + 2z^2)}.$$

Proof. The Generating function for the sequence $\{GM_n\}_{n\in\mathbb{N}}$ is given by

$$GM(z) = \sum_{j=0}^{\infty} GM_j z^j.$$

i.e.

$$GM(z) = GM_0 + GM_1 z^1 + GM_2 z^2 + \cdots + GM_n z^n + \cdots$$

Now,

(3.8) $GM(z) - 3zGM(z) + 2z^2 GM(z) = GM_0 + z(GM_1 - 3GM_0)$

Using the initial values of (3.1) in eqn. (3.8), we have

$$GM(z) = \frac{z + i\left(\frac{3}{2}z - \frac{1}{2}\right)}{(1 - 3z + 2z^2)}.$$

□

3.2. **Generalized k-Gaussian Mersenne numbers.**

Definition 3.7. *Let $k \in \mathbb{N}, n \in \mathbb{N}\cup\{0\}$ and $s, r \in \mathbb{N}\cup\{0\}$ the unique natural numbers such that $n = sk + r$, and $0 \leq r < k$. The generalized k-Gaussian Mersenne numbers $\{GM_n^{(k)}\}$ are defined by*

(3.9)

$$GM_n^{(k)} = \left(\lambda_1^s - \lambda_2^s + i(\lambda_1^{s-1} - \lambda_2^{s-1})\right)^{k-r}\left(\lambda_1^{s+1} - \lambda_2^{s+1} + i(\lambda_1^s - \lambda_2^s)\right)^r,$$

where $n = sk + r$, and λ_1 and λ_2 are the roots of the characteristic equation (1.2).

The relation between generalized k-Gaussian Mersenne numbers and Gaussian Mersenne numbers are (See the eqn. (3.2) and Definition 3.7)

$$GM_n^{(k)} = GM_s^{k-r} GM_{s+1}^r, \; n = sk + r. \tag{3.10}$$

If $k = 1$ then $r = 0$ and thus $m = n$. From eqn. (3.10) we have $GM_n^{(1)} = GM_n$. In particular for $k = 2, 3$, generalized k-Gaussian Mersenne and Gaussian Mersenne numbers are related as follows,

$$GM_{2s}^{(2)} = GM_s^2.$$
$$GM_{2s+1}^{(2)} = GM_s GM_{s+1}.$$
$$GM_{2s+1}^{(2)} = 3GM_{2s}^{(2)} - 2GM_{2s-1}^{(2)}.$$
$$GM_{3s}^{(3)} = GM_s^3.$$
$$GM_{3s+1}^{(3)} = GM_s^2 GM_{s+1}.$$
$$GM_{3s+1}^{(3)} = 3GM_{3s}^{(3)} - 2GM_{3s-1}^{(3)}.$$
$$GM_{3s+2}^{(3)} = GM_s GM_{s+1}^2.$$

Some generalized k-Gaussian Mersenne numbers are listed in the following table.

$GM_n^{(k)}$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
$GM_0^{(k)}$	$-i/2$	$-1/4$	$i/8$	$1/16$	$-i/32$
$GM_1^{(k)}$	1	$-i/2$	$-1/4$	$i/8$	$1/16$
$GM_2^{(k)}$	$3+i$	1	$-i/2$	$-1/4$	$i/8$
$GM_3^{(k)}$	$7+3i$	$3+i$	1	$-i/2$	$-i/4$
$GM_4^{(k)}$	$15+7i$	$8+6i$	$3+i$	1	$-i/2$
$GM_5^{(k)}$	$31+15i$	$18+16i$	$8+6i$	$3+i$	1

TABLE 3. List of some generalized k-Gaussian Mersenne numbers $GM_n^{(k)}$.

Proposition 3.8. *Let $k, s \in \mathbb{N}$, then we have $GM_{sk}^{(k)} = GM_s^k$.*

Theorem 3.9. *For $n, s \in \mathbb{N}$, we have $GM_{sn+1}^{(s)} = 3GM_{sn}^{(s)} - 2GM_{sn-1}^{(s)}$.*

Proof. From eqn. (3.10), we get

$$\begin{aligned} 3GM_{sn}^{(s)} - 2GM_{sn-1}^{(s)} &= 3GM_n^s - 2GM_{n-1}GM_n^{s-1} \\ &= GM_n^{s-1}GM_{n+1} \\ &= GM_{sn+1}^{(s)}. \end{aligned}$$

□

Theorem 3.10. *For $k, s \in \mathbb{N}$, we have $GM_{s+1}^k - GM_s^k = GM_{sk+k}^{(k)} - GM_{sk}^{(k)}$.*

Proof. From eqn. (3.10), we obtain

$$\begin{aligned} GM_{sk+k}^{(k)} - GM_{sk}^{(k)} &= [GM_s^{k-k}GM_{s+1}^k] - [GM_s^{k-0}GM_{s+1}^0] \\ &= GM_{s+1}^k - GM_s^k. \end{aligned}$$

□

Theorem 3.11. *Let $n, m \geq 0$ such that $n+m > 1$, then we have*

$$GM_{2(n+m-1)}^{(2)} - GM_{n+m}GM_{n+m-2} = (2^{n+m-1} - 2^{n+m-2}) + i3(2^{n+m-2}).$$

Proof. By eqn. (3.6) and Proposition 3.8, we get

$$\begin{aligned} GM_{2(n+m-1)}^{(2)} - GM_{n+m}GM_{n+m-2} &= GM_{(n+m-1)}^2 - GM_{n+m}M_{n+m-2} \\ &= (2^{n+m-1} - 2^{n+m-2}) + i3(2^{n+m-2}). \end{aligned}$$

□

Theorem 3.12. *Let $n, k \geq 2$, then the Cassini type identity for $GM_n^{(k)}$ is,*

$$(GM_{nk+a-1}^{(k)})^2 - GM_{nk+a}^{(k)} GM_{nk+a-2}^{(k)}$$
$$= \begin{cases} GM_n^{2k-2}\left[(2^{n-1} - 2^{n-2}) + i3(2^{n-2})\right], & a = 1 \\ 0, & a \neq 1. \end{cases}$$

Proof. If $a = 1$, then by eqn. (3.10)

$$\begin{aligned} (3.11) \qquad & GM_{nk+1}^{(k)} GM_{nk-1}^{(k)} - (GM_{nk}^{(k)})^2 \\ =& (GM_n^{k-1} GM_{n+1})(GM_{n-1} GM_n^{k-1}) - (GM_n^k)^2 \\ =& GM_n^{2k-2}[GM_{n+1}GM_{n-1} - GM_n^2] \\ =& GM_n^{2k-2}\left[(2^{n-1} - 2^{n-2}) + i3(2^{n-2})\right] \qquad \text{(using eqn. (3.6))} \end{aligned}$$

and if $a \neq 1$, then by eqn. (3.10)

$$\begin{aligned} & GM_{nk+a}^{(k)} GM_{nk+a-2}^{(k)} - (GM_{nk+a-1}^{(k)})^2 \\ =& (GM_n^{k-a} GM_{n+1}^{a})(GM_n^{k-a+2} GM_{n+1}^{a-2}) - (GM_n^{k-a+1} GM_{n+1}^{a-1})^2 \\ =& GM_n^{2k-2a+2}[GM_{n+1}^{2a-2} - GM_{n+1}^{2a-2}] \\ =& 0. \end{aligned}$$

□

3.3. **Gaussian Mersenne Polynomials.**

Definition 3.13. *The* Gaussian Mersenne polynomials $\{GM_k(x)\}$ *are defined by the recurrence relation,*

$$GM_{k+2}(x) = 3xGM_{k+1}(x) - 2GM_k(x), \qquad k \geq 0 \tag{3.12}$$

with $GM_0 = -i/2,\ GM_1 = 1$.

The first few Gaussian Mersenne polynomials are $-i/2,\ 1,\ 9x^2 - 2 + i3x,\ 27x^3 - 12x + i(9x^2 - 2)$. The Gaussian Mersenne polynomials and Mersenne polynomials are related as

$$GM_{k+2} = M_{k+2}(x) + iM_{k+1}(x),$$

where $M_k(x)$ is the k^{th}-Mersenne polynomial.

Theorem 3.14. *For every $n \in \mathbb{N}$, the Binet type formula for the Gaussian Mersenne polynomials is*

$$GM_n(x) = \left(\frac{\lambda_1^n(x) - \lambda_2^n(x)}{\lambda_1(x) - \lambda_2(x)}\right) + i\left(\frac{\lambda_1^{n-1}(x) - \lambda_2^{n-1}(x)}{\lambda_1(x) - \lambda_2(x)}\right), \tag{3.13}$$

where $\lambda_1(x)$ and $\lambda_2(x)$ are the roots of the characteristic equation. (2.4).

Proof. The general term of the Gaussian Mersenne polynomials can be obtained by,

$$GM_n(x) = a\lambda_1^n(x) + b\lambda_2^n(x). \tag{3.14}$$

Using initial conditions given in eqn. (3.12), we eliminate constants a and b. Since we have $GM_0 = a+b$ and $GM_1 = a\lambda_1(x)+b\lambda_2(x)$, we find $a = 1+i/2$ and $b = -1-i$. Thus, from eqn. (3.14), we have

$$GM_n(x) = \left(\frac{\lambda_1^n(x) - \lambda_2^n(x)}{\lambda_1(x) - \lambda_2(x)}\right) + i\left(\frac{\lambda_1^{n-1}(x) - \lambda_2^{n-1}(x)}{\lambda_1(x) - \lambda_2(x)}\right).$$

□

Theorem 3.15 (Cassini Type Identity)**.** *For $n \geq 1$, we have*

$$GM_{n+1}(x)GM_{n-1}(x) - GM_n^2(x) = (2^{n-2} - 2^{n-1}) - i3x(2^{n-2}). \tag{3.15}$$

Proof. We prove it using mathematical induction on n. For $n = 1$,

$$GM_2(x)GM_0(x) - GM_1^2(x) = (3x+i)(-i/2) - 1 = (2^{-1} - 1) - i3x(2^{-1}).$$

So, the statement is true for $n = 1$.

Assume that result is true for $n = k$, *i.e.*

$$GM_{k+1}(x)GM_{k-1}(x) - GM_k^2(x) = (2^{k-2} - 2^{k-1}) - i3x(2^{k-2}). \tag{3.16}$$

Now, for $n = k+1$, using eqn. (3.12) and eqn. (3.16), we have

$$\begin{aligned}
&GM_{k+2}(x)GM_k(x) - GM_{k+1}^2(x)\\
=&\left[(3xGM_{k+1}(x) - 2GM_k(x))\left(\frac{1}{3x}GM_{k+1}(x) + \frac{2}{3x}GM_{k-1}(x)\right)\right]\\
&- GM_{k+1}^2(x) \qquad\qquad \text{(using eqn. (3.12))}\\
=&2GM_{k+1}(x)GM_{k-1}(x) - \frac{2}{3x}GM_{k+1}(x)GM_k(x) - \frac{4}{3x}GM_k(x)GM_{k-1}(x)\\
=&2GM_k^2(x) + 2(2^{k-2} - 2^{k-1}) - i3x(2^{k-2}) - \frac{2}{3x}GM_{k+1}(x)GM_k(x)\\
&+ \frac{4}{3x}GM_{k-1}(x)GM_k(x)\\
=&(2^{k-1} - 2^k) - i3x(2^{k-1}).
\end{aligned}$$

□

3.4. Generalized k-Gaussian Mersenne polynomials.

Definition 3.16. *Let $k \in \mathbb{N}, n \in \mathbb{N}\cup\{0\}$ and $s, r \in \mathbb{N}\cup\{0\}$ be the unique natural numbers such that $n = sk + r$, $0 \leq r < k$. The generalized k-Gaussian Mersenne polynomials $\{GM_n^{(k)}(x)\}$ are defined by*

$$\begin{aligned}
GM_n^{(k)}(x) = &\left(\frac{(\lambda_1^s(x) - \lambda_2^s(x)) + i(\lambda_1^{s-1}(x) - \lambda_2^{s-1}(x))}{\lambda_1(x) - \lambda_2(x)}\right)^{k-r}\\
&\cdot\left(\frac{(\lambda_1^{s+1}(x) - \lambda_2^{s+1}(x)) + i(\lambda_1^s(x) - \lambda_2^s(x))}{\lambda_1(x) - \lambda_2(x)}\right)^r,
\end{aligned}$$

where $\lambda_1(x)$ and $\lambda_2(x)$ are the roots of the characteristic equation (2.4).

The relation between the generalized k-Gaussian Mersenne polynomials and Gaussian Mersenne polynomials is

$$GM_n^{(k)}(x) = GM_s^{k-r}(x)GM_{s+1}^r(x),\ n = sk + r. \tag{3.17}$$

If $k = 1$ then $r = 0$ and hence $m = n$. So, from eqn. (3.17) we have $GM_n^{(1)}(x) = GM_n(x)$. The following relations between the generalized k-Gaussian Mersenne polynomials and the Gaussian Mersenne polynomials hold for $k = 2, 3$.

$$\begin{aligned}
GM_{2s}^{(2)}(x) &= GM_s^2(x).\\
GM_{2s+1}^{(2)}(x) &= GM_s(x)GM_{s+1}(x).\\
GM_{2s+1}^{(2)}(x) &= 3xGM_{2s}^{(2)}(x) - 2GM_{2s-1}^{(2)}(x).\\
GM_{3s}^{(3)}(x) &= GM_s^3(x).\\
GM_{3s+1}^{(3)}(x) &= GM_s^2(x)GM_{s+1}(x).\\
GM_{3s+1}^{(3)}(x) &= 3xGM_{3s}^{(3)}(x) - 2GM_{3s-1}^{(3)}(x).\\
GM_{3s+2}^{(3)}(x) &= GM_s(x)GM_{s+1}^2(x).
\end{aligned}$$

The following table shows the list of some generalized k-Gaussian Mersenne polynomials,

$GM_n^{(k)}(x)$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
$GM_0^{(k)}(x)$	$-i/2$	$-1/4$	$i/8$	$1/16$	$-i/32$
$GM_1^{(k)}(x)$	1	$-i/2$	$-1/4$	$i/8$	$1/16$
$GM_2^{(k)}(x)$	$3x+i$	1	$-i/2$	$-1/4$	$i/8$
$GM_3^{(k)}(x)$	$9x^2-2+i3x$,	$3x+i$	1	$-i/2$	$-i/4$
$GM_4^{(k)}(x)$	$27x^3-12x$ $+i(9x^2-2)$	$(9x^2-1)+i6x$	$3x+i$	1	$-i/2$
$GM_5^k(x)$	$(81x^4-54x^2+4)$ $+i(27x^3-12x)$	$(27x^3-3x^2-6x)$ $+i(18x^2-2)$	$(9x^2-1)+i6x$	$3x+i$	1

TABLE 4. List of some generalized k-Gaussian Mersenne polynomials $GM_n^{(k)}(x)$.

Proposition 3.17. *Let $k, s \in \mathbb{N}$, then we have $GM_{ks}^{(k)}(x) = GM_s^k(x)$.*

Theorem 3.18. *For $n, s \in \mathbb{N}$, we have*

$$GM_{sn+1}^{(s)}(x) = 3xGM_{sn}^{(s)}(x) - 2GM_{sn-1}^{(s)}(x).$$

Proof. From eqn. (3.17),

$$\begin{aligned}
3xGM_{sn}^{(s)}(x) - 2GM_{sn-1}^{(s)}(x) &= 3xGM_n^s(x) - 2GM_{n-1}(x)GM_n^{s-1}(x)\\
&= GM_n^{s-1}(x)GM_{n+1}(x)\\
&= GM_{sn+1}^{(s)}(x).
\end{aligned}$$

□

Theorem 3.19. *For $k, s \in \mathbb{N}$, we have* $GM^k_{s+1}(x) - GM^k_s(x) = GM^{(k)}_{sk+k}(x) - GM^{(k)}_{sk}(x)$.

Proof. From eqn. (3.17), we obtain

$$\begin{aligned} GM^{(k)}_{sk+k}(x) - GM^{(k)}_{sk}(x) &= [GM^{k-k}_s(x)GM^k_{s+1}(x)] - [GM^{k-0}_s(x)GM^0_{s+1}(x)] \\ &= GM^k_{s+1}(x) - GM^k_s(x). \end{aligned}$$

□

Theorem 3.20. *Let $n, m \geq 0$ such that $n + m > 1$, then we have*

$$\begin{aligned} &GM^{(2)}_{2(n+m-1)}(x) - GM_{n+m}(x)GM_{n+m-2}(x) \\ =&(2^{n+m-1} - 2^{n+m-2}) + ix3(2^{n+m-2}) \end{aligned}$$

Proof. By eqn. (3.15) and Proposition 3.17, we get

$$\begin{aligned} &GM^{(2)}_{2(n+m-1)}(x) - GM_{n+m}(x)GM_{n+m-2}(x) \\ =&GM^2_{(n+m-1)}(x) - GM_{n+m}(x)GM_{n+m-2}(x) \\ =&(2^{n+m-1} - 2^{n+m-2}) + i3x(2^{n+m-2}). \end{aligned}$$

□

Theorem 3.21. *Let $n, k \geq 2$, then we can write the Cassini identity for $GM^{(k)}_n(x)$ as,*

$$\begin{aligned} &(GM^{(k)}_{nk+a-1})^2(x) - GM^{(k)}_{nk+a}(x)GM^{(k)}_{nk+a-2}(x) \\ =&\begin{cases} GM^{2k-2}_n(x)\left[(2^{n-1} - 2^{n-2}) + i3x(2^{n-2})\right], & a = 1 \\ 0, & a \neq 1. \end{cases} \end{aligned}$$

Proof. Let $a = 1$, then using eqn. (3.17) we have,

$$\begin{aligned} (3.18)\quad & GM^{(k)}_{nk+1}(x)GM^{(k)}_{nk-1}(x) - (GM^{(k)}_{nk})^2(x) \\ =& \left(GM^{k-1}_n(x)GM_{n+1}(x)\right)\left(GM_{n-1}(x)GM^{k-1}_n(x)\right) - \left(GM^k_n(x)\right)^2 \\ =&GM^{2k-2}_n(x)[GM_{n+1}(x)GM_{n-1}(x) - GM^2_n(x)] \\ =&GM^{2k-2}_n(x)(2^{n-1} - 2^{n-2}) + i3x(2^{n-2}). \qquad \text{(using eqn. 3.15)} \end{aligned}$$

and if $a \neq 1$, then by eqn. (3.17),

$$\begin{aligned}
&GM^{(k)}_{nk+a}(x)GM^{(k)}_{nk+a-2}(x) - (GM^{(k)}_{nk+a-1})^2(x)\\
=&\left(GM_n^{k-a}(x)GM^a_{n+1}(x)\right)\left(GM_n^{k-a+2}(x)GM^{a-2}_{n+1}(x)\right)\\
&-\left(GM_n^{k-a+1}(x)GM^{a-1}_{n+1}(x)\right)^2\\
=&GM_n^{2k-2a+2}(x)[GM^{2a-2}_{n+1}(x) - GM^{2a-2}_{n+1}(x)]\\
=&0.
\end{aligned}$$

□

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