Contributions to Discrete Mathematics

Volume 18, Number 1, Pages 90–104
ISSN 1715-0868

ON A GENERALIZED BASIC SERIES AND ROGERS-RAMANUJAN TYPE IDENTITIES - II

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Abstract. This paper is in continuation with our recent paper "On a generalized basic series and Rogers–Ramanujan type identities" [18]. Here, we consider two generalized basic series and interpret these basic series as the generating function of some restricted $(n+t)$-color partitions and restricted weighted lattice paths. The basic series discussed in the aforementioned paper, is now a mere particular case of one of the generalized basic series that are discussed in this paper. Besides, eight particular cases are also discussed which give combinatorial interpretations of eight Rogers-Ramanujan type identities which are combinatorially unexplored till date.

1. Introduction

The Rogers-Ramanujan identities are the most mysterious and celebrated results in the theory of partitions. Their remarkable applications appear in areas as distinct as enumerative combinatorics, number theory, representation theory, group theory, statistical physics, probability and complex analysis [7, 8]. These identities have been known to the world for one hundred and twenty-five years and are still the subject of attention and active research. The Rogers-Ramanujan identities are a pair of infinite "Sum-Product" basic series identities. Bailey [9, 10] systematically studied and generalized Rogers's work on Rogers-Ramanujan type identities. A large collection of such identities was produced by L.J. Slater [17]. The connection between Rogers-Ramanujan identities and ordinary partitions was established by MacMahon [15]. But there were several identities in Slater's list for which combinatorial interpretation using ordinary partitions was not possible. So, it demanded the generalization of ordinary partitions which is attributed to Agarwal and Andrews work [4] in 1987 . They named these new sets of partitions as $(n+t)$-color partitions. Agarwal and Bressoud [5]

Received by the editors January 5, 2022, and in revised form May 9, 2022.

2010 *Mathematics Subject Classification.* 05A15, 05A17, 05A19, 11P81.

Key words and phrases. generalized basic series, $(n+t)$-color partitions, weighted lattice paths, Rogers-Ramanujan type identities.

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introduced and studied weighted lattice paths for the graphical representation of Rogers-Ramanujan type identities. By now there are several combinatorial identities established for the Rogers-Ramanujan type identities using $(n+t)$-color partitions and weighted lattice paths, see for instance, [1, 3, 14]. In our recent paper [18], we studied a generalized basic series that generalizes several Rogers-Ramanujan type identities found in literature [11, 12, 17]. In this paper, we propose two new generalized combinatorial identities for Rogers-Ramanujan type identities with a minimum amount of algebraic manipulation. Our proofs involve the analysis of two distinct sets of combinatorial objects, viz., $(n+t)$-color partitions and weighted lattice paths using algebraic and constructive approaches. Our results are elementary and approachable. These results give a new insight into future advanced combinatorial generalizations of Rogers-Ramanujan type identities.

In this paper, the following two generalized basic series have been studied.

$$\sum_{n=0}^{\infty}\frac{q^{\alpha n^2+\beta n}}{(q^{\gamma};q^{2\gamma})_n(q^{\delta};q^{\delta})_n}, \tag{1.1}$$

$$\sum_{n=0}^{\infty}\frac{q^{\alpha n^2+tn}}{(q^{\gamma};q^{2\gamma})_{n+1}(q^{\delta};q^{\delta})_n}, \tag{1.2}$$

where $\alpha,\gamma,\delta,t\in\mathbb{Z}^+$ and $\beta\in\mathbb{Z}^+\cup\{0\}$ and

$$(a;q)_n=\prod_{r=0}^{n-1}(1-aq^r),$$

$$(a;q)_\infty=\lim_{n\to\infty}(a;q)_n,$$

where $(a_1,a_2,\ldots,a_n;z)_\infty=\prod_{r=1}^{n}(a_r;z)_\infty$ and $|q|<1$.

The purpose of this paper is to interpret the above generalized basic series (1.1)–(1.2) as generating function of certain restricted classes of $(n+t)$-color partitions and weighted lattice paths. Hence, these results provide an infinite set of combinatorial identities and also provide many Rogers-Ramanujan type combinatorial identities as their particular cases. The following series is the generalized basic series studied in [18]:

$$\sum_{n=0}^{\infty}\frac{q^{\alpha n^2+\beta n}}{(q^{s};q^{2s})_n(q^{2m};q^{2m})_n}. \tag{1.3}$$

If we compare the series (1.3) with the basic series (1.1), one can observe that the basic series (1.3) is a mere particular case of (1.1) when δ is even.

We split the proof of the main results into three virtually independent parts. In the first part, that is, the algebraic part, we use recurrence relations and q-functional equations to provide partition theoretic interpretations. In the second part, these basic series are interpreted in terms of weighted lattice paths using a constructive approach, and in the last part, we establish combinatorial identities by establishing direct bijections between restricted

$(n+t)$-color partitions and weighted lattice paths. Our proofs are elementary and completely self-contained. We conclude with the final remarks section.

The main results in this paper are stated as:

Theorem 1.1. *Let $A^{(\alpha,\beta)}_{(\gamma,\delta)}(\xi)$ denote the number of n-color partitions of ξ in such a way that*

1. *all subscripts are greater than or equal to α, and congruent to $\alpha \pmod{\gamma}$. For any part e_ℓ, $e \geq \ell + \beta$,*
2. *if m_i is the smallest or the only part in the partition then $m \equiv i + \beta \pmod{\delta}$,*
3. *the weighted difference of any two consecutive parts is non-negative and is congruent to $0 \pmod{\delta}$.*

Let $B^{(\alpha,\beta)}_{(\gamma,\delta)}(\xi)$ denote the number of lattice paths of weight ξ which start from $(0,0)$ and

1. *they have no valley above height 0,*
2. *the height of each peak is $\geq \alpha$ and is $\equiv \alpha \pmod{\gamma}$,*
3. *there is a plain of length congruent to β modulo δ at the beginning of the path and the lengths of the other plains, if any, are congruent to $0 \pmod{\delta}$.*

Then, we have $A^{(\alpha,\beta)}_{(\gamma,\delta)}(\xi) = B^{(\alpha,\beta)}_{(\gamma,\delta)}(\xi)$ *for all ξ and*

$$\sum_{\xi=0}^{\infty} A^{(\alpha,\beta)}_{(\gamma,\delta)}(\xi)\, q^{\xi} = \sum_{\xi=0}^{\infty} B^{(\alpha,\beta)}_{(\gamma,\delta)}(\xi)\, q^{\xi} = \sum_{n=0}^{\infty} \frac{q^{\alpha n^2+\beta n}}{(q^{\gamma};q^{2\gamma})_n (q^{\delta};q^{\delta})_n}, \tag{1.4}$$

where $\alpha, \gamma, \delta \in \mathbb{Z}^+$ and $\beta \in \mathbb{Z}^+ \cup \{0\}$.

Theorem 1.2. *Let $C^{(\alpha,t)}_{(\gamma,\delta)}(\xi)$ denote the number of $(n+t)$-color partitions of ξ in such a way that*

1. *the smallest or the singleton part is of the form i_{i+t} where $i \equiv 0 \pmod{\gamma}$,*
2. *for all other parts, subscripts are at least α and congruent to $\alpha \pmod{\gamma}$,*
3. *the weighted difference of any two consecutive parts is non-negative and congruent to $0 \pmod{\delta}$.*

Let $D^{(\alpha,t)}_{(\gamma,\delta)}(\xi)$ denote the number of lattice paths of weight ξ which start from $(0,t)$ and

1. *they have no valley above height 0,*
2. *the height of the first peak is at least t and congruent to $t \pmod{\gamma}$,*
3. *for all other peaks, the height is at least α and is congruent to $\alpha \pmod{\gamma}$,*
4. *the lengths of the plains, if any, are congruent to $0 \pmod{\delta}$,*

Then, we have $C^{(\alpha,t)}_{(\gamma,\delta)}(\xi) = D^{(\alpha,t)}_{(\gamma,\delta)}(\xi)$ *for all* ξ *and*

$$\sum_{\xi=0}^{\infty} C^{(\alpha,t)}_{(\gamma,\delta)}(\xi)\, q^{\xi} = \sum_{\xi=0}^{\infty} D^{(\alpha,t)}_{(\gamma,\delta)}(\xi)\, q^{\xi} = \sum_{n=0}^{\infty} \frac{q^{\alpha n^2+tn}}{(q^{\gamma};q^{2\gamma})_{n+1}(q^{\delta};q^{\delta})_n}, \tag{1.5}$$

where $\alpha, t, \gamma, \delta \in \mathbb{Z}^+$.

Before we proceed further, we first recall a few basic definitions.

Definition 1.3. [4] *A partition with "$n+t$ copies of n", $t \geq 0$, is a partition in which a part of size n, $n \geq 0$, can come in $n+t$ different colors denoted by subscripts: $n_1, n_2, n_3, \ldots, n_{n+t}$.*

For example, the partitions of 2 with "$n+1$ copies of n" are

2_1	2_10_1	1_11_1	$1_11_10_1$
2_2	2_20_1	1_21_1	$1_21_10_1$
2_3	2_30_1	1_21_2	$1_21_20_1$

Remark: Note that zeros are permitted if and only if t is greater than zero.

For $t = 0$, these partitions are known as n-color partitions [1].

Definition 1.4. *The weighted difference of two parts x_i and y_j, $(x \geq y)$, is defined by $x - y - i - j$ and is expressed by $((x_i - y_j))$.*

Definition 1.5. [5] *All weighted lattice paths will be of finite lengths and they lie in the first quadrant. They will start on the y-axis or the x-axis and end on the x-axis. Only three steps are allowed:*

- northeast: *from* (a, b) *to* $(a+1, b+1)$.
- southeast: *from* (a, b) *to* $(a+1, b-1)$, *only allowed if* $b > 0$.
- horizontal: *from* $(a, 0)$ *to* $(a+1, 0)$, *only allowed along x-axis.*

Every lattice path is either empty or ends with a southeast step: from $(a, 1)$ *to* $(a+1, 0)$.

To illustrate the lattice paths, the following terminology is used.

- Peak: *Either a vertex on the y-axis is followed by a southeast step or a vertex preceded by a northeast step and followed by a southeast step.*
- Valley: *A vertex preceded by a southeast step and followed by a northeast step. Remember that a southeast step followed by a horizontal step followed by a northeast step does not form a valley.*
- Mountain: *A section of the path which begins on either the x- or y- axis, which terminates on the x-axis, and which does not touch the x-axis throughout in between the endpoints. There is at least one peak in a mountain and the number of peaks may exceed one.*
- Plain: *A section of the path including only horizontal steps which begins either on the y-axis or at a vertex preceded by a southeast step and terminates at a vertex followed by a northeast step.*

The height *of a vertex is its y-coordinate, the* weight *of a vertex is its x-coordinate, and the* weight *of a lattice path is the sum of the weights of its peaks.*

2. Main Results

Firstly, we will prove Theorem 1.1 in three steps.

2.1. **Step I: Algebraic approach.** Let $A^{(\alpha,\beta)}_{(\gamma,\delta)}(\zeta,\xi)$ represent the number of partitions of ξ enumerated by $A^{(\alpha,\beta)}_{(\gamma,\delta)}(\xi)$ with the added restriction that there be exactly ζ parts. First of all, we will prove the following recurrence relation:

$$\begin{aligned}A^{(\alpha,\beta)}_{(\gamma,\delta)}(\zeta,\xi) &= A^{(\alpha,\beta)}_{(\gamma,\delta)}(\zeta,\xi-\delta\zeta) + A^{(\alpha,\beta)}_{(\gamma,\delta)}(\zeta-1,\xi-2\alpha\zeta+\alpha-\beta)\\ &\quad + A^{(\alpha,\beta)}_{(\gamma,\delta)}(\zeta,\xi-2\gamma\zeta+\gamma) - A^{(\alpha,\beta)}_{(\gamma,\delta)}(\zeta,\xi-\zeta(2\gamma+\delta)+\gamma).\end{aligned}$$

To prove this, we split the partitions enumerated by $A^{(\alpha,\beta)}_{(\gamma,\delta)}(\zeta,\xi)$ into three classes:

(i) those that do not contain $\lambda_{\lambda-\beta}$ as a part,
(ii) those that contain $(\alpha+\beta)_\alpha$ as a part, and
(iii) those that contain $\lambda_{\lambda-\beta}$, $\lambda > \alpha+\beta$ as a part.

We now transform the partitions in class (i) by subtracting δ from each part ignoring the subscripts. Obviously, this transformation will not disturb the inequalities between the parts and so the transformed partition will be of the type enumerated by $A^{(\alpha,\beta)}_{(\gamma,\delta)}(\zeta,\xi-\delta\zeta)$. Next, we transform the partitions in class (ii) by deleting the least part $(\alpha+\beta)_\alpha$ and then subtracting 2α from all the remaining parts ignoring the subscripts. The transformed partition will be of the type enumerated by $A^{(\alpha,\beta)}_{(\gamma,\delta)}(\zeta-1,\xi-2\alpha\zeta+\alpha-\beta)$. Finally, we transform the partitions in class (iii) by replacing $\lambda_{\lambda-\beta}$ by $(\lambda-\gamma)_{\lambda-\beta-\gamma}$ and then subtracting 2γ from all the remaining parts ignoring the subscripts. This will produce a partition of $\xi-2\gamma\zeta+\gamma$ into ζ parts. It is important to note here that by this transformation we get only those partitions of $\xi-2\gamma\zeta+\gamma$ into ζ parts which contain a part of the form $\lambda_{\lambda-\beta}$. Therefore, the actual number of partitions which belong to class (iii) is

$$A^{(\alpha,\beta)}_{(\gamma,\delta)}(\zeta,\xi-2\gamma\zeta+\gamma) - A^{(\alpha,\beta)}_{(\gamma,\delta)}(\zeta,\xi-\zeta(2\gamma+\delta)+\gamma),$$

where $A^{(\alpha,\beta)}_{(\gamma,\delta)}(\zeta,\xi-\zeta(2\gamma+\delta)+\gamma)$ is the number of partitions of $\xi-2\gamma\zeta+\gamma$ into ζ parts which are free from the parts like $\lambda_{\lambda-\beta}$. The above transformations are clearly reversible and so establish a bijection between the partitions enumerated by $A^{(\alpha,\beta)}_{(\gamma,\delta)}(\zeta,\xi)$ and those enumerated by

$$\begin{aligned}&A^{(\alpha,\beta)}_{(\gamma,\delta)}(\zeta,\xi-\delta\zeta) + A^{(\alpha,\beta)}_{(\gamma,\delta)}(\zeta-1,\xi-2\alpha\zeta+\alpha-\beta)\\ &+ A^{(\alpha,\beta)}_{(\gamma,\delta)}(\zeta,\xi-2\gamma\zeta+\gamma) - A^{(\alpha,\beta)}_{(\gamma,\delta)}(\zeta,\xi-\zeta(2\gamma+\delta)+\gamma).\end{aligned}$$

This generates the identity

$$\begin{aligned} A^{(\alpha,\beta)}_{(\gamma,\delta)}(\zeta,\xi) &= A^{(\alpha,\beta)}_{(\gamma,\delta)}(\zeta,\xi-\delta\zeta) + A^{(\alpha,\beta)}_{(\gamma,\delta)}(\zeta-1,\xi-2\alpha\zeta+\alpha-\beta) \\ &\quad + A^{(\alpha,\beta)}_{(\gamma,\delta)}(\zeta,\xi-2\gamma\zeta+\gamma) - A^{(\alpha,\beta)}_{(\gamma,\delta)}(\zeta,\xi-\zeta(2\gamma+\delta)+\gamma). \end{aligned} \tag{2.1}$$

Let

$$f^{(\alpha,\beta)}_{(\gamma,\delta)}(z;q) = \sum_{\xi=0}^{\infty}\sum_{\zeta=0}^{\infty} A^{(\alpha,\beta)}_{(\gamma,\delta)}(\zeta,\xi)z^{\zeta}q^{\xi}. \tag{2.2}$$

From (2.1) and (2.2), we have

$$\begin{aligned} f^{(\alpha,\beta)}_{(\gamma,\delta)}(z;q) &= f^{(\alpha,\beta)}_{(\gamma,\delta)}(zq^{\delta};q) + zq^{\alpha+\beta} f^{(\alpha,\beta)}_{(\gamma,\delta)}(zq^{2\alpha};q) + q^{-\gamma} f^{(\alpha,\beta)}_{(\gamma,\delta)}(zq^{2\gamma};q) \\ &\quad - q^{-\gamma} f^{(\alpha,\beta)}_{(\gamma,\delta)}(zq^{2\gamma+\delta};q). \end{aligned} \tag{2.3}$$

Since $f^{(\alpha,\beta)}_{(\gamma,\delta)}(z;q)$ is analytic function for $|q|<1$ and $|z|<|q|^{-1}$, we have

$$f^{(\alpha,\beta)}_{(\gamma,\delta)}(z;q) = \sum_{n=0}^{\infty} a_n(q)z^n. \tag{2.4}$$

Employing (2.4) into (2.3) and then comparing the coefficients of z^n on each side of the resulting identity, we deduce that

$$a_n(q) = \frac{q^{\alpha(2n-1)+\beta}a_{n-1}(q)}{(1-q^{\delta n})(1-q^{2\gamma(n-1)+\gamma})}.$$

On iterating and using $a_0(q)=1$, we obtain that

$$a_n(q) = \frac{q^{\alpha n^2+\beta n}}{(q^{\delta};q^{\delta})_n(q^{\gamma};q^{2\gamma})_n}.$$

Hence

$$f^{(\alpha,\beta)}_{(\gamma,\delta)}(z;q) = \sum_{n=0}^{\infty} \frac{q^{\alpha n^2+\beta n}}{(q^{\gamma};q^{2\gamma})_n(q^{\delta};q^{\delta})_n} z^n.$$

Therefore

$$\begin{aligned} \sum_{\xi=0}^{\infty} A^{(\alpha,\beta)}_{(\gamma,\delta)}(\xi)q^{\xi} &= \sum_{\xi=0}^{\infty}\sum_{\zeta=0}^{\infty} A^{(\alpha,\beta)}_{(\gamma,\delta)}(\zeta,\xi)q^{\xi} \\ &= f^{(\alpha,\beta)}_{(\gamma,\delta)}(1;q) \\ &= \sum_{n=0}^{\infty} \frac{q^{\alpha n^2+\beta n}}{(q^{\gamma};q^{2\gamma})_n(q^{\delta};q^{\delta})_n}. \end{aligned}$$

2.2. **Step II: Constructive approach.** In this step, we will prove that

$$\sum_{\xi=0}^{\infty} B_{(\gamma,\delta)}^{(\alpha,\beta)}(\xi)\, q^{\xi} = \sum_{n=0}^{\infty} \frac{q^{\alpha n^2+\beta n}}{(q^{\gamma};q^{2\gamma})_n (q^{\delta};q^{\delta})_n}. \tag{2.5}$$

In

$$\frac{q^{\alpha n^2+\beta n}}{(q^{\gamma};q^{2\gamma})_n (q^{\delta};q^{\delta})_n},$$

the factor $q^{\alpha n^2+\beta n}$ generates a lattice path from $(0,0)$ to $(\beta+2\alpha n,0)$ having n peaks each of height α and a plain of length β at the beginning of the path. For example, $\alpha=2,\beta=1,n=4$, the path begins as

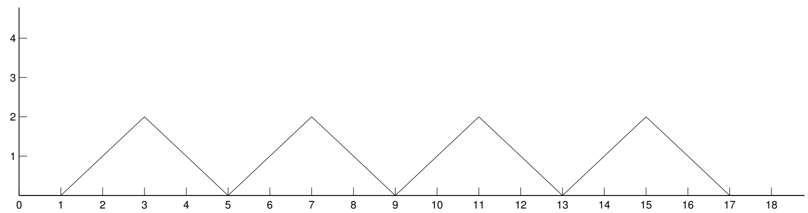


FIGURE 1. Four peaks each of height 2, three valleys each at height zero and a plain of length 1 at the beginning of the path

In Figure 1, we take two consecutive peaks say, j^{th} and $(j+1)^{\text{th}}$ and denote them by P_1 and P_2 respectively.

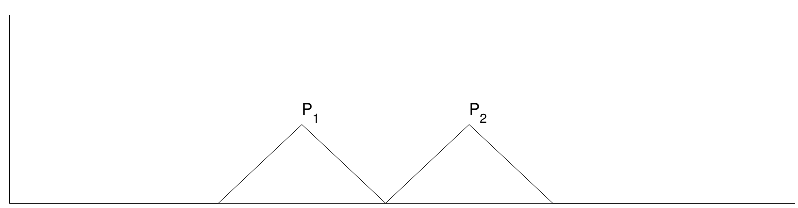


FIGURE 2. Two peaks of same height

Clearly, in Figure 2

$$P_1 \equiv (\beta+\alpha(2j-1),\alpha) \quad \text{and} \quad P_2 \equiv (\beta+\alpha(2j+1),\alpha).$$

The factor

$$\frac{1}{(q^{\delta};q^{\delta})_n},$$

generates n nonnegative multiples of δ, say, $u_1 \geq u_2 \geq \ldots \geq u_n \geq 0$ which are encoded by inserting u_n horizontal steps in front of the first mountain and $u_j - u_{j+1}$ horizontal steps in front of the $(n-j+1)^{\text{th}}$ mountain for $1 \leq j \leq n-1$. Thus the x-coordinate of the j^{th} peak is increased by u_{n-j+1} and the x-coordinate of the $(j+1)^{\text{th}}$ peak is increased by u_{n-j}. Figure 2 now turns into Figure 3.

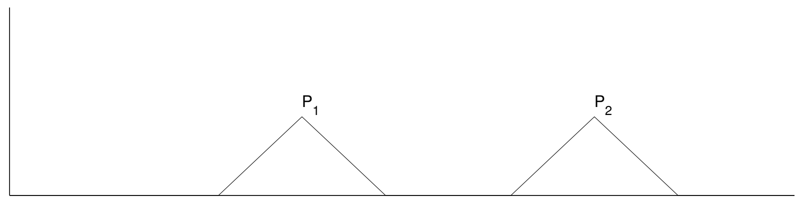


FIGURE 3. Two peaks separated by a plain of length multiple of δ

Thus two consecutive peaks P_1 and P_2 becomes

$$P_1 \equiv (\beta + \alpha(2j-1) + u_{n-j+1}, \alpha) \quad \text{and} \quad P_2 \equiv (\beta + \alpha(2j+1) + u_{n-j}, \alpha).$$

The factor $\dfrac{1}{(q^{\gamma}; q^{2\gamma})_n}$ generates n nonnegative odd multiples of γ, say $v_1 \times \gamma, v_2 \times 3\gamma, v_3 \times 5\gamma, \ldots, v_n \times (2n-1)\gamma$. These can be encoded by raising the height of j^{th} peak by γv_{n-j+1}, $1 \leq j \leq n$. So, j^{th} peak grows to height $\gamma v_{n-j+1} + \alpha$. Each increase by one in height of a given peak increases its weight by one and the weight of each subsequent peak by two. Figure 3 is altered to Figure 4 or Figure 5 depending on if $v_{n-j} > v_{n-j+1}$ or $v_{n-j} < v_{n-j+1}$. In case when $v_{n-j} = v_{n-j+1}$, the new Figure looks like Figure 3.

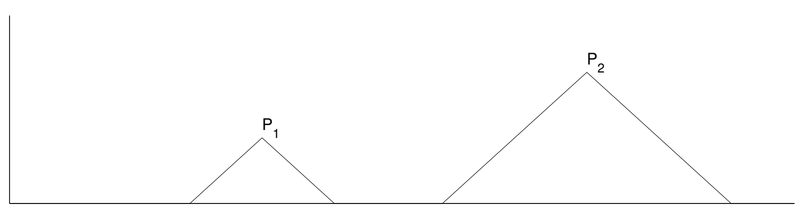


FIGURE 4. P_2 has more height than P_1 for $v_{n-j} > v_{n-j+1}$

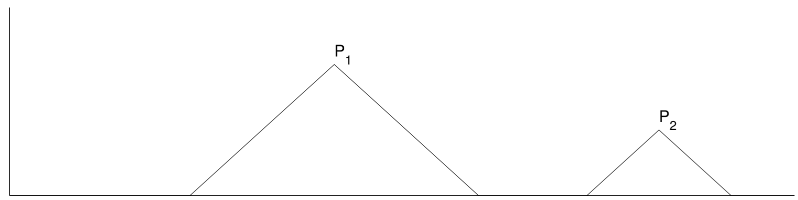


FIGURE 5. P_1 has more height than P_2 for $v_{n-j} < v_{n-j+1}$

By this way, we can uniquely generate each lattice path enumerated by $B^{(\alpha,\beta)}_{(\gamma,\delta)}(\xi)$. This demonstrates (2.5).

2.3. **Step III: Direct bijections.** We now establish a bijection between the lattice paths enumerated by $B^{(\alpha,\beta)}_{(\gamma,\delta)}(\xi)$ and the n-color partitions enumerated by $A^{(\alpha,\beta)}_{(\gamma,\delta)}(\xi)$. We do this by encoding every path as the sequence of the weights of the peaks with each weight subscripted by the height of the respective peak. Therefore, if in the final figure, we represent the j^{th} and $(j+1)^{\text{th}}$ peaks by G_x and H_y, $(H \geq G)$, respectively, then

$$G = \beta + \alpha(2j-1) + u_{n-j+1} + 2\gamma(v_n + v_{n-1} + \ldots + v_{n-j+2}) + \gamma v_{n-j+1},$$
$$H = \beta + \alpha(2j+1) + u_{n-j} + 2\gamma(v_n + v_{n-1} + \ldots + v_{n-j+1}) + \gamma v_{n-j},$$
$$x = \gamma v_{n-j+1} + \alpha,$$
$$y = \gamma v_{n-j} + \alpha.$$

Now, the weighted difference of H_y and G_x is equal to

$$((H_y - G_x)) = H - G - x - y = u_{n-j} - u_{n-j+1}.$$

Clearly weighted difference is ≥ 0 and it is $\equiv 0 \pmod{\delta}$. Now for every peak,

$$G - x = \beta + 2\alpha(j-1) + u_{n-j+1} + 2\gamma(v_n + v_{n-1} + \ldots + v_{n-j+2}),$$

hence $G - x \geq \beta$, and thus $G \geq x + \beta$.

Next, say G_x is the first peak, then it will correspond to the least part in the corresponding n-color partition or to the singleton part if the n-color partition contains only one part and in both of the cases $G - x = \beta + u_n \equiv \beta \pmod{\delta}$. This gives $G \equiv x + \beta \pmod{\delta}$.

Moreover, $x = \gamma v_{n-j+1} + \alpha$ and $y = \gamma v_{n-j} + \alpha$. It is clear that each subscript is at least α and is congruent to $\alpha \pmod{\gamma}$. In addition, all parts are at least $\alpha + \beta$.

To check the reverse implication, we take two n-color parts of a partition enumerated by $A^{(\alpha,\beta)}_{(\gamma,\delta)}(\xi)$, say M_r and N_s with $N \geq M, N \geq s+\beta, M \geq r+\beta$, $r \geq \alpha$ and $s \geq \alpha$. Let $Q_1 \equiv (M, r)$ and $Q_2 \equiv (N, s)$ be the associated peaks in the corresponding lattice path.

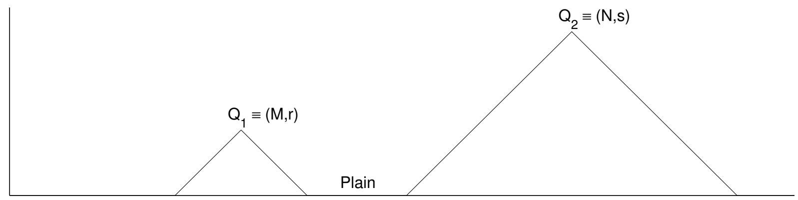


FIGURE 6. Two peaks separated by a plain

The length of the plain between the two peaks is $N - M - r - s = ((N_s - M_r)) \equiv 0 \pmod{\delta}$.

Also, there can not be a valley above height 0. This can be proved by contradiction. Let us assume a valley V at height h ($h > 0$) between the peaks Q_1 and Q_2.

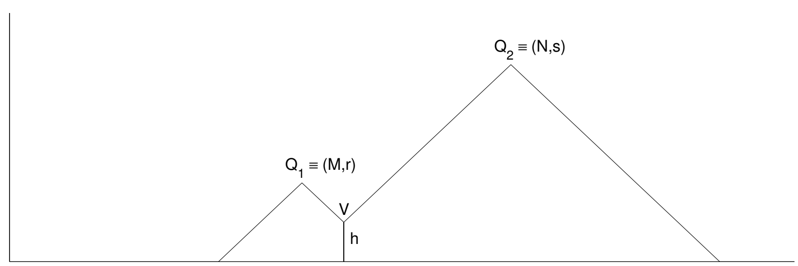


FIGURE 7. Two peaks along a valley at height h

Clearly, there is a descent of $r - h$ from Q_1 to V and an ascent of $s - h$ from V to Q_2. This implies

$$N = M + (r - h) + (s - h) \implies N - M - r - s = -2h,$$

hence $((N_s - M_r)) = -2h$.

Now, $((N_s - M_r)) \geq 0 \implies -2h \geq 0 \implies h = 0$. This confirms, there is no valley above height 0.

Now in (1.4), the extra factor $q^{\beta n}$ puts β horizontal steps in front of the first peak. This makes the length of the plain (which is at the beginning of the path) congruent to $\beta \pmod{\delta}$. This completes the proof of Theorem 1.1.

2.4. **Proof of Theorem 1.2.** Since the proof of Theorem 1.2 is similar to that of Theorem 1.1, we omit the details and give only the identities analogous to (2.1) and (2.3).

$$C^{(\alpha,t)}_{(\gamma,\delta)}(\zeta - 1, \xi - (2\alpha - t)(\zeta - 1) - \alpha) = A^{(\alpha,0)}_{(\gamma,\delta)}(\zeta, \xi) - A^{(\alpha,0)}_{(\gamma,\delta)}(\zeta, \xi - \delta\zeta),$$

and

$$zq^{\alpha}\phi^{(\alpha,t)}_{(\gamma,\delta)}(zq^{2\alpha-t};q) = f^{(\alpha,0)}_{(\gamma,\delta)}(z;q) - f^{(\alpha,0)}_{(\gamma,\delta)}(zq^{\delta};q),$$

where,

$$\phi^{(\alpha,t)}_{(\gamma,\delta)}(z;q) = \sum_{\xi=0}^{\infty}\sum_{\zeta=0}^{\infty} C^{(\alpha,t)}_{(\gamma,\delta)}(\zeta,\xi) z^{\zeta} q^{\xi} = \sum_{n=0}^{\infty} \frac{q^{\alpha n^2+tn}}{(q^{\gamma};q^{2\gamma})_{n+1}(q^{\delta};q^{\delta})_n} z^n.$$

Now comparing the identity (1.2) with identity (1.1), when $\beta = 0$, we see that there are two extra factors, viz., q^{tn} and $(1-q^{(2n+1)\gamma})^{-1}$. The extra factor q^{tn} puts t southeast steps: $(0,t)$ to $(1,t-1),\cdots,(t-1,1)$ to $(t,0)$. So, there are $n+1$ peaks starting from $(0,t)$ and the extra factor $\dfrac{1}{1-q^{(2n+1)\gamma}}$ generates a nonnegative multiple of $(2n+1)\gamma$ say $v_{n+1}\times(2n+1)\gamma$. This can be inserted by raising the height of the first peak by γv_{n+1} i.e. first peak grows to height of $\gamma v_{n+1}+t$ in the northeast direction. Clearly, $(\gamma v_{n+1})_{\gamma v_{n+1}+t}$ which is of the form i_{i+t} will be the colored part corresponding to the first peak.

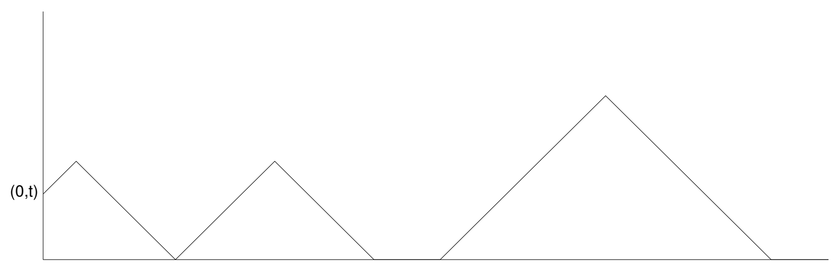


FIGURE 8. First peak corresponding to the height $\gamma v_{n+1}+t$

3. Rogers-Ramanujan type identities

For some particular values of α,β,γ, δ and t, the generalized series (1.1)–(1.2) yields the following eight Rogers-Ramanujan type identities. These identities are also found in Bailey [11], Chu and Zhang [12] and Slater [17]

$$\sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q^2;q^4)_{n+1}(q^2;q^2)_n} = \frac{(q^8,q^{20},q^{28};q^{28})_\infty}{(q^2;q^2)_\infty}, \tag{3.1}$$

$$\sum_{n=0}^{\infty} \frac{q^{3n^2}}{(q^3;q^6)_n(q^3;q^3)_n} = \frac{(q^{18},q^{24},q^{42};q^{42})_\infty}{(q^3;q^3)_\infty}, \tag{3.2}$$

$$\sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q^4;q^8)_n(q^4;q^4)_n} = \frac{(-q^4;q^4)_\infty(q^8,q^{20},q^{28};q^{28})_\infty}{(q^4;q^4)_\infty(-q^4,-q^{24};q^{28})_\infty}, \tag{3.3}$$

$$\sum_{n=0}^{\infty} \frac{q^{4n^2+4n}}{(q^2;q^4)_{n+1}(q^4;q^4)_n} = \frac{(-q^2,-q^{14},q^{16};q^{16})_\infty}{(q^4;q^4)_\infty}, \tag{3.4}$$

$$\sum_{n=0}^{\infty} \frac{q^{2n^2+4n}}{(q^2;q^4)_{n+1}(q^2;q^2)_n} = \frac{(q^4,q^{24},q^{28};q^{28})_\infty}{(q^2;q^2)_\infty}, \tag{3.5}$$

$$\sum_{n=0}^{\infty} \frac{q^{6n^2+3n}}{(q^3;q^6)_{n+1}(q^6;q^6)_n} = \frac{(q^6,q^{18},q^{24};q^{24})_\infty(q^{12},q^{36};q^{48})_\infty}{(q^3;q^3)_\infty}, \tag{3.6}$$

$$\sum_{n=0}^{\infty} \frac{q^{5n^2+5n}}{(q^5;q^{10})_{n+1}(q^5;q^5)_n} = \frac{(q^{20},q^{50},q^{70};q^{70})_\infty}{(q^5;q^5)_\infty}, \tag{3.7}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+3n}}{(q^2;q^4)_{n+1}(q^2;q^2)_n} = \frac{(-q^2;q^2)_\infty(q^6,q^8,q^{14};q^{14})_\infty(q^2,q^{26};q^{28})_\infty}{(q^2;q^2)_\infty}. \tag{3.8}$$

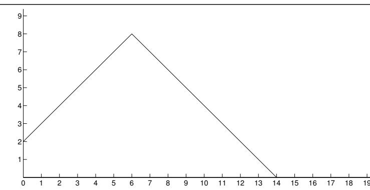
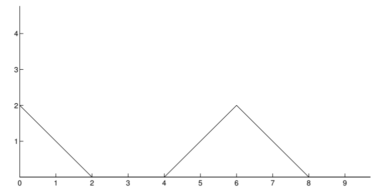
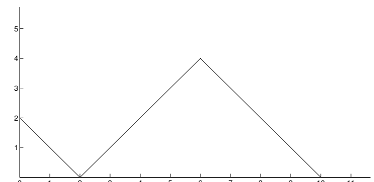
The basic series (3.1)–(3.8) have their combinatorial counterparts in form of the following theorems, respectively.

Theorem 3.1. *Let $X_1(\xi)$ represent the number of ordinary partitions of ξ into parts congruent to $\pm 2, \pm 4, \pm 6, \pm 10, \pm 12, 14 \pmod{28}$. Then*

$$X_1(\xi) = C^{(2,2)}_{(2,2)}(\xi) = D^{(2,2)}_{(2,2)}(\xi), \quad for\ all\ \xi,$$

where $C^{(2,2)}_{(2,2)}(\xi)$ and $D^{(2,2)}_{(2,2)}(\xi)$ are as described in Theorem 1.2.

The below-mentioned table describes the particular case (3.1) more precisely.

Partitions enum. by $X_1(6)$	Partitions enum. by $C^{(2,2)}_{(2,2)}(6)$	Lattice paths enum. by $D^{(2,2)}_{(2,2)}(6)$
6, 4+2, 2+2+2	6_8	
	$6_2 0_2$	
	$6_4 0_2$	

Theorem 3.2. *Let $X_2(\xi)$ represent the number of ordinary partitions of ξ into parts congruent to* $\pm 3, \pm 6, \pm 9, \pm 12, \pm 15, 21 \pmod{42}$. *Then*

$$X_2(\xi) = A^{(3,0)}_{(3,3)}(\xi) = B^{(3,0)}_{(3,3)}(\xi), \quad for\ all\ \xi,$$

where $A^{(3,0)}_{(3,3)}(\xi)$ and $B^{(3,0)}_{(3,3)}(\xi)$ are as described in Theorem 1.1.

Theorem 3.3. *Let $Y_3(\xi)$ represent the number of ordinary partitions of ξ into parts congruent to* $\pm 4, \pm 12 \pmod{28}$ *and $Z_3(\xi)$ represent the number of ordinary partitions of ξ into distinct parts congruent to* $0, \pm 8, \pm 12 \pmod{28}$. *Then*

$$X_3(\xi) = \sum_{i=0}^{\xi} Y_3(\xi - i) Z_3(i) = A^{(2,2)}_{(4,4)}(\xi) = B^{(2,2)}_{(4,4)}(\xi), \quad for\ all\ \xi,$$

where $A^{(2,2)}_{(4,4)}(\xi)$ and $B^{(2,2)}_{(4,4)}(\xi)$ are as described in Theorem 1.1.

Theorem 3.4. *Let $X_4(\xi)$ represent the number of ordinary partitions of ξ into parts congruent to* $\pm 2, \pm 8, \pm 12, \pm 14 \pmod{32}$. *Then*

$$X_4(\xi) = C^{(4,4)}_{(2,4)}(\xi) = D^{(4,4)}_{(2,4)}(\xi), \quad for\ all\ \xi,$$

where $C^{(4,4)}_{(2,4)}(\xi)$ and $D^{(4,4)}_{(2,4)}(\xi)$ are as described in Theorem 1.2.

Theorem 3.5. *Let $X_5(\xi)$ represent the number of ordinary partitions of ξ into parts congruent to* $\pm 2, \pm 6, \pm 8, \pm 10, \pm 12, 14 \pmod{28}$. *Then*

$$X_5(\xi) = C^{(2,4)}_{(2,2)}(\xi) = D^{(2,4)}_{(2,2)}(\xi), \quad for\ all\ \xi,$$

where $C^{(2,4)}_{(2,2)}(\xi)$ *and* $D^{(2,4)}_{(2,2)}(\xi)$ *are as described in Theorem 1.2.*

Theorem 3.6. *Let* $X_6(\xi)$ *represent the number of ordinary partitions of* ξ *into parts congruent to* $\pm 3, \pm 9, \pm 15, \pm 21 \pmod{48}$*. Then*

$$X_6(\xi) = C^{(6,3)}_{(3,6)}(\xi) = D^{(6,3)}_{(3,6)}(\xi), \quad for\ all\ \xi,$$

where $C^{(6,3)}_{(3,6)}(\xi)$ *and* $D^{(6,3)}_{(3,6)}(\xi)$ *are as described in Theorem 1.2.*

Theorem 3.7. *Let* $X_7(\xi)$ *represent the number of ordinary partitions of* ξ *into parts congruent to* $\pm 5, \pm 10, \pm 15, \pm 25, \pm 30, 35 \pmod{70}$*. Then*

$$X_7(\xi) = C^{(5,5)}_{(5,5)}(\xi) = D^{(5,5)}_{(5,5)}(\xi), \quad for\ all\ \xi,$$

where $C^{(5,5)}_{(5,5)}(\xi)$ *and* $D^{(5,5)}_{(5,5)}(\xi)$ *are as described in Theorem 1.2.*

Theorem 3.8. *Let* $Y_8(\xi)$ *represent the number of ordinary partitions of* ξ *into parts congruent to* $\pm 4, \pm 10, \pm 12 \pmod{28}$*, and let* $Z_8(\xi)$ *represent the number of ordinary partitions of* ξ *into distinct parts congruent to* $0, \pm 2, \pm 4, \pm 6, \pm 8, \pm 10, \pm 12, 14 \pmod{28}$*. Then*

$$X_8(\xi) = \sum_{i=0}^{\xi} Y_8(\xi - i) Z_8(i) = C^{(1,3)}_{(2,2)}(\xi) = D^{(1,3)}_{(2,2)}(\xi), \quad for\ all\ \xi,$$

where $C^{(1,3)}_{(2,2)}(\xi)$ *and* $D^{(1,3)}_{(2,2)}(\xi)$ *are as described in Theorem 1.2.*

4. Conclusion

In the present paper, the interpretation of two generalized basic series in terms of $(n+t)$-color partitions and weighted lattice paths enables us to provide two infinite classes of combinatorial identities. Our results not only generalize the results we found in the literature (Agarwal [2, 3], Agarwal and Goyal [6, 13], Sareen and Rana [16]), but also provide combinatorial interpretations of entirely new Rogers-Ramanujan type identities. So, the obvious question that arises here is, can we obtain such generalizations by using other combinatorial objects, viz., associated lattice paths, F-partitions, Bender and Knuth matrices, anti-hook differences etc.

5. Acknowledgment

We are thankful to the anonymous referees for their valuable suggestions which lead to the better representation of the research work.

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