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(G,s)-TRANSITIVE GRAPHS OF VALENCY 15

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ABSTRACT. Let X be a finite simple undirected graph and $G \leq \mathrm{Aut}(X)$. If G is transitive on the set of s-arcs but not on the set of $(s+1)$-arcs of X, then X is called (G,s)-transitive. In this paper, we determine the structure of the vertex-stabilizer G_v when X is a connected (G,s)-transitive graph of valency 15. We also give some examples to show that each type of G_v with $s \geq 2$, can be realized.

1. Introduction

In this paper, all graphs are finite, undirected, and simple, i.e. without loops or multiple edges. For a graph X, we use $V(X)$, $E(X)$, and $\mathrm{Aut}(X)$ to denote its vertex set, edge set, and full automorphism group, respectively. For $u,v \in V(X)$, $\{u,v\}$ is the edge incident to u and v in X. The set of all vertices adjacent to v is denoted by $X_1(v)$. Let $G \leq \mathrm{Aut}(X)$. We denote the vertex stabilizer of $v \in V(X)$ in G by G_v. Denote by $G_v^{X_1(v)}$ the constituent of G_v acting on $X_1(v)$ and by G_v^* the kernel of G_v acting on $X_1(v)$. Then, $G_v^{X_1(v)} \cong G_v/G_v^*$. For an edge $\{u,v\} \in E(X)$, we write $G_{uv} = G_u \cap G_v$ and $G_{uv}^* = G_u^* \cap G_v^*$.

For each integer $s \geq 0$, an *s-arc* of X is a sequence $(v_0, v_1, \ldots, v_{s-1}, v_s)$ of vertices such that v_{i-1} is adjacent to v_i and $v_{i-1} \neq v_{i+1}$ for all admissible i. For a subgroup $G \leq \mathrm{Aut}(X)$, X is said to be *(G,s)-arc-transitive* if G is transitive on the set of s-arcs in X. A (G,s)-arc-transitive graph which is not $(G,s+1)$-arc-transitive is called *(G,s)-transitive*. A graph X is called *s-arc-transitive* or *s-transitive* if it is $(\mathrm{Aut}(X),s)$-arc-transitive or $(\mathrm{Aut}(X),s)$-transitive, respectively. In particular, X is said to be *vertex-transitive* or *symmetric* if it is $(\mathrm{Aut}(X),0)$-arc-transitive or $(\mathrm{Aut}(X),1)$-arc-transitive, respectively.

As we all know, a graph X is G-arc-transitive if and only if G is transitive on $V(X)$ and G_v is transitive on $X_1(v)$. So the structure of G_v plays an important role in the study of such graphs. A G-arc-transitive graph X

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is called *G-locally-primitive* if G_v acts on $X_1(v)$ primitively, that is, the induced permutation group $G_v^{X_1(v)}$ is primitive of degree $|X_1(v)|$.

Interest in s-transitive graphs stems from a beautiful result of Tutte [20] in 1947 who proved that for any s-transitive cubic graph, $s \leq 5$. Tutte's Theorem was generalized in 1981 by Weiss [22] who proved that there exists no finite s-transitive graph for $s = 6$ and $s \geq 8$. This is a very deep result, the proof of which depends on the classification of the finite simple groups. Note that the only connected graphs of valency two are cycles which are s-arc-transitive for any positive integer s. So the valency of a s-transitive graph is greater than 2. Let X be a connected (G, s)-transitive graph for some $s \geq 1$. Up to now, we know the structure of G_v when X has prime or twice a prime valency [5, 24, 17, 9, 11, 12, 10, 13, 16]. Note that in general, the order of G_v may be unbounded for some valencies. For instance, there are infinite families of connected 1-arc-transitive graphs of valency 4 with arbitrary large vertex-stabilizers [2, 18]. Although, when the valency of X is prime or $s \geq 2$, it is well-known that $|G_v|$ is bounded above.

Let p be a prime and n a positive integer. The symmetric and alternating group on n letters are denoted by S_n and A_n, respectively. We use both n and $\mathbb{Z}_n$ to denote the cyclic group of order n. The expression $[n]$ denotes an unspecified group of order n while p^n denotes an elementary abelian group of order p^n. In particular, for $q = p^f$, some times we also let q^m be p^{fm} with $m \geq 1$. For two groups M and N, we denote by $N.M$ an extension of N by M, and $N : M$ stands for a semidirect product of N by M. We use $\mathbf{O}_p(M)$ to denote the largest normal p-subgroup of M. The remaining notation is standard and hopefully self-explanatory. For group and graph theoretic concepts not defined here, we refer the reader to [4, 8]. Our main result is as follows.

Theorem 1.1. *Let X be a finite connected (G, s)-transitive graph of valency 15 for some $G \leq \mathrm{Aut}(X)$ and $s \geq 1$. Let $v \in V(X)$. Then $s \leq 3$ and one of the following holds.*

- (i) *$s = 1$, G_v is a $\{2, 3, 5\}$-group. In particular, if X is a G-locally-primitive graph, then $G_v \cong A_6$, S_6, $\mathbf{O}_2(G_v).N.A_6$ or $\mathbf{O}_2(G_v).N.S_6$ where $N \trianglelefteq S_3$.*
- (ii) *$s = 2$, $G_v \cong A_7$, $\mathrm{PSL}_4(2)$, A_{15}, S_{15}, $2^r : \mathrm{SL}_4(2)$ where $r \in \{4, 5, 6\}$, $[2^{11}] : \mathrm{SL}_4(2)$ or $[2^{14}] : \mathrm{SL}_4(2)$.*
- (iii) *$s = 3$, $G_v \cong A_7 \times \mathrm{PSL}_2(7)$, $\mathrm{PSL}_4(2) \times (2^3 : \mathrm{PSL}_3(2))$, $A_{15} \times A_{14}$, $S_{15} \times S_{14}$, $(A_{15} \times A_{14}) : 2$ with $A_{15} : 2 = S_{15}$ and $A_{14} : 2 = S_{14}$ or $2^{12} : (\mathrm{SL}_4(2) \times \mathrm{SL}_3(2))$.*

The paper is organized as follows. In section 2, we collect some background results which are required in the paper. The proof of Theorem 1.1 is sketched in section 3 and completed in section 4. Finally, in section 5 we give some examples to show that each type of G_v with $s \geq 2$ in Theorem 1.1 can be realized.

Several facts motivate us to characterize the structure of vertex stabilizers of symmetric graphs of special valency. When dealing with symmetric graphs, the goal is to gain as much information as possible about the structural properties of their vertex-stabilizers. The results of this paper would be useful to classify the symmetric graphs of valency 15 and study other graph properties of symmetric graphs.

2. Preliminaries

In this section, we collect some preliminary results which will be needed throughout the paper. Let H be a transitive permutation group on a set Ω and let $\alpha \in \Omega$. If the stablizer H_α is transitive on $\Omega \backslash \{\alpha\}$ then H is called *2-transitive* on Ω. The following proposition is about sufficient and necessary conditions for symmetric graphs. Its proof is straightforward and left to the reader.

Proposition 2.1. *Let X be a graph and $G \leq \mathrm{Aut}(X)$. Then we have;*

(i) *X is G-arc-transitive if and only if X is G-vertex-transitive and G_v is transitive on $X_1(v)$ for each $v \in V(X)$.*
(ii) *X is $(G,2)$-arc-transitive if and only if X is G-vertex-transitive and G_v is 2-transitive on $X_1(v)$ for each $v \in V(X)$.*

The next proposition is from [25], Lemma 2.7.

Proposition 2.2. *Let X be a connected symmetric graph. Let $\{u,v\} \in E(X)$ and $A = \mathrm{Aut}(X)$. Assume that $H \leq A_u$ is transitive on $X_1(u)$ and $K \leq A_v$ is transitive on $X_1(v)$. Then $\langle H, K\rangle \leq A$ is transitive on $E(X)$.*

In view of [7], Lemma 2.1, we have the following lemma.

Lemma 2.3. *Let X be a connected G-vertex-transitive graph for some $G \leq \mathrm{Aut}(X)$. If a prime p divides $|G_v|$ for some $v \in V(X)$, then p divides $|G_v^{X_1(v)}|$.*

The proof of the next lemma is straightforward and left to the reader.

Lemma 2.4. *Let X be a (G,s)-arc-transitive graph for some $G \leq \mathrm{Aut}(X)$ and $s \geq 1$. Let $\{u,v\} \in E(X)$. Then we have;*

(i) $G_v \cong G_v^*.G_v^{X_1(v)} \cong (G_{uv}^*.(G_v^*)^{X_1(u)}).G_v^{X_1(v)}$.
(ii) ${G_v^*}^{X_1(u)} \trianglelefteq G_{uv}^{X_1(u)} \cong G_{uv}^{X_1(v)}$.

Combining a result from [6], Corollary 2.3 and [16], Theorem 4.3, we have the following lemma.

Lemma 2.5. *Let X be a connected graph, $\{u,v\} \in E(X)$ and $G \leq \mathrm{Aut}(X)$. Suppose that X is a G-locally-primitive arc-transitive graph. Then G_{uv}^* is a p-group for some prime p. Moreover, either $G_{uv}^* = 1$ or $\mathbf{O}_p((G_v^*)^{X_1(u)}) \neq 1$ and $\mathbf{O}_p(({G_v}^{X_1(v)})_u) \neq 1$.*

The next theorem is formulated from [21, 22, 23].

Theorem 2.6. *Let X be a connected (G,s)-transitive graph with $s \geq 2$ and let $\{u,v\} \in E(X)$. Then one of the following holds.*

(i) $s \leq 3$, $G_{uv}^* = 1$ *and* $G_v^* \cong G_v^{*X_1(u)} \trianglelefteq G_{uv}^{X_1(u)} \cong G_{uv}^{X_1(v)}$.

(ii) G_{uv}^* *is a nontrivial p-group,* $\mathrm{PSL}_d(q) \trianglelefteq G_v^{X_1(v)}$, $q = p^f$, $|X_1(v)| = \frac{q^d-1}{q-1}$ *,and either $d \geq 3$ and $s \in \{2,3\}$, or $d = 2$, $s \geq 4$ and one of the following holds.*

(a) $s = 4$ *and* $G_v \cong q^2 : \frac{1}{(3,q-1)} GL_2(q).[e]$ *with* $e|(3, q-1)f$.

(b) $s = 5$, $p = 2$, *and* $G_v \cong q^3 : GL_2(q).\mathbb{Z}_b$ *with* $b|f$.

(c) $s = 7$, $p = 3$ *and* $G_v \cong (q^2 \times q^{1+2}) : GL_2(q).\mathbb{Z}_b$ *with* $b|f$.

We finish this section with the following lemma which will be used repeatedly in section 3.

Lemma 2.7. *Let X be a $(G,2)$-arc-transitive graph and let $\{u,v\} \in E(X)$. Set $C = C_{G_v}(G_v^*)$. Let $G_{uv}^* = 1$ and $G_v^* \neq 1$. Then the following statements hold.*

(i) $G_u^* \leq C$ *and* $C \nleq G_v^*$.

(ii) $C^{X_1(v)} \trianglelefteq G_v^{X_1(v)}$.

(iii) *If* $Z(G_v^*) = 1$, *then* $G_v^{X_1(v)}$ *has a normal subgroup isomorphic to* C *and* $C \times G_v^* \trianglelefteq G_v$.

(iv) *If* $G_v^{X_1(v)}$ *is a nonabelian simple group and* $Z(G_v^*) = 1$, *then* $C \cong G_v^{X_1(v)}$ *and* $G_v \cong C \times G_v^*$.

Proof. (i) Let $x, u \in X_1(v)$, $\alpha \in G_v^*$ and $\beta \in G_u^*$. Then $x^{\alpha^{-1}\beta^{-1}\alpha\beta} = x^{\beta^{-1}\alpha\beta} = (x^{\beta^{-1}})^{\alpha\beta} = (x^{\beta^{-1}})^{\beta} = x$. Thus $[G_u^*, G_v^*] \leq G_v^*$. By the same argument we conclude that $[G_u^*, G_v^*] \leq G_u^*$. Thus $[G_u^*, G_v^*] \leq G_{uv}^* = 1$, which implies that $G_u^* \leq C$.

Suppose that $C \leq G_v^*$. Since $G_u^* \leq C$ and $G_u^* \cong G_v^*$, we get that $G_u^* = G_v^*$, a contradiction. Thus $C \nleq G_v^*$ and (i) is proved.

(ii) Since $C \trianglelefteq G_v$, we have $C^{X_1(v)} \cong C/C \cap G_v^* \cong CG_v^*/G_v^* \trianglelefteq G_v/G_v^* \cong G_v^{X_1(v)}$ which proves (ii).

(iii) Since $Z(G_v^*) = C \cap G_v^* = 1$, by (ii) we can easily conclude (iii).

(iv) As $G_v^{X_1(v)}$ is a simple group, by (iii) $C = 1$ or $C \cong G_v^{X_1(v)}$. But according to (i), the former can not be occurred. Therefore $C \cong G_v^{X_1(v)}$. Now by (ii), $G_v \cong C \times G_v^*$ which completes the proof of (iv). □

3. Main results

In this section, we give the main results of this paper. In fact, we investigate the truth of Theorem 1.1 by a sequence of lemmas and theorems.

Let X be a connected (G,s)-transitive graph of valency 15 for some $G \leq \mathrm{Aut}(X)$ and $s \geq 1$. Since X has valency 15, $G_v^{X_1(v)} \cong G_v/G_v^* \leq S_{15}$ is a transitive permutation group of degree 15. We first prove a reduced form of Theorem 1.1 in the case that $s = 1$.

Theorem 3.1. *Let X be a finite connected $(G, 1)$-transitive graph of valency 15 for some $G \leq \mathrm{Aut}(X)$. Then G_v is a $\{2, 3, 5\}$-group. In particular, if X is a G-locally-primitive graph, then $G_v \cong A_6$, S_6, $\mathbf{O}_2(G_v).N.A_6$ or $\mathbf{O}_2(G_v).N.S_6$ where $N \trianglelefteq S_3$.*

Proof. By Proposition 2.1, $G_v^{X_1(v)}$ is a transitive but not a 2-transitive permutation group of degree 15. Thus $15||G_v^{X_1(v)}|$. Let p be a prime factor of $|G_v|$. By Lemma 2.3, $p||G_v^{X_1(v)}|$. Thus $p \leq 13$. Since by [14], we know the order of all transitive permutation groups of degree 15, it is easy to check that G_v is a $\{2, 3, 5\}$-group.

Now assume that $G_v^{X_1(v)}$ is a primitive permutation group of degree 15 and let $\{u, v\} \in E(X)$. By [15], $G_v^{X_1(v)} \cong A_6$ or S_6. Let $G_v^{X_1(v)} \cong A_6$. Then $G_{uv}^{X_1(v)\setminus\{u\}} \cong S_4$. By Lemma 2.4(ii), $(G_v^*)^{X_1(u)} \cong 1, 2^2, A_4$ or S_4. If $(G_v^*)^{X_1(u)} = 1$, then $G_{uv}^* = G_v^*$. Thus $G_v^* \leq G_u^*$ and hence $G_v^* = G_u^*$ because the transitivity of G on $V(X)$ implies $|G_v^*| = |G_u^*|$. Since $G_v^* \trianglelefteq G_v$ and $G_u^* \trianglelefteq G_u$, we have $G_v^* \trianglelefteq \langle G_u, G_v\rangle$. By Proposition 2.2, $\langle G_u, G_v\rangle$ is transitive on $E(X)$. Now since G_v^* fixes the edge $\{u, v\}$, we may deduce that G_v^* fixes each edge in X and so $G_v^* = 1$. Therefore $G_v \cong A_6$. If $(G_v^*)^{X_1(u)} \cong 2^2$, A_4 or S_4, then by Lemma 2.5 G_{uv}^* is a 2-group. Note that $G_v/G_v^* \cong A_6$. It follows that $\mathbf{O}_2(G_v) = \mathbf{O}_2(G_v^*) \cong G_{uv}^*.\mathbf{O}_2((G_v^*)^{X_1(u)}) \cong G_{uv}^*.2^2$. By Lemma 2.4(i), we conclude that $G_v \cong \mathbf{O}_2(G_v).N.A_6$ where $N \trianglelefteq S_3$.

Finally, let $G_v^{X_1(v)} \cong S_6$. Then $G_{uv}^{X_1(v)\setminus\{u\}} \cong S_4 \times 2$ and $(G_v^*)^{X_1(u)} \cong 1$, 2, 2^2, 2^3, A_4, S_4, $A_4 \times 2$ or $S_4 \times 2$. If $(G_v^*)^{X_1(u)} \cong 1$, then $G_v^* = 1$ and so $G_v \cong S_6$. Now assume that $(G_v^*)^{X_1(u)} \neq 1$. It follows that $(G_v^*)^{X_1(u)} \cong 2$, 2^2, 2^3, A_4, S_4, $A_4 \times 2$, or $S_4 \times 2$. By Lemma 2.4, G_{uv}^* is a 2-group. Note that $G_v/G_v^* \cong S_6$. It follows that $\mathbf{O}_2(G_v) = \mathbf{O}_2(G_v^*) \cong G_{uv}^*.\mathbf{O}_2((G_v^*)^{X_1(u)})$. Recall that by Lemma 2.5(i), $G_v \cong (G_{uv}^*.(G_v^*)^{X_1(u)}).S_6$. Now we consider each possibilities cases of $(G_v^*)^{X_1(u)}$ and determine the structure of G_v. If $(G_v^*)^{X_1(u)} \cong 2$ then $\mathbf{O}_2(G_v^*) \cong G_{uv}^*.2$ and $G_v \cong \mathbf{O}_2(G_v).S_6$. If $(G_v^*)^{X_1(u)} \cong 2^2$ then $\mathbf{O}_2(G_v^*) \cong G_{uv}^*.2^2$ and $G_v \cong \mathbf{O}_2(G_v).S_6$. If $(G_v^*)^{X_1(u)} \cong 2^3$ then $\mathbf{O}_2(G_v^*) \cong G_{uv}^*.2^3$ and $G_v \cong \mathbf{O}_2(G_v).S_6$. If $(G_v^*)^{X_1(u)} \cong A_4$ then $\mathbf{O}_2(G_v^*) \cong G_{uv}^*.2^2$ and $G_v \cong \mathbf{O}_2(G_v).\mathbb{Z}_3.S_6$. If $(G_v^*)^{X_1(u)} \cong S_4$ then $\mathbf{O}_2(G_v^*) \cong G_{uv}^*.2^2$ and $G_v \cong \mathbf{O}_2(G_v).S_3.S_6$. If $(G_v^*)^{X_1(u)} \cong A_4 \times 2$ then $\mathbf{O}_2(G_v^*) \cong G_{uv}^*.2^3$ and $G_v \cong \mathbf{O}_2(G_v).\mathbb{Z}_3.S_6$. If $(G_v^*)^{X_1(u)} \cong S_4 \times 2$ then $\mathbf{O}_2(G_v^*) \cong G_{uv}^*.2^3$ and $G_v \cong \mathbf{O}_2(G_v).S_3.S_6$. So we can summarize that $G_v \cong \mathbf{O}_2(G_v).N.S_6$ where $N \trianglelefteq S_3$. This completes the proof. □

Remark: By [3], Proposition 4.1, we know that the order of $\mathbf{O}_2(G_v)$ is bounded above.

In the remainder of this section, we assume that X is a (G, s)-transitive graph for some $s \geq 2$. Thus by Proposition 2.1(ii), $G_v^{X_1(v)}$ is a 2-transitive permutation group of degree 15. In view of [15] and [4], Appendix B, we have the following observation.

Proposition 3.3. *Let X be a $(G,2)$-arc-transitive graph of valency 15 and let $\{u,v\} \in E(X)$. Then we have;*

(i) ${G_v}^{X_1(v)} \cong A_7$, $\mathrm{PSL}_4(2)$, A_{15} *or* S_{15}.
(ii) $G_{uv}^{X_1(v)\setminus\{u\}} \cong \mathrm{PSL}_2(7)$, $2^3 : \mathrm{PSL}_3(2)$, A_{14} *or* S_{14}.

By Theorem 2.5, G^*_{uv} is a p-group. Thus according to Theorem 2.6, we may split the proof into three cases: $G^*_v = 1$, $G^*_{uv} = 1 \neq G^*_v$ and $G^*_{uv} \neq 1$.

Theorem 3.4. *Let X be a (G,s)-transitive graph of valency 15 with $s \geq 2$ and $G \leq \mathrm{Aut}(X)$. Suppose that $G^*_v = 1$. Then $s = 2$ and $G_v \cong A_7$, $\mathrm{PSL}_4(2)$, A_{15} or S_{15}.*

Proof. Since $G^*_v = 1$, we have $G_v^{X_1(v)} \cong G_v/G^*_v \cong G_v$. By Proposition 3.3, $G_v \cong A_7, \mathrm{PSL}_4(2), A_{15}$ or S_{15}. Now let u and w be two distinct vertices in $X_1(v)$. Then we have, $G_{uv} \cong \mathrm{PSL}_2(7), 2^3 : \mathrm{PSL}_3(2), A_{14}$ or S_{14} and $G_{uvw} \cong A_4, 2^3 : A_4, A_{13}$ or S_{13}. So G_{uvw} cannot act transitively on $X_1(u)\setminus\{v\}$ because $|X_1(u)\setminus\{v\}| = 14$. It follows that X is not a $(G,3)$-arc transitive graph. Therefore $s = 2$ which completes the proof. □

In the next two lemmas, suppose that $G^*_{uv} = 1 \neq G^*_v$. By Proposition 3.3 and Theorem 2.6(i), if $G_v^{X_1(v)} \cong A_7$ or A_{15}, then $G^*_v \cong \mathrm{PSL}_2(7)$ or A_{14}. Thus in what follows, we determine G^*_v for the cases $G_v^{X_1(v)} \cong \mathrm{PSL}_4(2)$ or S_{15}. Recall that according to Lemma 2.7, for simplicity of notation, we write C instead of $C_{G_v}(G^*_v)$.

Lemma 3.5. *Let $G_v^{X_1(v)} \cong \mathrm{PSL}_4(2)$. Then $G^*_v \cong 2^3 : \mathrm{PSL}_3(2)$.*

Proof. Since $G_{uv}^{X_1(v)\setminus\{u\}} \cong 2^3 : \mathrm{PSL}_3(2)$, we have by Lemma 2.7(i), $G^*_v \cong 2^3$ or $2^3 : \mathrm{PSL}_3(2)$. We claim that $G^*_v \cong 2^3 : \mathrm{PSL}_3(2)$. If not, then G^*_v acts on $X_1(u)\setminus\{v\}$ as 2^3. It means that $G^*_v \cong G^*_u \cong G^*_u/G^*_{uv} \cong {G^*_u}^{X_1(v)\setminus\{u\}} \cong 2^3$. As $\mathrm{PSL}_4(2)$ is a nonabelian simple group, by Lemma 2.7(ii), we deduce that $C^{X_1(v)} \cong G_v^{X_1(v)}$. Thus $C_u^{X_1(v)\setminus\{u\}} \cong G_{uv}^{X_1(v)\setminus\{u\}} \cong 2^3 : \mathrm{PSL}_3(2)$. Since $G^*_u < C_u$, we may take an element $h \in C_u \setminus G^*_u$ such that h fixes u with a 3-cycle on $X_1(v)\setminus\{u\}$. Since h commutes with every element in G^*_v, we have $[G^*_v, \langle h\rangle] = [G^*_v G^*_u/G^*_u, \langle h\rangle\, G^*_u/G^*_u] = 1$. It means that $\langle h\rangle\, G^*_u/G^*_u$ centralizes $G^*_v G^*_u/G^*_u$ in G_u/G^*_u. But 2^3 is an abelian group which implies that 2^3 is self-centralizing in $\mathrm{PSL}_4(2)$. Therefore $\langle h\rangle\, G^*_u/G^*_u \leq G^*_v G^*_u/G^*_u \cong G^*_v \cong 2^3$. It follows that $h = h_v h_u$ with $h_v \in G^*_v$ and $h_u \in G^*_u$. Since h induces a 3-cycle on $X_1(v)\setminus\{u\}$ and h_v fixes each vertex in $X_1(v)$, therefore h_u induces a 3-cycle on $X_1(v)\setminus\{u\}$. It follows that ${G^*_u}^{X_1(v)\setminus\{u\}}$ has an element of order 3, which is impossible because ${G^*_u}^{X_1(v)\setminus\{u\}} \cong 2^3$. Therefore $G^*_v \cong 2^3 : \mathrm{PSL}_3(2)$ and lemma is proved. □

Lemma 3.6. *Let $G_v^{X_1(v)} \cong S_{15}$. Then the following statements hold.*

(i) *If $G^*_v \cong S_{14}$, then $C \cong S_{15}$.*
(ii) *If $G^*_v \cong A_{14}$, then $C \cong A_{15}$.*

Proof. Let $G_v^{X_1(v)} \cong S_{15}$. Then $G_{uv}^{X_1(v)\setminus\{u\}} \cong S_{14}$. By Theorem 2.6(i), $G_v^* \cong S_{14}$ or A_{14} and by Lemma 2.7(ii), $C \cong S_{15}$ or A_{15}. Suppose that $G_v^* \cong S_{14}$. Since $S_{14} \cong G_v^* \cong G_u^* \leq C$, we get that $C \cong S_{15}$ and (i) is proved.

Suppose that $G_v^* \cong A_{14}$. If $C \cong S_{15}$, that is, $C \cong G_v^{X_1(v)}$, then $C_u \cong G_{uv}^{X_1(v)\setminus\{u\}} \cong S_{14}$. Now $A_{14} \cong G_u^* \leq C \cong S_{15}$ and $S_{14} \cong C_u \leq C \cong S_{15}$, which implies that there exist an element $h \in C_u \setminus G_u^*$. Since $h \in C_u$, thus $u^h = u$ and h commutes with every element of G_v^*, that is $[G_v^*, \langle h \rangle] = 1$. Now by isomorphism we have $[G_v^* G_u^*/G_u^*, \langle h \rangle G_u^*/G_u^*] = 1$. It means that $\langle h \rangle G_u^*/G_u^*$ centralizes $G_v^* G_u^*/G_u^*$ in G_u/G_u^* which is impossible because $G_v^* G_u^*/G_u^* \cong G_v^* \cong A_{14}$ and $G_u/G_u^* \cong S_{15}$. Thus $C \cong A_{15}$ and (ii) is proved. □

Now we are ready to determine the structure of G_v when $G_{uv}^* = 1 \neq G_v^*$. Recall that $s \geq 2$ and $G_v^{X_1(v)} \cong A_7$, $\mathrm{PSL}_4(2)$, A_{15} or S_{14}.

Theorem 3.7. *Let X be a (G, s)-transitive graph of valency 15 where $G \leq \mathrm{Aut}(X)$ and $s \geq 2$. Let $\{u, v\} \in E(X)$ and $G_{uv}^* = 1 \neq G_v^*$. Then $s = 3$ and $G_v \cong A_7 \times \mathrm{PSL}_2(7)$, $\mathrm{PSL}_4(2) \times (2^3 : \mathrm{PSL}_3(2))$, $A_{15} \times A_{14}$, $S_{15} \times S_{14}$ or $(A_{15} \times A_{14}) : 2$ with $A_{15} : 2 = S_{15}$ and $A_{14} : 2 = S_{14}$.*

Proof. Suppose that $G_v^{X_1(v)} \cong A_7$. Then $G_{uv}^{X_1(v)\setminus\{u\}} \cong \mathrm{PSL}_2(7)$. By Lemma 2.6(i), $G_v^* \cong \mathrm{PSL}_2(7)$. By Lemma 2.7(iv), we have $C \cong A_7$ and $G_v \cong A_7 \times \mathrm{PSL}_2(7)$.

Suppose that $G_v^{X_1(v)} \cong \mathrm{PSL}_4(2)$. Then $G_{uv}^{X_1(v)\setminus\{u\}} \cong 2^3 : \mathrm{PSL}_3(2)$. By Lemma 3.5, $G_v^* \cong 2^3 : \mathrm{PSL}_3(2)$. By Lemma 2.7(iv), $C \cong \mathrm{PSL}_4(2)$ and $G_v \cong \mathrm{PSL}_4(2) \times (2^3 : \mathrm{PSL}_3(2))$.

Suppose that $G_v^{X_1(v)} \cong A_{15}$. Then $G_{uv}^{X_1(v)\setminus\{u\}} \cong A_{14}$ and $G_v^* \cong A_{14}$. Therefore $C \cong A_{15}$ and $G_v \cong A_{15} \times A_{14}$.

Finally, suppose that $G_v^{X_1(v)} \cong S_{15}$. Then $G_{uv}^{X_1(v)\setminus\{u\}} \cong S_{14}$ and $G_v^* \cong A_{14}$ or S_{14}. If $G_v^* \cong S_{14}$, then by Lemma 3.6(i), $C \cong G_v^{X_1(v)} \cong S_{15}$. Now according to Lemma 2.7(ii), $G_v \cong S_{15} \times S_{14}$. If $G_v^* \cong A_{14}$, then by Lemma 3.6(ii), $C \cong A_{15}$. By Lemma 2.7(iii), $CG_v^* \cong C \times G_v^* \cong A_{15} \times A_{14} \trianglelefteq G_v$. Note that $A_{15} \cong CG_v^*/G_v^* \trianglelefteq G_v/G_v^* \cong S_{15}$. Therefore $|G_v : C \times G_v^*| = 2$ and $G_v \cong (A_{15} \times A_{14}).2$. Now we find an involution $g \in G_v$ such that $C \langle g \rangle = S_{15}$ and $G_v^* \langle g \rangle = S_{14}$. Since $G_v^{X_1(v)} \cong S_{15}$, there exists an element $g_1 \in G_{uv}$ such that g_1 induces a transposition on $X_1(v)$. Recall that $G_v^* \cong A_{14}$ and G_v^* acts faithfully on $X_1(u)$ because $G_{uv}^* = 1$. So there exists an element $g_2 \in G_v^*$ such that $g_1 g_2$ induces the identity or a transposition on $X_1(u) \setminus \{v\}$. For the former, $g_1 g_2 \in G_u^*$ and $g_1 g_2$ induces the same transposition as g_1 on $X_1(v)$, contrary to the fact that $G_u^* \cong A_{14}$ acting faithfully on $X_1(v)$. Set $g = g_1 g_2$. Thus, $g \in G_{uv}$ induces a transposition on both $X_1(u)$ and $X_1(v)$. Furthermore, $g^2 \in G_{uv}^* = 1$ and hence g is an involution. It follows that $C \langle g \rangle = S_{15}$ and $G_v^* \langle g \rangle = S_{14}$ because $G_v^* \trianglelefteq G_v$ and $C \trianglelefteq G_v$. Thus $G_v \cong (A_{15} \times A_{14}) : 2$ with $A_{14} : 2 = S_{14}$ and $A_{15} : 2 = S_{15}$.

Now we determine the transitivity of G_v. Note that in all of the cases G_v^* acts transitively on $X_1(u)\setminus\{v\}$. Recall that $G_v^* \neq 1$ and $G_{uv}^* = 1$. So we have $G_u^* \neq G_v^*$. This implies that $(X_1(u)\setminus\{v\}) \cap (X_1(v)\setminus\{u\}) = \emptyset$, that is, X has no 3-cycles. Let (v_0, v_1, v_2, v_3) and (u_0, u_1, u_2, u_3) be two 3-arcs in X. Since $s \geq 2$, there exists an element $g \in G$ such that $(v_0, v_1, v_2)^g = (u_0, u_1, u_2)$. Clearly $(v_0, v_1, v_2, v_3)^g = (u_0, u_1, u_2, v_3^g)$ is a 3-arc. Note that $G_{u_1}^*$ fixes u_0, u_1 and u_2 and acts on $X_1(u_2)\setminus\{u_1\}$ transitively. Thus, there exists an element $h \in G_{u_1}^*$ such that $v_3^{gh} = u_3$, that is, $(v_0, v_1, v_2, v_3)^{gh} = (u_0, u_1, u_2, u_3)$. It follows that X is $(G, 3)$-arc-transitive. Since by Theorem 2.6(i), $s \leq 3$, we get that $s = 3$ which completes the proof. □

Finally, in the last theorem, we deal with the case $G_{uv}^* \neq 1$.

Theorem 3.8. *Let X be a (G, s)-transitive graph of valency 15 where $G \leq \mathrm{Aut}(X)$ and $s \geq 2$. Let $\{u, v\} \in E(X)$ and $G_{uv}^* \neq 1$. Then one of the following holds.*

(i) $s = 2$, $G_v \cong 2^r : \mathrm{SL}_4(2)$ *where* $r \in \{4, 5, 6\}$, $[2^{11}] : \mathrm{SL}_4(2)$ *or* $[2^{14}] : \mathrm{SL}_4(2)$.
(ii) $s = 3$, $G_v \cong 2^{12} : (\mathrm{SL}_4(2) \times \mathrm{SL}_3(2))$.

Proof. By Theorem 2.6(ii) and Proposition 3.3, we have $G_v^{X_1(v)} \cong G_v/G_v^* \cong \mathrm{PSL}_4(2) \cong \mathrm{SL}_4(2)$ and G_{uv}^* is a 2-group. Let $G_v^{[2]}$ be the pointwise stabilizer in G of the set of vertices of X which are at a distance at most 2 from v. By ([19], p.314, lines 24-25), $O_2(G_v)/G_v^{[2]}$ is elementary abelian and $s = 2$ or 3. Thus for the case $G_v^{[2]} = 1$, we get that $O_2(G_v)$ is elementary abelian.

Suppose that $s = 2$. If $G_v^{[2]} = 1$, then by [19], p.315, lines 13-16, we have $G_v^* = O_2(G_v)$ and $G_v/O_2(G_v)$ is isomorphic to a quotient group of $\mathrm{SL}_4(2)$. By [19], p.316, lines 28-32, $O_2(G_v) \cong 2^4$ or 2^6. Since $G_v/G_v^* \cong \mathrm{SL}_4(2)$ acts naturally on $X_1(v)$ inducing the projective special linear group, we get that $G_v \cong 2^4 : \mathrm{SL}_4(2)$ or $G_v \cong 2^6 : \mathrm{SL}_4(2)$. If $G_v^{[2]} \neq 1$, then by [19], p.319-321, (d), (e), (f), we get that $G_v \cong 2^5 : \mathrm{SL}_4(2)$, $[2^{11}] : \mathrm{SL}_4(2)$ or $[2^{14}] : \mathrm{SL}_4(2)$ and (i) is proved.

Suppose that $s = 3$. By [19], p.318, line 24, $G_v^{[2]} = 1$. By [19], p.316, lines 17-19, $O_2(G_v) \cong 2^{12}$, $H_1 \cong \mathrm{SL}_4(2)$, $H_2 \cong \mathrm{SL}_3(2)$ and $H = O_2(G_v) : (H_1 \times H_2) \trianglelefteq G_v$. It follows that $H^{X_1(v)} \cong H/H \cap G_v^* \cong HG_v^*/G_v^* \trianglelefteq G_v/G_v^* \cong {G_v}^{X_1(v)} \cong \mathrm{PSL}_4(2)$. Since $\mathrm{PSL}_4(2)$ is a simple group, $H^{X_1(v)} = 1$ or $H^{X_1(v)} = {G_v}^{X_1(v)}$. But clearly the former can not be occurred, thus $H^{X_1(v)} = {G_v}^{X_1(v)} \cong \mathrm{SL}_4(2)$. Now to prove $G_v = H$, it is sufficient to show that $H_v^* = H \cap G_v^* = G_v^*$. Recall that $G_{uv}^* \leq O_2(G_v) \leq H$ which implies $G_{uv}^* = H_{uv}^*$. Note that $H_v^* \cong 2^{12} : \mathrm{SL}_3(2)$. Thus $\mathrm{SL}_3(2) \cong N \leq H_v^*/H_{uv}^* = H_v^*/G_{uv}^* \trianglelefteq G_v^*/G_{uv}^* \cong G_v^*G_u^*/G_u^* \trianglelefteq G_{uv}/G_u^* \cong G_{uv}^{X_1(v)\setminus\{u\}} \cong 2^3 : \mathrm{PSL}_3(2)$. It forces $H_v^*/H_{uv}^* = H_v^*/G_{uv}^* = G_v^*/G_{uv}^* \cong 2^3 : \mathrm{PSL}_3(2)$. Therefore $H_v^* = G_v^*$ as required and $G_v \cong 2^{12} : (\mathrm{SL}_4(2) \times \mathrm{SL}_3(2))$. □

4. Proof of Theorem 1.1

We now draw all our strings together and prove Theorem 1.1.

Proof of Theorem 1.1. Let X be a connected (G, s)-transitive graph of valency 15 for some $G \leq \mathrm{Aut}(X)$ and $s \geq 1$. Since X has valency 15, $G_v^{X_1(v)} \cong G_v/G_v^* \leq S_{15}$ is a transitive permutation group of degree 15. By Theorem 3.1, part (i) holds. Thus in what follows, we may assume that $s \geq 2$. By Proposition 3.3, $G_v^{X_1(v)} \cong A_7, \mathrm{PSL}_4(2), A_{15}$ or S_{15}. Clearly, By Theorem 2.6 we get that $s = 2$ or 3. Now combining Theorems 3.4, 3.7 and 3.8, it is easily seen that parts (ii) and (iii) hold. This completes the proof. □

5. Realization

Let X be a connected (G, s)-transitive graph of valency 15 and let $v \in V(X)$. In this section, we show that each type of G_v with $s \geq 2$ in Theorem 1.1 can be realized. Let n be a positive integer. The first example is a connected $(G, 2)$-transitive graph of valency 15 with $G_v \cong A_{15}$ or S_{15}.

Example 5.1. *Let $X = K_{16}$ be the complete graph of order 16. Then, $A = \mathrm{Aut}(X) = S_{16}$. Clearly, A has an arc-transitive subgroup B isomorphic to A_{16}. Thus the vertex stabilizers A_v and B_v are isomorphic to S_{15} and A_{15}, respectively.*

The following example is a connected $(G, 2)$-arc-transitive graph with $G_v \cong \mathrm{PSL}_4(2) \times (2^3 : \mathrm{PSL}_3(2))$, $S_{15} \times S_{14}$, $A_{15} \times A_{14}$ or $(A_{15} \times A_{14}) : 2$ with $A_{14} : 2 = S_{14}$ and $A_{15} : 2 = S_{15}$.

Example 5.2. *Let $X = K_{15,15}$ be the complete bipartite graph of order 30 with bipartite sets $\{1, 3, \dots, 29\}$ and $\{2, 4, \dots, 30\}$. Then $A = \mathrm{Aut}(x) \cong S_{15} wr S_2$. Clearly, A has a 3-transitive subgroup $B \cong A_{15} wr S_2$. Thus, $A_1 \cong S_{15} \times S_{14}$ and $B_1 = A_{15} \times A_{14}$. Now let $D = \langle B, a \rangle$ with $a = (1, 3)(2, 30)$. Then D is 3-transitive and $D_1 \cong (A_{15} \times A_{14}) : 2$ with $A_{15} : 2 = S_{15}$ and $A_{14} : 2 = S_{14}$. Since $\mathrm{PSL}_4(2)$ is a transitive subgroup of S_{15}, we have $M = \mathrm{PSL}_4(2) wr S_2$ is an arc-transitive subgroup of A. Therefore it is easy to see that $M_1 \cong \mathrm{PSL}_4(2) \times (2^3 : \mathrm{PSL}_3(2))$.*

The next coset graph is extracted from [19], Example 3.3, which G_v is isomorphic to $2^{12} : (\mathrm{SL}_4(2) \times \mathrm{SL}_3(2))$.

Example 5.3. *Let $G = \mathrm{Aut}(\mathrm{PSL}_7(2)) \cong \mathrm{PSL}_7(2) : 2$ and $T = \mathrm{Soc}(G) \cong \mathrm{PSL}_7(2)$. Then by [1], T has maximal subgroup $H \cong 2^{12} : (\mathrm{SL}_4(2) \times \mathrm{SL}_3(2))$ and we have that H has a subgroup $K \cong 2^{12} : ((2^3 : \mathrm{PSL}_3(2)) \times \mathrm{SL}_3(2))$, such that $L = N_G(K) \cong 2^{12} : ((2^3 : \mathrm{PSL}_3(2)) \times \mathrm{SL}_3(2)) : 2$. Thus there exists an element $g \in L \backslash K$ of order 2 such that $K = H \cap H^g$, $HgH = Hg^{-1}H$ and $|H : H \cap H^g| = 15$. Since H is maximal in T and $g \notin T$, we get $\langle H, g \rangle = G$. Thus, $\mathrm{Cos}(G, H, HgH)$ is a connected $(G, 3)$-transitive graph of valency 15 with $H \cong 2^{12} : (\mathrm{SL}_4(2) \times \mathrm{SL}_3(2))$ as a vertex stabilizer.*

The following example gives a $(G, 2)$-transitive graph with a vertex stabilizer G_v isomorphic to $2^4 : \mathrm{SL}_4(2)$ ([19], Example 3.4).

Example 5.4. *Let* $G = \mathrm{Aut}(\mathrm{PSL}_5(2)) \cong \mathrm{PSL}_5(2) : 2$ *and* $T = \mathrm{Soc}(G) \cong \mathrm{PSL}_5(2)$. *Then by* [1], T *has maximal subgroup* $H \cong 2^4 : \mathrm{SL}_4(2)$ *and we have that* H *has a subgroup* $K \cong 2^4 : (2^3 : \mathrm{PSL}_3(2))$, *such that* $L = N_G(K) \cong 2^4 : (2^3 : \mathrm{PSL}_3(2)) : 2$. *Thus there exists an element* $g \in L \backslash K$ *of order* 2 *such that* $K = H \cap H^g$, $HgH = Hg^{-1}H$ *and* $|H : H \cap H^g| = 15$. *Since* H *is maximal in* T *and* $g \notin T$, *we get* $\langle H, g\rangle = G$. *Thus,* $\mathrm{Cos}(G, H, HgH)$ *is a connected* $(G, 2)$*-transitive graph of valency* 15 *with* $H \cong 2^4 : \mathrm{SL}_4(2)$ *as a vertex stabilizer.*

By a similar argument we can see other examples in [19], examples 3.5, 6.1, 7.1, and 7.2.

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