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HANKEL DETERMINANTS OF CERTAIN SEQUENCES OF BERNOULLI POLYNOMIALS: A DIRECT PROOF OF AN INVERSE MATRIX ENTRY FROM STATISTICS

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ABSTRACT. We calculate the Hankel determinants of certain sequences of Bernoulli polynomials. This corresponding Hankel matrix comes from statistically estimating the variance in nonparametric regression. Besides its entries' natural and deep connection with Bernoulli polynomials, a special case of the matrix can be constructed from a corresponding Vandermonde matrix. As a result, instead of asymptotic analysis, we give a direct proof of calculating an entry of its inverse. Further extensions also include an identity of Stirling numbers of both kinds.

1. INTRODUCTION

The *Hankel determinant* of a given sequence $\mathbf{c} = (c_0, c_1, \ldots)$, denoted by $H_n(\mathbf{c})$ or $H_n(c_k)$, is defined as the determinant of the *Hankel matrix*, or *persymmetric matrix*, given by

$$(c_{i+j})_{0\le i,j\le n} = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_n \\ c_1 & c_2 & c_3 & \cdots & c_{n+1} \\ c_2 & c_3 & c_4 & \cdots & c_{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n+1} & c_{n+2} & \cdots & c_{2n} \end{pmatrix}. \tag{1.1}$$

Hankel determinants of various classes of sequences have been extensively studied, partly due to their close relationship with the classical orthogonal polynomials; see, e.g., [15, Ch. 2]; And for numerous results see, e.g., the very extensive treatments in [17, 18, 19], and the numerous references provided there.

Applications and appearance of Hankel matrices and determinants also involve statistics. Dai et al. [5] studied the following Hankel determinant

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from nonparametric regression. Define

$$I_k := \sum_{c=1}^{r} c^k \tag{1.2}$$

and the Hankel matrix

$$V_n := \begin{pmatrix} I_0 & I_2 & \cdots & I_{2n} \\ I_2 & I_4 & \cdots & I_{2n+2} \\ \vdots & \vdots & \ddots & \vdots \\ I_{2n} & I_{2n+2} & \cdots & I_{4n} \end{pmatrix}.$$

We shall show when V_k is invertible as follows.

Proposition 1.1. *V_n is invertible iff $n < r$.*

In fact, a more specific result gives the left-upper entry of the inverse of V_{r-1}, namely for the case $n = r-1$, is given. One can find a proof in [6], by asymptotic analysis, for the following result.

Proposition 1.2. $(V_{r-1}^{-1})_{1,1} = 2\left(\binom{4r}{2r}/\binom{2r}{r}^2 - 1\right)$, *where the left-hand side is the $(1,1)$ entry of the inverse matrix of V_{r-1}.*

It is the original purpose of this paper to give a direct computational proof of Proposition 1.2. Thanks to Dr. Christian Krattenthaler for his email, it turns out both Propositions 1.1 and 1.2 are direct corollaries of the following expressions.

Theorem 1.3. *Let $(a)_n = a(a+1)\cdots(a+n-1)$ be the Pochhammer symbol and $B_n(x)$ be the nth Bernoulli polynomial.*

$$\begin{aligned} &H_n\left(\frac{B_{2k+5}\left(\frac{x+1}{2}\right)}{2k+5}\right) \\ &= \frac{\prod_{i=1}^{n} \frac{(2i+3)!^2(2i+2)!^2}{(4i+5)!(4i+4)!} \prod_{\ell=0}^{n} (x-2n-1+2\ell)_{4n-4\ell+3}}{5\cdot 2^{n+2}} \\ &\quad \times \sum_{i=1}^{n+2} \frac{(2i-1)\left(n+\frac{5}{2}\right)_{i-1}\left(\frac{x+1}{2}\right)_{n+2}\left(\frac{x-2n-3}{2}\right)_{n+2}}{\left(n-i+\frac{5}{2}\right)_i (n+2-i)!(n+1+i)!\,(x^2-(2i-1)^2)} \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} \det V_n &= 2^{2n^2-2n-1} \prod_{i=1}^{n} \frac{(2i)!^4}{(4i)!(4i+1)!} \prod_{\ell=0}^{n} (r-\ell)_{2\ell+1} \prod_{\ell=0}^{n-1} \left(r+\frac{1}{2}-\ell\right)_{2\ell+1} \\ &\quad \times \sum_{i=1}^{n+1} \frac{(2n+2i)!(2n+2-2i)!(r+1)_{n+1}}{(n+i)!^2(n+1-i)!^2(r+i)}. \end{aligned} \tag{1.4}$$

We shall provide a different proof of (1.3), aside from the techniques in [11] and [12]; meanwhile, (1.4) follows from (1.3) naturally. More precisely, besides (1.3), we shall also prove the following Hankel determinants.

Proposition 1.4.

$$H_n\left(\frac{B_{2k+1}\left(\frac{x+1}{2}\right)}{2k+1}\right)$$
$$=\left(\frac{x}{2}\right)^{n+1}\times\prod_{\ell=1}^{n}\left(\frac{(2\ell)^2(2\ell-1)^2(x^2-(2\ell-1)^2)(x^2-(2\ell)^2)}{16(4\ell-3)(4\ell-1)^2(4\ell+1)}\right)^{n+1-\ell}, \tag{1.5}$$

and

$$H_n\left(\frac{B_{2k+3}\left(\frac{x+1}{2}\right)}{2k+3}\right)$$
$$=\left(\frac{x^3-x}{24}\right)^{n+1}\times\prod_{\ell=1}^{n}\left(\frac{(2\ell)^2(2\ell+1)^2(x^2-(2\ell+1)^2)(x^2-(2\ell)^2)}{16(4\ell-1)(4\ell+1)^2(4\ell+3)}\right)^{n+1-\ell}. \tag{1.6}$$

Note that these three sequences in (1.5), (1.6), and (1.3) were not studied in the recent work [7, 8, 9], by Dilcher and the first author, on the Hankel determinants of other sequences related to Bernoulli and Euler polynomial. Although the Hankel determinant of $H_n\left(B_{2k+1}(\frac{x+1}{2})\right)$ is obtained in [7, Thm. 1.1], the expression of

$$H_n\left(\frac{B_{2k+1}\left(\frac{x+1}{2}\right)}{2k+1}\right)$$

is quite different, which, in general, is true for $H_n(a_k)$, $H_n(a_k/k)$, and $H_n(ka_{k-1})$. Take the Euler numbers E_k as an example: we have [1, Eq. (4.2)]

$$H_n(E_k)=(-1)^{\binom{n+1}{2}}\prod_{\ell=1}^{n}\ell!^2,$$

and [8, Cor. 3.4],

$$H_n\left(kE_{k-1}\right)=\begin{cases}0, & k=2m;\\ (-1)^{m+1}2^{4m(m+1)}\prod_{\ell=1}^{m}\ell!^8, & k=2m+1.\end{cases}$$

Namely, the latter also depends on the parity of the dimension.

This paper is structured as follows. In Section 2, we first quote important results in Bernoulli polynomials, orthogonal polynomials, continued fractions, and polygamma functions, required in later sections. In Section 3, we give the proofs of three Hankel determinants (1.5), (1.6), and (1.3). In Section 4, besides the proof of (1.4), some further results on V_k are given, including alternative proofs of Props. 1.1 and 1.2, rather than direct corollaries from (1.4). This approach leads to an identity involving Stirling numbers, stated and proven finally in Section 5, together with some further remarks.

2. Preliminaries

All the necessary background stated here in this section can be found in [7, 8, 9], in a concise form. We repeat this material here for easy reference, and to make this paper self-contained.

2.1. **Bernoulli numbers, Bernoulli polynomials, and the connection with V_n.** The Bernoulli numbers B_n and Bernoulli polynomials $B_n(x)$ are usually defined by their exponential generating functions

$$\frac{t}{e^t-1}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!} \quad \text{and} \quad \frac{te^{tx}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}.$$

They are well studied and widely applied in number theory, combinatorics, numerical analysis, and other areas. The direct connection between $B_n(x)$ and I_k, defined in (1.2), can be derived from

$$B_{n+1}(x+1)-B_{n+1}(x)=(n+1)x^n;$$

(see e.g., [16, Entry (24.4.1)]), so that

$$I_k=\frac{B_{k+1}(r+1)-B_{k+1}(1)}{k+1}.$$

Note that $B_{2k+1}(1)=0$ (see e.g., [16, Entries (24.2.4), (24.4.3)]) for all positive integers k. Hence,

$$I_{2k}=\begin{cases}\frac{B_{2k+1}(r+1)}{2k+1}, & k>0;\\ B_1(r+1)-B_1(1)=r, & k=0.\end{cases} \tag{2.1}$$

Namely, $\det V_n$ is almost the same as $H_{n-1}\left(\frac{B_{2k+1}(r+1)}{2k+1}\right)$, (please note the difference in the dimensions) except for $I_0=r$ and $B_1(r+1)=r+1/2$.

2.2. **Orthogonal polynomials.** Given a sequence $\mathbf{c}=(c_0,c_1,c_2,\ldots)$ of numbers, which may also satisfy the normalization property that $c_0=1$; then we can define a linear functional L on polynomials by

$$L(x^k)=c_k, \quad k=0,1,2,\ldots. \tag{2.2}$$

We now summarize several well-known facts and state them as a lemma with two corollaries; see, e.g., [15, Ch. 2] and [3, pp. 7–10].

Lemma 2.1. *Let L be the linear functional in* (2.2)*. If (and only if) $H_n(c_k)\neq 0$ for all $n=0,1,2,\ldots$, there exists a unique sequence of monic polynomials $P_n(y)$ of degree n, $n=0,1,\ldots$, and a sequence of positive numbers $(\zeta_n)_{n\geq 1}$, with $\zeta_0=1$, such that*

$$L\left(P_m(y)P_n(y)\right)=\zeta_n\delta_{m,n}, \tag{2.3}$$

where $\delta_{m,n}$ is the Kronecker delta function. Furthermore, for all $n \geq 1$ we have $\zeta_n = H_n(\mathbf{c})/H_{n-1}(\mathbf{c})$, and for $n \geq 1$,

$$P_n(y) = \frac{1}{H_{n-1}(\mathbf{c})} \det \begin{pmatrix} c_0 & c_1 & \cdots & c_n \\ c_1 & c_2 & \cdots & c_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_n & \cdots & c_{2n-1} \\ 1 & y & \cdots & y^n \end{pmatrix}, \tag{2.4}$$

where the polynomials $P_n(y)$ satisfy the 3-term recurrence relation $P_0(y) = 1$, $P_1(y) = y + s_0$, and

$$P_{n+1}(y) = (y + s_n)P_n(y) - t_n P_{n-1}(y) \qquad (n \geq 1), \tag{2.5}$$

for some sequences $(s_n)_{n\geq 0}$ and $(t_n)_{n\geq 1}$.

We now multiply both sides of (2.4) by y^r and replace y^j by c_j, which includes replacing the constant term by $c_0 = 1$, for $r = 0$. Then for $0 \leq r \leq n-1$ the last row of the matrix in (2.4) is identical with one of the previous rows, and thus the determinant is 0. When $r = n$, the determinant is $H_n(\mathbf{c})$. We therefore have the following result.

Corollary 2.2. *With the sequence (c_k) and the polynomials $P_n(y)$ as above, we have*

$$y^r P_n(y)\Big|_{y^k = c_k} = \begin{cases} 0, & 0 \leq r \leq n-1; \\ H_n(\mathbf{c})/H_{n-1}(\mathbf{c}), & r = n. \end{cases} \tag{2.6}$$

The polynomials $P_n(y)$ are known as "the monic orthogonal polynomials belonging to the sequence $\mathbf{c} = (c_0, c_1, \ldots)$", or "the polynomials orthogonal with respect to $\mathbf{c}$".

The next result establishes a connection with certain continued fractions (called *J-fractions*). It can be found in various relevant publications, for instance in [17, p. 20].

Lemma 2.3. *Let $\mathbf{c} = (c_k)_{k\geq 0}$ be a sequence of numbers with $c_0 \neq 0$, and suppose that its generating function is written in the form*

$$\sum_{k=0}^{\infty} c_k z^k = \cfrac{c_0}{1 + s_0 z - \cfrac{t_1 z^2}{1 + s_1 z - \cfrac{t_2 z^2}{1 + s_2 z - \ddots}}}$$

where both sides are considered as formal power series. Then the sequences (s_n) and (t_n) are the same as in (2.5); and we have

$$H_n(\mathbf{c}) = c_0^{n+1} t_1^n t_2^{n-1} \cdots t_{n-1}^2 t_n \qquad (n \geq 0). \tag{2.7}$$

Following [21], we consider the infinite band matrix

$$J := \begin{pmatrix} -s_0 & 1 & 0 & 0 & \cdots \\ t_1 & -s_1 & 1 & 0 & \cdots \\ 0 & t_2 & -s_2 & 1 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{2.8}$$

Furthermore, for each $n \geq 0$ let J_n be the $(n+1)$th leading principal submatrix of J and let

$$D_n := \det J_n, \tag{2.9}$$

so that $D_0 = -s_0$. We also set $D_{-1} = 1$ by convention, and furthermore, using elementary determinant operations, we get from (2.8) the recurrence relation

$$D_{n+1} = -s_{n+1} D_n - t_{n+1} D_{n-1}. \tag{2.10}$$

We can now quote the following results.

Lemma 2.4. [21, Prop. 1.2] *With notation as above, for a given sequence* $\mathbf{c} = (c_0, c_1, \ldots)$, *we have*

$$H_n(c_{k+1}) = H_n(c_k) \cdot D_n, \tag{2.11}$$

and

$$H_n(c_{k+2}) = H_n(c_k) \cdot \left(\prod_{\ell=1}^{n+1} t_\ell \right) \cdot \sum_{\ell=-1}^{n} \frac{D_\ell^2}{\prod_{j=1}^{\ell+1} t_j}. \tag{2.12}$$

Lemma 2.5. [13, Eq. (2.4)] *For a given sequence* $\mathbf{c} = (c_0, c_1, \ldots)$ *and* (s_n) *as defined above, we have*

$$s_n = -\frac{1}{H_{n-1}(c_{k+1})} \left(\frac{H_{n-1}(c_k) H_n(c_{k+1})}{H_n(c_k)} + \frac{H_n(c_k) H_{n-2}(c_{k+1})}{H_{n-1}(c_k)} \right). \tag{2.13}$$

2.3. **Continued fractions.** Following the usage in books such as [4] or [20], we write

$$b_0 + \mathop{\mathbf{K}}_{m=1}^{\infty} (a_m/b_m) = b_0 + \mathbf{K}(a_m/b_m) = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots}} \tag{2.14}$$

for an *infinite continued fraction.* The nth *approximant* is expressed by

$$b_0 + \mathop{\mathbf{K}}_{m=1}^{n} (a_m/b_m) = b_0 + \cfrac{a_1}{b_1 + \ddots + \frac{a_n}{b_n}} = \frac{A_n}{B_n}, \tag{2.15}$$

and A_n, B_n are called the nth *numerator* and *denominator*, respectively. The continued fraction (2.14) is said to *converge* if the sequence of the approximants in (2.15) converges. In this case, the limit is called the *value* of the continued fraction (2.14).

Two continued fractions are said to be *equivalent* if and only if they have the same sequences of approximants. In other words, we have

$$b_0 + \mathop{\mathbf{K}}_{m=1}^{n}(a_m/b_m) = d_0 + \mathop{\mathbf{K}}_{m=1}^{n}(c_m/d_m)$$

if and only if there exists a sequence of nonzero complex numbers $(r_m)_{m\geq 0}$ with $r_0 = 1$, such that for $m \geq 0$,

$$d_m = r_m b_m \quad \text{and} \quad c_{m+1} = r_{m+1} r_m a_{m+1}; \tag{2.16}$$

(see [4, Eq. (1.4.2)]). We also require the following special case of the more general concept of a contraction; see, e.g., [4, p. 16].

Definition 2.6. *Let A_n, B_n denote the nth numerator and denominator, respectively, of a continued fraction* $\mathrm{cf}_1 := b_0 + \mathbf{K}(a_m/b_m)$*; and let C_n, D_n be the corresponding quantities of* $\mathrm{cf}_2 := d_0 + \mathbf{K}(c_m/d_m)$. *Then* cf_2 *is called an* even canonical contraction *of* cf_1 *if*

$$C_n = A_{2n} \qquad D_n = B_{2n} \qquad (n \geq 0);$$

and is called an odd canonical contraction *of cf1 if*

$$C_0 = \frac{A_1}{B_1}, \quad D_0 = 1, \quad C_n = A_{2n+1}, \quad D_n = B_{2n+1} \qquad (n \geq 0).$$

We will now state three identities that will be used in later sections; see [4, pp. 16–18], [20, pp. 83–85], or [22, pp. 21–21] for proofs and further details.

Lemma 2.7. *An even canonical contraction of $b_0 + \mathbf{K}(a_m/b_m)$ exists if and only if $b_{2k} \neq 0$ for $k \geq 1$, and we have*

$$b_0 + \mathop{\mathbf{K}}_{m=1}^{\infty}(a_m/b_m) = b_0 + \cfrac{a_1 b_2}{a_2 + b_1 b_2 - \cfrac{a_2 a_3 \frac{b_4}{b_2}}{a_4 + b_3 b_4 + a_3 \frac{b_4}{b_2} - \cfrac{a_4 a_5 \frac{b_6}{b_4}}{a_6 + b_5 b_6 + a_5 \frac{b_6}{b_4} - \cfrac{a_6 a_7 \frac{b_8}{b_6}}{\ddots}}}}$$

In particular, with $b_0 = 0$, $b_k = 1$ for $k \geq 1$, $a_1 = 1$, and $a_k = \alpha_{k-1} t$ $(k \geq 1)$, for some variable t, we have

$$\cfrac{1}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \ddots}}} = \cfrac{1}{1 - \alpha_1 t - \cfrac{\alpha_1 \alpha_2 t^2}{1 - (\alpha_2 + \alpha_3)t - \cfrac{\alpha_3 \alpha_4 t^2}{1 - (\alpha_4 + \alpha_5)t - \cfrac{\alpha_5 \alpha_6 t^2}{\ddots}}}} \tag{2.17}$$

Similarly, an odd canonical contraction gives

$$1 + \cfrac{\alpha_1 t}{1 - (\alpha_1 + \alpha_2)t - \cfrac{\alpha_2 \alpha_3 t^2}{1 - (\alpha_3 + \alpha_4)t - \cfrac{\alpha_4 \alpha_5 t^2}{1 - (\alpha_5 + \alpha_6)t - \cfrac{\alpha_6 \alpha_7 t^2}{\ddots}}}}, \tag{2.18}$$

for the continued fraction on the left-hand side of (2.17).

2.4. **Polygamma functions.** We shall use the *polygamma function of order m*, defined by $\psi^{(m)}(z) := \frac{d^{m+1}}{dz^{m+1}} \log\Gamma(z)$, with

$$\psi(z) = \psi^{(0)}(z) = \frac{d}{dz}(\log\Gamma(z)) = \frac{\Gamma'(z)}{\Gamma(z)}.$$

Among all the properties, we first recall the following well-known complete asymptotic expansion, valid for $|\arg z| < \pi$: (see e.g., [10, p. 48, Eq. (12)])
(2.19)

$$\log\Gamma(z+x) = \left(z + x - \frac{1}{2}\right)\log z - z + \frac{\log(2\pi)}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}B_{n+1}(x)}{n(n+1)z^n},$$

which, by differentiating both sides with respect to z, yields the asymptotic expansion of $\psi(z)$:

$$\psi(z+x) = \log z + \frac{x - \frac{1}{2}}{z} - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}B_{n+1}(x)}{(n+1)z^{n+1}}. \tag{2.20}$$

Meanwhile, the series expansion formula (see e.g., [16, Entry 5. 7. 6])

$$\psi(z) = -\gamma - \frac{1}{z} + \sum_{k=1}^{\infty} \frac{z}{k(k+z)} = -\gamma + \sum_{k=0}^{\infty}\left(\frac{1}{k+1} - \frac{1}{k+z}\right), \tag{2.21}$$

later shall lead us to the continued fraction expressions that are crucial to our proofs.

3. Hankel Determinants

We begin with our three main Hankel determinants.

3.1. $B_{2k+1}\left(\frac{x+1}{2}\right)/(2k+1)$.

Proof of (1.5). From (2.20), we have

$$\begin{aligned}
&\psi\left(z + \frac{1+x}{2}\right) - \psi\left(z + \frac{1-x}{2}\right)\\
=&\frac{x}{z} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+1)z^{n+1}}\left(B_{n+1}\left(\tfrac{1-x}{2}\right) - B_{n+1}\left(\tfrac{1+x}{2}\right)\right)\\
=&2\sum_{n=0}^{\infty} \frac{B_{2n+1}\left(\frac{1+x}{2}\right)}{(2n+1)z^{2n+1}},
\end{aligned}$$

where we used the reflection formula (see e.g., [16, Entry 24. 4. 23]) so that

$$B_{n+1}\left(\frac{1-x}{2}\right) = B_{n+1}\left(1 - \frac{1+x}{2}\right) = (-1)^n B_{n+1}\left(\frac{1+x}{2}\right)$$

and $B_1(x) = x - 1/2$. This, finally by the change of variables $z \mapsto 1/z$, implies

$$\sum_{n=0}^{\infty} \frac{B_{2n+1}(\frac{1+r}{2})}{2n+1} z^{2n} = \frac{\psi\left(\frac{1}{z} + \frac{1+x}{2}\right) - \psi\left(\frac{1}{z} + \frac{1-x}{2}\right)}{2z}.$$

We denote the left-hand side of the above equation by $F(z)$, where the independence on x is implied. By the series expansion (2.21), we see

$$zF(z) = \sum_{k=0}^{\infty} \left(\frac{1}{\frac{2}{z} - x + 2k + 1} - \frac{1}{\frac{2}{z} + x + 2k + 1} \right).$$

Now we use the following continued fraction due to Ramanujan (see e.g., [2, p. 149, Entry 30]) for either n is an integer or $Re(t) > 0$:

$$\sum_{k=0}^{\infty} \left(\frac{1}{t - n + 2k + 1} - \frac{1}{t + n + 2k + 1} \right) = \cfrac{n}{t + \cfrac{1(1-n^2)}{3t + \cfrac{2^2(2^2-n^2)}{5t + \cfrac{3^2(3^2-n^2)}{7t + \ddots}}}},$$

by letting $t = 2/z$ and $n = x$, to get

$$zF(z) = \cfrac{x}{\frac{2}{z} + \cfrac{1(1-x^2)}{\frac{6}{z} + \cfrac{2^2(2^2-x^2)}{\frac{10}{z} + \cfrac{3^2(3^2-x^2)}{\frac{14}{z} + \ddots}}}}.$$

Using equivalence of continued fractions (2.16), with

$$r_m = \begin{cases} 1, & m = 0; \\ \frac{z}{2(2m-1)}, & m \geq 1, \end{cases}$$

$$a_m = \begin{cases} x, & m = 1; \\ (m-1)^2((m-1)^2 - x^2), & m \geq 2, \end{cases}$$

and $b_m = \frac{2(2m-1)}{z}$, we could get

$$d_m = b_m r_m = \frac{2(2m-1)}{z} \cdot \frac{z}{2(2m-1)} = 1,$$

$$c_{m+1} = r_{m+1} r_m a_{m+1} = \frac{m^2(m^2 - r^2)z^2}{4(2m+1)(2m-1)}.$$

Hence, after dividing both sides by $xz/2$, we have

$$\frac{F(z)}{\frac{x}{2}} = \cfrac{1}{1 + \cfrac{\frac{(1-x^2)z^2}{12}}{1 + \cfrac{\frac{4(4-x^2)z^2}{15}}{1 + \ddots}}}. \tag{3.1}$$

For simplification, we define

$$\alpha_m = \frac{m^2(x^2 - m^2)}{4(2m+1)(2m-1)}, \tag{3.2}$$

and apply the even canonical contraction (2.17) on (3.1), to obtain

$$\tau_m^{(0)} = \alpha_{2m-1}\alpha_{2m} = \frac{(2m-1)^2(2m)^2(x^2-(2m-1)^2)(x^2-(2m)^2)}{16(4m-3)(4m-1)^2(4m+1)},$$
$$\sigma_m^{(0)} = \alpha_{2m} + \alpha_{2m+1} = \frac{(8m^2+4m-1)x^2-(32m^4+32m^3+8m^2-1)}{4(4m+3)(4m-1)},$$

such that

$$F(z) = \cfrac{\frac{x}{2}}{1+\sigma_0^{(0)}z^2 - \cfrac{\tau_1^{(0)}z^4}{1+\sigma_1^{(0)}z^2 - \cfrac{\tau_2^{(0)}z^4}{1+\sigma_1^{(0)}z^2 - \ddots}}}.$$

Then use Lemma 2.3 with $c_0 = \frac{x}{2}$, $s_j = -\sigma_j^{(0)}$, and $t_j = \tau_j^{(0)}$, we have the desired result (1.5). □

3.2. $B_{2k+3}\left(\frac{x+1}{2}\right)/(2k+3)$.

Proof of (1.6). Similarly, let

$$G(z) = \sum_{n=0}^{\infty} \frac{B_{2n+3}(\frac{1+x}{2})}{2n+3} z^{2n+2} = F(z) - B_1\left(\frac{1+x}{2}\right) = F(z) - \frac{x}{2}.$$

Then, by (3.1),

$$\frac{G(z)}{\frac{x}{2}} = \frac{F(z)}{\frac{x}{2}} - 1 = \cfrac{1}{1+\cfrac{\frac{(1-x^2)z^2}{12}}{1+\cfrac{\frac{4(4-x^2)z^2}{15}}{1+\ddots}}} - 1. \tag{3.3}$$

Focus on the first continued fractions above. Using odd canonical contraction (2.18) with α_m defined in (3.2), we would have

$$G(z) = \cfrac{\frac{x^3-x}{24}}{1-\sigma_0^{(1)}z^2 - \cfrac{\tau_1^{(1)}z^4}{1-\sigma_1^{(1)}z^2 - \cfrac{\tau_2^{(1)}z^4}{1-\ddots}}},$$

where

$$\tau_m^{(1)} = \alpha_{2m}\alpha_{2m+1} = \frac{(2m)^2(2m+1)^2(x^2-(2m))(x^2-(2m+1)^2)}{16(4m-1)(4m+1)^2(4m+3)}, \tag{3.4}$$

$$\begin{aligned}\sigma_m^{(1)} &= \alpha_{2m+1} + \alpha_{2m+2} \\ &= \frac{(2m+1)^2(x^2-(2m+1)^2)}{4(4m+3)(4m+1)} + \frac{(2m+2)^2(x^2-(2m+2)^2)}{4(4m+5)(4m+3)}.\end{aligned} \tag{3.5}$$

Then use Lemma 2.3 with $c_0 = \frac{x^3-x}{24} = \frac{x}{2}\alpha_1$, $s_j = -\sigma_j^{(1)}$, and $t_j = \tau_j^{(1)}$, we have the desired result. □

Remark: Noting that $c_k = b_{k+1}$, so the "left-shifted" sequence formula applies here and leads to the same result (1.6). In fact, we can easily derive that

$$D_n^{(0)} := \det\begin{pmatrix} \sigma_0^{(0)} & 1 & 0 & 0 & \cdots & 0 \\ \tau_1^{(0)} & \sigma_1^{(0)} & 1 & 0 & \cdots & 0 \\ 0 & \tau_2^{(0)} & \sigma_2^{(0)} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \tau_n^{(0)} & \sigma_n^{(0)} \end{pmatrix} = \frac{1}{4^{n+1}} \prod_{\ell=1}^{n+1} \frac{(x^2-(2\ell+1)^2)(2\ell-1)^2}{(4\ell-1)(4\ell-3)}, \tag{3.6}$$

which yields

$$H_n\left(\frac{B_{2k+3}\left(\frac{x+1}{2}\right)}{2k+3}\right) = H_n\left(\frac{B_{2k+1}\left(\frac{x+1}{2}\right)}{2k+1}\right) D_n^{(0)}. \tag{3.7}$$

3.3. $B_{2k+5}\left(\frac{x+1}{2}\right)/(2k+5)$**.** The following proof contains some tedious calculation and simplification, the details of which are omitted.

Proof of (1.3)*.* Similarly as the remark above, it suffices to show

$$\frac{H_n\left(\frac{B_{2k+5}\left(\frac{x+1}{2}\right)}{2k+5}\right)}{H_n\left(\frac{B_{2k+3}\left(\frac{x+1}{2}\right)}{2k+3}\right)} =: D_n^{(1)} = \det\begin{pmatrix} \sigma_0^{(1)} & 1 & 0 & 0 & \cdots & 0 \\ \tau_1^{(1)} & \sigma_1^{(1)} & 1 & 0 & \cdots & 0 \\ 0 & \tau_2^{(1)} & \sigma_2^{(1)} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \tau_n^{(1)} & \sigma_n^{(1)} \end{pmatrix},$$

or equivalently, by (2.10), to show

$$D_{n+1}^{(1)} = \sigma_{n+1}^{(1)} D_n^{(1)} - \tau_{n+1}^{(1)} D_{n-1}^{(1)}, \tag{3.8}$$

where $H_n\left(B_{2k+5}\left(\frac{x+1}{2}\right)/(2k+5)\right)$ is given by (1.3), and $\tau_n^{(1)}$ and $\sigma_n^{(1)}$ are given by (3.4) and (3.5), respectively.

(1) First of all, by simplification, we see

$$\left(\frac{x}{2}+\frac{1}{2}\right)_{n+2}\left(\frac{x}{2}-n-\frac{3}{2}\right)_{n+2} = \frac{1}{2^{2n+4}}\prod_{i=1}^{n+2}(x-(2i-1)^2).$$

(2) In addition, let

$$f(n,x) := \frac{1}{5\cdot 2^{n+2}}\prod_{i=1}^{n}\frac{(2i+3)!^2(2i+2)!^2}{(4i+5)!(4i+4)!}\prod_{l=0}^{n}(x-2n-1+2l)_{4n-4l+3},$$

and the recurrence, upon simplification,

$$\frac{f(n+1,x)}{f(n,x)} = \frac{x(x^2-1)}{2}\cdot\frac{(2n+5)!^2(2n+4)!^2}{(4n+9)!(4n+8)!} \times \prod_{l=1}^{n+1}\left[x^2-(2l+1)^2\right]\left[x^2-(2l)^2\right]$$

indicates

$$\frac{H_{n+1}\left(\frac{B_{2k+5}\left(\frac{x+1}{2}\right)}{2k+5}\right)}{H_n\left(\frac{B_{2k+3}\left(\frac{x+1}{2}\right)}{2k+3}\right)} = \frac{f(n+1,x)}{f(n,x)}\cdot\frac{4(4n+9)(4n+7)}{(2n+5)^2(2n+4)^2}.$$

(3) Due to $(a)_n = \Gamma(a+n)/\Gamma(a)$ and $\Gamma\left(\frac{1}{2}+n\right) = (2n-1)!!\sqrt{\pi}/2^n$, we see

$$\begin{aligned}&\frac{\left(n+\frac{5}{2}\right)_{i-1}}{\left(n-i+\frac{5}{2}\right)_i (n+2-i)!(n+1+i)!}\\ =&\frac{\Gamma\left(n+i+\frac{3}{2}\right)\Gamma\left(n-i+\frac{5}{2}\right)}{\Gamma^2\left(n+\frac{5}{2}\right)\Gamma(n+3-i)\Gamma(n+2+i)}\\ =&\frac{4^{n+2}}{(2n+3)!!^2\pi}\frac{\Gamma\left(n+i+\frac{3}{2}\right)\Gamma\left(n-i+\frac{5}{2}\right)}{\Gamma(n+3-i)\Gamma(n+2+i)}.\end{aligned}$$

Therefore, it suffices to show that

$$D_n^{(1)} = \frac{2(n+1)!^2}{(4n+5)!!\pi}\prod_{i=1}^{n+2}(x^2-(2i-1)^2) \times \sum_{i=1}^{n+2}\frac{(2i-1)\Gamma\left(n+i+\frac{3}{2}\right)\Gamma\left(n-i+\frac{5}{2}\right)}{\Gamma(n+3-i)\Gamma(n+2+i)\left(x^2-(2i-1)^2\right)},$$

satisfies (3.8). Although the product term $\prod_{i=1}^{n+2}(x^2-(2i-1)^2)$ has degree $2n+4$, the sum with $x^2-(2i-1)^2$ in the denominator will always cancel one factor. Therefore, $D_{n+1}^{(1)}$ is a polynomials in x of degree at most $2n+2$. Also noting that $\tau_n^{(1)}$ is of degree 4; while $\sigma_n^{(1)}$ of degree 2, in x, we see (3.8) is basically showing two polynomials of degree at most $2(n+2)$ are identical. Therefore, as long as the left-hand side matches the right-hand side at $2n+5$ different values of x, (3.8) holds for any x, and the proof is complete.

(1) Consider $x = \pm(2j-1)$, where $j = 1, 2, \dots, n+1$ (i.e., $2n+2$ different points). In this case, all the terms in the summation of $D_n^{(1)}$ will vanish, except for exactly the one with $i = 2j-1$, since the factor $x^2-(2i-1)^2$ is canceled with the previous product, making

it

$$\left(\prod_{i=1}^{j-1}((2j-1)^2-(2i-1)^2)\right)\left(\prod_{i=j+1}^{n+2}((2j-1)^2-(2i-1)^2)\right)$$

Then, by canceling the common product term and after simplification, we see the left-hand side of (3.8) is given by,

$$\left(\prod_{i=n+2}^{n+3}((2j-1)^2-(2i-1)^2)\right)$$
$$\times\frac{2(n+2)!^2(2j-1)}{4^{2n+5}(4n+9)!!}\binom{2n+4+2j}{n+2+j}\binom{2n+6-2j}{n+3-j},$$

while the right-hand side is

$$\left(\frac{(2n+4)^2((2j-1)^2-(2n+4)^2)}{4(4n+7)(4n+9)}+\frac{(2n+3)^2((2j-1)^2-(2n+3)^2)}{4(4n+7)(4n+5)}\right)$$
$$\times\left(((2n-1)^2-(2n+3)^2)\right)\frac{2(n+1)!^2(2j-1)}{4^{2n+3}(4n+5)!!}$$
$$\times\binom{2n+2+2j}{n+1+j}\binom{2n+4-2j}{n+2-j}$$
$$-\frac{(2n+3)^2(2n+2)^2((2j-1)^2-(2n+3)^2)((2j-1)^2-(2n+2)^2)}{16(4n+7)(4n+3)(4n+5)^2}$$
$$\times\frac{2(n!)^2(2j-1)}{4^{2n+1}(4n+1)!!}\binom{2n+2j}{n+j}\binom{2n+2-2j}{n+1-j}.$$

Further simplification shows that (3.8) with $x=\pm(2j-1)$, for $j=1,2,\ldots,n+1$ is equivalent to

$$\left((2j-1)^2-(2n+5)^2\right)\frac{(n+2)^2}{4(4n+9)}\frac{(2n+3+2j)(2n+5-2j)}{(n+2+j)(n+3-j)}$$
$$=\left(\frac{(2n+4)^2((2j-1)^2-(2n+4)^2)}{4(4n+9)}+\frac{(2n+3)^2((2j-1)^2-(2n+3)^2)}{4(4n+5)}\right)$$
$$+\frac{(2n+3)^2(n+1+j)(n+2-j)}{(4n+5)},$$

which is trivial to verify.

(2) Let $x=\pm(2n+3)$, i.e., $x=\pm(2j-1)$ for $j=n+2$. Note that in this case, the summation in $D_{n-1}^{(1)}$ will not reduce to a single term; but $\tau_{n+1}^{(1)}=0$, and (3.8) reduces to

$$\left((2n+3)^2-(2n+5)^2\right)\frac{2(n+2)!^2(2n+3)}{4^{2n+5}(4n+9)!!}\binom{4n+8}{2n+4}\binom{2}{1}$$
$$=\frac{(2n+4)^2((2n+3)^2-(2n+4)^2)}{4(4n+7)(4n+9)}\frac{2(n+1)!^2 2n+3}{4^{2n+3}(4n+5)!!}\binom{4n+6}{2n+3}\binom{0}{0},$$

which is equivalent to the trivial identity

$$(n+2)\binom{4n+8}{2n+4} = (4n+7)\binom{4n+6}{2n+3}.$$

(3) Now, we have already checked $2n+4$ different points, so one more is adequate. Note that $x = 2n+5$ does not work, since in this case, the summation in both $D_n^{(1)}$ and $D_{n-1}^{(1)}$ remain. Instead, we take $x = 2n+2$, so that

$$\sigma_{n+1}^{(1)}\Big|_{x=2n+2} = -\frac{(2n+3)(6n+13)}{4(4n+9)} \quad \text{and} \quad \tau_{n+1}^{(1)}\Big|_{x=2n+2} = 0.$$

Therefore, it suffices to show

$$\begin{aligned} &-3(n+2)^2 \sum_{i=1}^{n+3} \frac{(2i-1)\Gamma\left(n+i+\frac{5}{2}\right)\Gamma\left(n-i+\frac{7}{2}\right)}{\Gamma(n+4-i)\Gamma(n+3+i)\left((2n+2)^2-(2i-1)^2\right)} \\ &= -\frac{(2n+3)(6n+13)}{4} \sum_{i=1}^{n+2} \frac{(2i-1)\Gamma\left(n+i+\frac{3}{2}\right)\Gamma\left(n-i+\frac{5}{2}\right)}{\Gamma(n+3-i)\Gamma(n+2+i)\left((2n+2)^2-(2i-1)^2\right)}, \end{aligned}$$

i.e.,

$$\begin{aligned} &\frac{(n+2)^2\pi(2n+5)\binom{4n+10}{2n+5}}{4^{2n+5}\left(-(4n+7)\right)} \\ &= \sum_{i=1}^{n+2} \frac{(2i-1)\Gamma\left(n+i+\frac{3}{2}\right)\Gamma\left(n-i+\frac{5}{2}\right)}{(n+2-i)!(n+1+i)!\left((2n+2)^2-(2i-1)^2\right)} \\ &\quad \times \left(\frac{(2n+3)(6n+13)}{4} - \frac{3(n+2)^2\left(n+i+\frac{3}{2}\right)\left(n-i+\frac{5}{2}\right)}{(n+3-i)(n+2+i)}\right). \end{aligned}$$

We use the WZ-method, i.e., the fastZeil.m[1] package to show that the sum on the right-hand side is

$$\begin{aligned} &-\frac{(n+2)^2(2n+5)(4n+9)\sqrt{\pi}\Gamma\left(2n+\frac{7}{2}\right)}{4(2n+5)!} \\ &= \frac{3(n+2)^2(2n+5)\Gamma\left(2n+\frac{11}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(2n+6)\left(-3(4n+7)\right)}, \end{aligned}$$

which is exactly the left-hand side.

Therefore, we have proven (3.8), which is equivalent to (1.3). □

4. Results on V_k.

Due to (2.1), in this section, we always let $x = 2r+1$, which makes $(x+1)/2 = r+1$.

We first begin with the following simple lemma.

[1] https://risc.jku.at/sw/fastzeil/

Lemma 4.1.

$$\det V_n = H_n\left(\frac{B_{2k+1}(r+1)}{2k+1}\right) - \frac{1}{2}H_{n-1}\left(\frac{B_{2k+5}(r+1)}{2k+5}\right). \tag{4.1}$$

Proof. By the cofactor expansion, we can see that

$$\begin{aligned}&\det V_n - I_0 H_{n-1}\left(\frac{B_{2k+5}(r+1)}{2k+5}\right)\\ &= H_n\left(\frac{B_{2k+1}(r+1)}{2k+1}\right) - \left(r+\frac{1}{2}\right)H_{n-1}\left(\frac{B_{2k+5}(r+1)}{2k+5}\right),\end{aligned}$$

which gives the desired result. □

Proof of (1.4). Now, it is apparent to combine (1.3) and (4.1), in order to prove (1.4). The calculation and simplification are straightforward, but tedious, so are omitted here. □

Now, we shall give alternative proofs for Propositions 1.1 and 1.2, without directly using (1.4). This approach eventually leads to an identity involving Stirling numbers, as Corollary 5.1 in the next section.

Lemma 4.2. *We have*

$$V_{r-1} = VS(r)^T VS(r), \tag{4.2}$$

where $VS(n)$ is the $n\times n$ Vandermonde matrix of the first n squares, namely,

$$VS(r) = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1\\ 1 & 2^2 & 2^4 & \cdots & 2^{2(r-1)}\\ 1 & 3^2 & 3^4 & \cdots & 3^{2(r-1)}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & n^2 & n^4 & \cdots & n^{2(r-1)}\end{pmatrix}.$$

Proof. This directly follows from

$$\sum_{\ell=1}^{r}\left(\ell^2\right)^{k-1}\cdot\left(\ell^2\right)^{j-1} = \sum_{\ell=1}^{r}\ell^{2(k-1)+2(j-1)} = I_{2(k+j)-4},$$

which is the (k,j) (and also (j,k)) entry of V_{r-1}. Here, please note that the $(1,1)$ entry is I_0, i.e., with both horizontal and vertical indices shifted. □

Corollary 4.3. *For $n<r$,* $\det V_n \neq 0$.

Proof. Note that (4.2) indicates

$$\det V_{r-1} = \prod_{1\le \ell<j\le r}(j^2-\ell^2)^2 \neq 0.$$

We see for all $n<r-1$, V_n is one of the principal submatrices of V_{r-1}, which is real, symmetric, and positive-definite, due to again (4.2). Hence V_n is invertible. □

Corollary 4.4. *For $n>r$,* $\det V_k = 0$.

Proof. First of all, for $n > r$ in (1.5), the term $(2r+1)^2 - (2r+1)^2$ will appear in the product, so that

$$H_n\left(\frac{B_{2k+1}(r+1)}{2k+1}\right) = 0.$$

Similarly, for $n > r-1$, in (1.6), the product becomes 0, which also, by (2.11), leads to

$$H_n\left(\frac{B_{2k+5}(r+1)}{2k+5}\right) = 0.$$

Therefore, for $n > r$, i.e., $n-1 > r-1$, $\det V_k = 0$, by (4.1). □

Now, we can give an alternative proof of Proposition 1.2.

Proof of Proposition 1.2. Let $n = r-1$ in (4.1) to see

$$H_{r-2}\left(\frac{B_{2k+5}(r+1)}{2k+5}\right) = 2\left(H_{r-1}\left(\frac{B_{2k+1}(r+1)}{2k+1}\right) - \det V_{r-1}\right).$$

Meanwhile, by the formula of cofactors to compute the inverse of a matrix,

$$\left(V_{r-1}^{-1}\right)_{1,1} = \frac{H_{r-2}\left(\frac{B_{2k+5}(r+1)}{2k+5}\right)}{\det V_{r-1}} = 2\left(\frac{H_{r-1}\left(\frac{B_{2k+1}(r+1)}{2k+1}\right)}{\det V_{r-1}} - 1\right).$$

Therefore, it is equivalent to show

$$\frac{H_{r-1}\left(\frac{B_{2k+1}(r+1)}{2k+1}\right)}{\det V_{r-1}} = \frac{\binom{4r}{2r}}{\binom{2r}{r}^2},$$

which can be done by simplifying the left-hand side. Details are omitted here. □

Finally, we shall complete the proof of Proposition 1.1. By Corollaries 4.3 and 4.4, we only need to show $\det V_r = 0$. The following sequence plays the essential role.

Definition 4.5. *The sequence $T(n,k)$ (A204579*[2]*) can be defined by the recurrence:*

(4.3)
$$T(n,k) = \begin{cases} 1, & n=k=1; \\ T(n-1,k-1) - (n-1)^2 T(n-1,k), & n>1, 1 \le k \le n; \\ 0, & \textit{otherwise.} \end{cases}$$

Lemma 4.6. *Let* $\mathbf{0}$ *be the zero vector. We have*

$$V_r \cdot \begin{pmatrix} T(r+1,1) & T(r+1,2) & \cdots & T(r+1,r+1) \end{pmatrix}^T = \mathbf{0}. \tag{4.4}$$

[2]https://oeis.org/A204579

Proof. First of all, (4.4) is equivalent to the following $r+1$ identities: for any $j=0,1,\dots,r$,

$$\sum_{k=0}^{r} I_{2j+2k}T(r+1,k+1)=0. \tag{4.5}$$

By definition and simplification, we have

$$\sum_{k=0}^{r} I_{2j+2k}T(r+1,k+1)=\sum_{c=1}^{r} c^{2j}\sum_{k=0}^{r} c^{2k}T(r+1,k+1).$$

Now, we claim, for $c=1,2,\dots,r$,

$$\sum_{k=0}^{r} c^{2k}T(r+1,k+1)=0, \tag{4.6}$$

which implies (4.5), for any j.

When $r=1$, the only case is $c=1$. Since $T(2,1)=-1$, and $T(2,2)=1$, we have (4.6) for $r=1$. Assume (4.6) holds for $r=m-1$. For $r=m$, by the recurrence (4.3), we have

$$\begin{aligned}\sum_{k=0}^{m} c^{2k}T(m+1,k+1)&=\sum_{k=0}^{m} c^{2k}\left(T(m,k)-m^2T(m,k+1)\right)\\&=\sum_{k=0}^{m} c^{2k}T(m,k)-m^2\sum_{k=0}^{m} c^{2k}T(m,k+1)\\&=\sum_{k=1}^{m} c^{2k}T(m,k)-m^2\sum_{k=0}^{m-1} c^{2k}T(m,k+1),\end{aligned}$$

where in the last step, we used $T(m,0)=T(m,m+1)=0$. Finally, by shifting the summation index of the first sum, we have

$$\begin{aligned}\sum_{k=0}^{m} c^{2k}T(m+1,k+1)&=c^2\sum_{k=0}^{m-1} c^{2k}T(m,k+1)-m^2\sum_{k=0}^{m-1} c^{2k}T(m,k+1)\\&=(c^2-m^2)\sum_{k=0}^{m-1} c^{2k}T(m,k+1)=0,\end{aligned}$$

for $c=0,1,\dots,m-1$, (by the inductive assumption) and $c=m$ (due to the first factor). □

Lemma 4.7. $\det V_r=0$*, for all* $r\in\mathbb{N}$.

Proof. Note that $T(m,m)=T(m-1,m-1)=\cdots=T(1,1)=0$, so the vector in (4.4) is not a zero vector. Therefore V_r is not invertible. □

5. Final remarks

5.1. **An identity on Stirling numbers.** Let $s(n,k)$ be the Stirling numbers of the first kind and $S(n,k)$ be the Stirling numbers of the second kind. $T(n,k)$ also has an alternative expression:

$$T(r+1,k+1)=\sum_{i=0}^{2k+2}(-1)^{r+1+i}s(r+1,i)s(r+1,2k+2-i).$$

Meanwhile, recall the well-known connection between Bernoulli numbers and $S(n,k)$:

$$B_m=\sum_{\ell=0}^{m}\frac{(-1)^{\ell}\ell!}{\ell+1}S(m,\ell).$$

(See [16, Entry (24.15.6)].) Then, (2.1) indicates

$$I_{2j+2k}=\frac{1}{2j+2k+1}\sum_{m=0}^{2j+2k}\binom{2j+2k+1}{m}r^{2j+2k+1-m}\sum_{\ell=0}^{m}\frac{(-1)^{\ell}\ell!}{\ell+1}S(m,\ell),$$

where the expansion

$$B_n(x)=\sum_{m=0}^{n}\binom{n}{m}x^{n-m}B_m$$

(see [16, Entry (24.2.5)]) is also applied. Therefore, (4.5) yields the following identity of Stirling numbers.

Corollary 5.1. *For any $r\in\mathbb{N}$ and $j=0,1,\ldots,r$, we have*

$$\begin{aligned}&\sum_{k=0}^{r}\frac{1}{2j+2k+1}\left(\sum_{m=0}^{2j+2k}\binom{2j+2k+1}{m}((r+1)^{2j+2k+1-m}-1)\right.\\ &\times\left.\sum_{\ell=0}^{m}\frac{(-1)^{\ell}\ell!}{\ell+1}S(m,\ell)\sum_{i=0}^{2k+2}(-1)^{r+1+i}s(r+1,i)s(r+1,2k+2-i)\right)\\ &=0.\end{aligned}$$

5.2. **Continued fraction approach in general.** It is natural to consider the continued fraction to the generating function of $B_{2k+5}(\frac{x+1}{2})/(2k+5)$:

$$H(z)=\sum_{n=0}^{\infty}\frac{B_{2n+5}(\frac{1+x}{2})}{2n+5}z^{2n}=\frac{G(z)-B_3\left(\frac{1+x}{2}\right)}{z^2}=\frac{G(z)-\frac{x^3-x}{24}}{z^2}$$

Then,

$$\frac{z^2H(z)}{\frac{x^3-x}{24}}=\frac{G(z)}{\frac{x^3-x}{24}}-1.$$

In the proof of (1.6), we actually have shown that

$$\frac{G(z)}{\frac{x^3-x}{24}}=\cfrac{1}{1-(\alpha_1+\alpha_2)z^2-\cfrac{\alpha_2\alpha_3z^4}{1-(\alpha_3+\alpha_4)z^2-\cfrac{\alpha_4\alpha_5z^4}{1-\ddots}}}.$$

We then define another sequence $(\beta_n)_{n\geq 1}$ by $\beta_1=\alpha_1+\alpha_2$, and for $m\geq 1$,

$$\beta_{2m-1}\beta_{2m}=\alpha_{2m}\alpha_{2m+1}, \tag{5.1}$$

$$\beta_{2m}+\beta_{2m+1}=\alpha_{2m+1}+\alpha_{2m+2}. \tag{5.2}$$

This not only allows us to recursively solve β_n; but also indicate, by the inverse even contraction (2.17) and the odd contraction (2.18), that

$$\frac{G(z)}{\frac{x^3-x}{24}}=\cfrac{1}{1-\cfrac{\beta_1 z^2}{1-\cfrac{\beta_2 z^2}{1-\ddots}}}=1+\cfrac{\beta_1 z^2}{1-(\beta_1+\beta_2)z^2-\cfrac{\beta_2\beta_3 z^4}{1-(\beta_3+\beta_4)z^2-\cfrac{\beta_4\beta_5 z^4}{\ddots}}},$$

which eventually leads to

$$H(z)=\cfrac{\frac{x^3-x}{24}\beta_1}{1-(\beta_1+\beta_2)z^2-\cfrac{\beta_2\beta_3 z^4}{1-(\beta_3+\beta_4)z^2-\cfrac{\beta_4\beta_5 z^4}{\ddots}}}.$$

Note that

$$\frac{x^3-x}{24}\beta_1=\frac{x^3-x}{24}\left(\frac{x^2-1}{12}+\frac{x^2-4}{15}\right)=\frac{(3x^2-7)x(x^2-1)}{480}=\frac{B_5\left(\frac{x+1}{2}\right)}{5}.$$

Hence,

$$H_n\left(\frac{B_{2k+5}\left(\frac{x+1}{2}\right)}{2k+5}\right)=\left(\frac{(3x^2-7)x(x^2-1)}{480}\right)^{n+1}\prod_{\ell=1}^{n}\left(\beta_{2\ell}\beta_{2\ell+1}\right)^{n+1-\ell}.$$

In fact, it is not hard to see $D_n^{(1)}=\prod_{\ell=0}^{n}\beta_{2\ell+1}$.

Remark: The anonymous referee suggests here, to get the continued fraction of H from the continued fraction G, one can consider the Chop contraction and Haircut contraction, see, e.g., [14, Thm. 2.1]. For more general applications, one can also consider the proof of Theorem 4.3, in [14]. We did not continue to pursue this direction.

5.3. **Further results.** We list some nice partial results with $x=2r+1$ that can be easily calculated from the results above.

- $\alpha_{2r+1}=0$, which implies $\tau_r^{(1)}=0$.
- $D_{r-1}^{(1)}=r!^2$ and $D_r^{(1)}=-\frac{(r+1)!^2}{4r+5}$.
- $\beta_{2r}=0$ and $\beta_{2r+1}=-\frac{(r+1)^2}{4r+5}$.
- $$\frac{H_{r-1}\left(\frac{B_{2k+5}(r+1)}{2k+5}\right)}{H_r\left(\frac{B_{2k+1}(r+1)}{2k+1}\right)}=2 \quad\text{and}\quad \frac{H_{r-1}\left(\frac{B_{2k+5}(r+1)}{2k+5}\right)}{H_{r-1}\left(\frac{B_{2k+3}(r+1)}{2k+3}\right)}=(r!)^2.$$

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