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HANKEL DETERMINANTS OF SHIFTED SEQUENCES OF BERNOULLI AND EULER NUMBERS

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Abstract. Hankel determinants of sequences related to Bernoulli and Euler numbers have been studied before, and numerous identities are known. However, when a sequence is shifted by one unit, the situation often changes significantly. In this paper we use classical orthogonal polynomials and related methods to prove a general result concerning Hankel determinants for shifted sequences. We then apply this result to obtain new Hankel determinant evaluations for a total of 14 sequences related to Bernoulli and Euler numbers, one of which concerns Euler polynomials.

1. Introduction

The *Hankel determinant* of a sequence $\mathbf{c} = (c_0, c_1, \ldots)$ of numbers or polynomials is defined as the determinant of the *Hankel matrix*, or *persymmetric matrix*, given by

$$(1.1) \qquad \left(c_{i+j}\right)_{0\le i,j\le n} = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_n \\ c_1 & c_2 & c_3 & \cdots & c_{n+1} \\ c_2 & c_3 & c_4 & \cdots & c_{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n+1} & c_{n+2} & \cdots & c_{2n} \end{pmatrix}.$$

Hankel determinants of various classes of sequences have been extensively studied, partly due to their close relationship with classical orthogonal polynomials; see, e.g., [10, Ch. 2]. In fact, many evaluations of Hankel determinants come from this connection and a related connection with continued fractions. For numerous results see, e.g., the very extensive treatments in [14, 15, 16], and the numerous references provided there.

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In the recent paper [6] we used the connection with orthogonal polynomials and continued fractions to find new evaluations of Hankel determinants of certain subsequences of Bernoulli and Euler polynomials. This was followed in [7] by evaluations of Hankel determinants of various other sequences related to Bernoulli and Euler numbers and polynomials.

We recall that the *Bernoulli numbers* B_n and *Bernoulli polynomials* $B_n(x)$ are usually defined by the generating functions

$$\frac{t}{e^t-1}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!}\qquad\text{and}\qquad\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}. \tag{1.2}$$

We have $B_0=1$, $B_1=-1/2$, and $B_{2j+1}=0$ for $j\geq 1$; a few further values are listed in Table 1. The *Euler numbers* E_n and *polynomials* $E_n(x)$ are defined by the generating functions

$$\frac{2}{e^t+e^{-t}}=\sum_{n=0}^{\infty}E_n\frac{t^n}{n!}\qquad\text{and}\qquad\frac{2e^{xt}}{e^t+1}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}. \tag{1.3}$$

The first few values are again given in Table 1. Comparing the generating functions in (1.2) and in (1.3), respectively, we get

$$B_n(x)=\sum_{j=0}^{n}\binom{n}{j}B_jx^{n-j},\qquad 2^nE_n(x)=\sum_{j=0}^{n}\binom{n}{j}E_j(2x-1)^{n-j}, \tag{1.4}$$

and thus, for all $n=0,1,\ldots,$

$$B_n=B_n(0),\qquad E_n=2^nE_n\big(\tfrac{1}{2}\big). \tag{1.5}$$

The four sequences in (1.2) and in (1.3) are among the most important special number and polynomial sequences in mathematics, with numerous applications in number theory, combinatorics, numerical analysis, and other areas. Further properties can be found, e.g., in [18, Ch. 24].

n	B_n	E_n	$B_n(x)$	$E_n(x)$	$E_n(1)$
0	1	1	1	1	1
1	$-1/2$	0	$x-\frac{1}{2}$	$x-\frac{1}{2}$	$1/2$
2	$1/6$	-1	$x^2-x+\frac{1}{6}$	x^2-x	0
3	0	0	$x^3-\frac{3}{2}x^2+\frac{1}{2}x$	$x^3-\frac{3}{2}x^2+\frac{1}{4}$	$-1/4$
4	$-1/30$	5	$x^4-2x^3+x^2-\frac{1}{30}$	x^4-2x^3+x	0
5	0	0	$x^5-\frac{5}{2}x^4+\frac{5}{3}x^3-\frac{1}{6}x$	$x^5-\frac{5}{2}x^4+\frac{5}{2}x^2-\frac{1}{2}$	$1/2$
6	$1/42$	-61	$x^6-3x^5+\frac{5}{2}x^4-\frac{1}{2}x^2+\frac{1}{42}$	$x^6-3x^5+5x^3-3x$	0

TABLE 1. $B_n, E_n, B_n(x)$, $E_n(x)$, and $E_n(1)$ for $0\leq n\leq 6$.

The paper [7] also contains a list of all "Hankel-Bernoulli/Euler" identities known to us. It turned out that the majority of such identities, when written in a standard way, are of a very specific form. Indeed, if we set

$$H_n(\mathbf{c}) = H_n(c_k) = \det_{0\le i,j\le n}\big(c_{i+j}\big), \tag{1.6}$$

then for a large number of sequences $\mathbf{c}$ related to Bernoulli and Euler numbers and polynomials we have

$$H_n(c_k) = (-1)^{\varepsilon(n)}\cdot a^{n+1}\cdot\prod_{\ell=1}^{n} b(\ell)^{n+1-\ell}, \tag{1.7}$$

where $\varepsilon(n)$ is either 0 or a polynomial in n of degree at most 2, a is a positive rational number, and $b(\ell)$ is a rational function in ℓ having only linear factors in the numerator and the denominator. (In a few cases where c_k is a polynomial sequence, a and $b(\ell)$ also have some linear factors in x). For instance, if $c_k = B_{2k+2}$, then

$$\varepsilon(n) = 0,\quad a = \frac{1}{6},\quad b(\ell) = \frac{\ell^3(\ell+1)(2\ell-1)(2\ell+1)^3}{(4\ell-1)(4\ell+1)^2(4\ell+3)}, \tag{1.8}$$

and when $c_k = B_{2k+4}$, we have

$$\varepsilon(n) = n+1,\quad a = \frac{1}{30},\quad b(\ell) = \frac{\ell(\ell+1)^3(2\ell+1)^3(2\ell+3)}{(4\ell+1)(4\ell+3)^2(4\ell+5)}. \tag{1.9}$$

These two expressions were adapted from the identities (3.59) and (3.60), respectively, in [14].

On the other hand, while there is such a formula for $c_k = B_{2k}(1/2)$, no general identity has been known for the seemingly more natural sequence $c_k = B_{2k}$. In fact, the four smallest nontrivial Hankel determinants $H_n(B_{2k})$, for $n = 1, 2, 3, 4$, factor as

$$\frac{-11}{2^2\cdot 3^2\cdot 5},\quad \frac{137}{2\cdot 3^2\cdot 5^3\cdot 7^2},\quad \frac{-2^2\cdot 3}{5\cdot 7^4\cdot 13},\quad \frac{2^{10}\cdot 7129}{5\cdot 7^2\cdot 11^4\cdot 13^3\cdot 17},$$

respectively. This indicates that we cannot expect an identity such as (1.7). However, further numerical experiments revealed that for all $n\ge 1$ we might conjecture

$$H_n(B_{2k}) = (-1)^n\frac{(4n+3)!}{(n+1)\cdot(2n+1)!^3}\cdot\mathcal{H}_{2n+1}\cdot H_n(B_{2k+2}), \tag{1.10}$$

where $H_n(B_{2k+2})$ is given by (1.7) and (1.8), and $\mathcal{H}_n$ is the nth harmonic number

$$\mathcal{H}_n = \sum_{j=1}^{n}\frac{1}{j}. \tag{1.11}$$

It is the purpose of this paper to prove the identity (1.10) and a number of other similar and apparently new identities. When $(b_k)_{k\ge 0}$ is a given

$b_k, k \geq 1$	b_0	Prop.	$b_k, k \geq 1$	b_0	Prop.
B_{k-1}	0	3.1	$E_{k+3}(1)$	$(\frac{-1}{4})$	5.2
B_{2k}	(1)	6.1	$E_{2k-1}(1)$	0	3.3
$(2k+1)B_{2k}$	(1)	6.2	$E_{2k+5}(1)$	$(\frac{1}{2})$	5.1
$(2^{2k}-1)B_{2k}$	(0)	3.4	$E_k(1)/k!$	(1)	3.6
$(2k+1)E_{2k}$	0	3.5	$E_{2k-1}(1)/(2k-1)!$	0	6.3
E_{2k-2}	0	7.3	$E_{2k-2}(\frac{x+1}{2})$	0	7.2
$E_{k-1}(1)$	0	3.2	$E_{k-1}^{(p)}$	$\alpha \in \mathbb{R}$	8.1

TABLE 2. Summary of results.

sequence, the identities are all of the form

$$H_n(b_k) = F_n H_n(b_{k+1}), \tag{1.12}$$

where $H_n(b_{k+1})$ has a known evaluation. The sequence F_n is most often a sequence of harmonic or related numbers, but factorials and in one case a recurrence sequence also occur.

This paper is structured as follows. In Section 2 we provide some necessary background, mainly related to orthogonal polynomials, and we prove a lemma that will be the basis for all further results. In Section 3 we state and prove six different Hankel determinant evaluations that follow more or less directly from this lemma. Then, in Section 4, we state without proof several further known auxiliary results, which will then be used in the remaining sections to prove several more Hankel determinant evaluations. Section 7 is different from the previous ones in that it deals with sequences of *polynomials*, and Section 8 deals with a situation where the results are particularly simple. The paper ends with a few additional remarks.

We conclude this introduction with a summary of Hankel determinant identities that are proved in this paper. See Table 2, were the sequence (b_k) is as in (1.12), and those values of b_0 that are consistent with the general b_k are given in parentheses.

2. Orthogonal polynomials and a fundamental lemma

We begin this section with some necessary background on the connection between orthogonal polynomials and Hankel determinants. All this is well-known and can also be found in concise form in [6] and [7]. We repeat this material here for easy reference, and to make this paper self-contained. The second part of this section is new, and will be the basis for much of what follows.

2.1. **Orthogonal polynomials.** Suppose we are given a sequence $\mathbf{c} = (c_0, c_1, \ldots)$ of numbers; then we can define a linear functional L on polynomials by

$$L(x^k) = c_k, \quad k = 0, 1, 2, \ldots. \tag{2.1}$$

We may also normalize the sequence such that $c_0 = 1$. We now summarize several well-known facts and state them as a lemma with two corollaries; see, e.g., [10, Ch. 2] and [3, pp. 7–10].

Lemma 2.1. *Let L be the linear functional in* (2.1). *If (and only if) $H_n(c_k) \neq 0$ for all $n = 0, 1, 2, \ldots$, there exists a unique sequence of monic polynomials $P_n(y)$ of degree n, $n = 0, 1, \ldots$, and a sequence of positive numbers $(\zeta_n)_{n\geq 1}$, with $\zeta_0 = 1$, such that*

$$L\left(P_m(y)P_n(y)\right) = \zeta_n \delta_{m,n}, \tag{2.2}$$

where $\delta_{m,n}$ is the Kronecker delta function. Furthermore, for all $n \geq 1$ we have $\zeta_n = H_n(\mathbf{c})/H_{n-1}(\mathbf{c})$, and for $n \geq 1$,

$$P_n(y) = \frac{1}{H_{n-1}(\mathbf{c})} \det \begin{pmatrix} c_0 & c_1 & \cdots & c_n \\ c_1 & c_2 & \cdots & c_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_n & \cdots & c_{2n-1} \\ 1 & y & \cdots & y^n \end{pmatrix}, \tag{2.3}$$

where the polynomials $P_n(y)$ satisfy the 3*-term recurrence relation $P_0(y) = 1$, $P_1(y) = y + s_0$, and*

$$P_{n+1}(y) = (y + s_n)P_n(y) - t_n P_{n-1}(y) \qquad (n \geq 1), \tag{2.4}$$

for some sequences $(s_n)_{n\geq 0}$ and $(t_n)_{n\geq 1}$.

We now multiply both sides of (2.3) by y^r and replace y^j by c_j, which includes replacing the constant term 1 by c_0 for $r = 0$. Then for $0 \leq r \leq n-1$ the last row of the matrix in (2.3) is identical with one of the previous rows, and thus the determinant is 0. When $r = n$, the determinant is $H_n(\mathbf{c})$. We therefore have the following result.

Corollary 2.2. *With the sequence (c_k) and the polynomials $P_n(y)$ as above, we have*

$$y^r P_n(y)\Big|_{y^k = c_k} = \begin{cases} 0 & \text{when } 0 \leq r \leq n-1, \\ H_n(\mathbf{c})/H_{n-1}(\mathbf{c}) & \text{when } r = n. \end{cases} \tag{2.5}$$

The polynomials $P_n(y)$ are known as "the monic orthogonal polynomials belonging to the sequence $\mathbf{c} = (c_0, c_1, \ldots)$", or "the polynomials orthogonal with respect to $\mathbf{c}$". Another important consequence of Lemma 2.1 is the main reason for the specific form of the general formula (1.7).

Corollary 2.3. *With the sequence (t_n) as in* (2.4), *we have*

$$H_n(\mathbf{c}) = t_1^n t_2^{n-1} \cdots t_{n-1}^2 t_n \qquad (n \geq 0). \tag{2.6}$$

The next lemma, which will also be required later in this paper, deals with the case where $\mathbf{c}$ is a sequence of functions in a single variable x. It was proved as Lemma 5.7 in [7].

Lemma 2.4. *Let $c_k(x)$ be a sequence of C^1 functions, and let $P_n(y;x)$ be the corresponding monic orthogonal polynomials. If $c_k(x_0)=0$ for some $x_0\in\mathbb{C}$ and for all $k\geq 0$, then $P_n(y;x_0)$ are the monic orthogonal polynomials with respect to the sequence of derivatives $c_k'(x_0)$, as long as $H_n(c_k'(x_0))$ are all nonzero.*

2.2. **A fundamental lemma.** We are now ready to state and prove a general lemma which will be used in most of the proofs that follow.

Lemma 2.5. *Let $\mathbf{c}=(c_0,c_1,\dots)$ be a sequence such that the unique sequence $P_n(y)$, $n\geq 0$, of polynomials orthogonal with respect to $\mathbf{c}$ exists. Let α be a constant and define the sequence $\mathbf{b}=(b_0,b_1,\dots)$ by*

$$b_k := \begin{cases} \alpha, & k=0,\\ c_{k-1}, & k\geq 1. \end{cases} \tag{2.7}$$

Then for all $n\geq 2$ we have

$$\frac{H_{n+1}(b_k)}{H_n(c_k)} = -s_n\frac{H_n(b_k)}{H_{n-1}(c_k)} - t_n\frac{H_{n-1}(b_k)}{H_{n-2}(c_k)}, \tag{2.8}$$

where the sequences (s_n) and (t_n) are as in (2.4).

Some care must be taken when $\alpha=0$. In this case $H_0(b_k)=0$ and thus $P_1(y)$, as given by (2.3), does not exist. However, as long as $H_n(b_k)\neq 0$ for $k\geq 1$, due to uniqueness the terms $P_n(y)$, for $n\geq 2$, are still given by (2.3); meanwhile, for $n=1$, we can compute the Hankel determinant directly.

Proof of Lemma 2.5. By the definitions (1.1), (1.6), and (2.7) we have

$$\begin{aligned}
H_n(b_k) &= \det\begin{pmatrix} \alpha & c_0 & c_1 & \cdots & c_{n-1}\\ c_0 & c_1 & c_2 & \cdots & c_n\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ c_{n-1} & c_n & c_{n+1} & \cdots & c_{2n-1}\end{pmatrix}\\
&= (-1)^n\det\begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_n\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ c_{n-1} & c_n & c_{n+1} & \cdots & c_{2n-1}\\ \alpha & c_0 & c_1 & \cdots & c_{n-1}\end{pmatrix}\\
&= (-1)^n H_{n-1}(c_k)\left(\left.\frac{P_n(y)-P_n(0)}{y}\right|_{y^k=c_k} + \alpha P_n(0)\right),
\end{aligned}$$

where in the last equation we have used (2.3). Hence

$$\left.\frac{P_n(y)-P_n(0)}{y}\right|_{y^k=c_k} + \alpha P_n(0) = (-1)^n\frac{H_n(b_k)}{H_{n-1}(c_k)}. \tag{2.9}$$

Next, setting $y = 0$ in (2.4), we get

$$P_{n+1}(0) = s_n P_n(0) - t_n P_{n-1}(0). \tag{2.10}$$

We then subtract (2.10) from (2.4), divide both sides by y, and add to the resulting equation once again (2.10), multiplied by α. This gives

$$\begin{aligned} &\frac{P_{n+1}(y) - P_{n+1}(0)}{y} + \alpha P_{n+1}(0) \\ =&P_n(y) + s_n \left(\frac{P_n(y) - P_n(0)}{y} + \alpha P_n(0) \right) \\ &- t_n \left(\frac{P_{n-1}(y) - P_{n-1}(0)}{y} + \alpha P_{n-1}(0) \right). \end{aligned} \tag{2.11}$$

Finally, if we evaluate $y^k = c_k$, then by (2.5) the first term on the right-hand side of (2.11), namely $P_n(y)$, vanishes, and (2.11) with (2.9) yields the desired identity (2.8). □

3. Hankel determinant identities, I

In this section we deal with those identities whose proofs follow most directly from Lemma 2.5. In addition to the harmonic numbers $\mathcal{H}_n$ in (1.11), we require the following related sequences: For $n \geq 1$ we denote

$$\mathcal{H}_{1,n}^{-} := \sum_{j=1}^{n} \frac{(-1)^{j-1}}{j}, \qquad \mathcal{H}_{2,n}^{-} := \sum_{j=1}^{n} \frac{(-1)^{j-1}}{j^2}. \tag{3.1}$$

We can now state and then prove the following six results.

Proposition 3.1. *If the sequence* $(b_0, b_1, \ldots)$ *is defined by*

$$b_k := \begin{cases} 0, & k = 0, \\ B_{k-1}, & k \geq 1, \end{cases}$$

then

$$H_n(b_k) = \frac{2 \cdot (2n+1)!}{n!^3} \cdot \mathcal{H}_{2,n}^{-} \cdot H_n(B_k). \tag{3.2}$$

The known Hankel determinant on the right of (3.2), and in all further results, will be given in the relevant proofs below.

Proposition 3.2. *If the sequence* $(b_0, b_1, \ldots)$ *is defined by*

$$b_k := \begin{cases} 0, & k = 0, \\ E_{k-1}(1), & k \geq 1, \end{cases}$$

then

$$H_n(b_k) = (-1)^{n-1} \frac{2^{n+1}}{n!} \cdot \mathcal{H}_{1,n}^{-} \cdot H_n\big(E_k(1)\big). \tag{3.3}$$

Here it should be mentioned that

$$E_k(1) = \frac{2}{k+1}\left(2^{k+1}-1\right)B_{k+1} \quad (k \geq 1); \tag{3.4}$$

see, e.g., [18, Eq. 24.4.26]. The following result can be seen as the odd-index analogue of Proposition 3.2.

Proposition 3.3. *If the sequence* $(b_0, b_1, \ldots)$ *is defined by*

$$b_k := \begin{cases} 0, & k = 0, \\ E_{2k-1}(1), & k \geq 1, \end{cases}$$

then

$$H_n(b_k) = (-1)^n \frac{2^{2n+1}}{(2n+1)!} \cdot \mathcal{H}_n \cdot H_n\big(E_{2k+1}(1)\big). \tag{3.5}$$

Once again, (3.4) could be used to rewrite this result in terms of Bernoulli numbers. The next result is related to the right-hand side of (3.4).

Proposition 3.4. *For all* $n \geq 1$ *we have*

$$H_n\big((2^{2k}-1)B_{2k}\big) = \frac{(-1)^n}{n!(n+1)!} \cdot \mathcal{H}_n \cdot H_n\big((2^{2k+2}-1)B_{2k+2}\big). \tag{3.6}$$

The following result is somewhat different from the previous ones. While its proof is similar to that of Proposition 3.4, the statement involves a recurrence sequence in place of a harmonic or related sequence.

Proposition 3.5. *If the sequence* $(b_0, b_1, \ldots)$ *is defined by*

$$b_k := \begin{cases} 0, & k = 0, \\ (2k+1)E_{2k}, & k \geq 1, \end{cases}$$

then for all $n \geq 1$*,*

$$H_n(b_k) = \frac{(-1)^n}{(2^n n!)^4} \cdot h_n \cdot H_n\big((2k+1)E_{2k}\big), \tag{3.7}$$

where the sequence $(h_n)_{n\geq 0}$ *is defined by* $h_0 = 0, h_1 = 1$*, and for* $n \geq 1$

$$h_{n+1} = (8n^2+8n+3)h_n - (2n)^4 h_{n-1}. \tag{3.8}$$

The first few terms of the sequence (h_n), starting with $n = 1$, are

$$1,\ 19,\ 713,\ 45\,963,\ 4\,571\,521,\ 651\,249\,603,\ 125\,978\,555\,961.$$

The final result in this section is again different from the previous ones in that it contains neither a harmonic-type sequence, nor a recurrence sequence.

Proposition 3.6. *For all* $n \geq 1$ *we have*

$$H_n\big(E_k(1)/k!\big) = \frac{(2n+2)!}{(n+1)!} \cdot H_n\big(E_{k+1}(1)/(k+1)!\big). \tag{3.9}$$

The remainder of this section contains the proofs of Propositions 3.1–3.6.

Proof of Proposition 3.1. The Hankel determinants $H_n(B_k)$ of the Bernoulli numbers were first determined by Al-Salam and Carlitz [1]; here we use a standard format as given in [7, Sect. 7.1], namely

$$H_n(B_k) = (-1)^{n(n+1)/2} \prod_{\ell=1}^{n} \left(\frac{\ell^4}{4(2\ell+1)(2\ell-1)} \right)^{n+1-\ell}, \tag{3.10}$$

so that

$$\frac{H_n(B_k)}{H_{n-1}(B_k)} = (-1)^n \prod_{\ell=1}^{n} \frac{\ell^4}{4(2\ell+1)(2\ell-1)} = \frac{(-1)^n n!^6}{(2n)!(2n+1)!}.$$

Comparing this with (3.2), we see that we need to show that for all $n \geq 1$ we have

$$\frac{H_n(b_k)}{H_{n-1}(B_k)} = (-1)^n \frac{2 \cdot n!^3}{(2n)!} \cdot \mathcal{H}_{2,n}^{-} =: r_n. \tag{3.11}$$

Touchard [19] was the first to show, in a slightly different normalization, that (2.4) holds for $c_k = B_k$ with

$$s_n = \frac{1}{2}, \qquad t_n = \frac{-n^4}{4(2n+1)(2n-1)}.$$

Hence by Lemma 2.5 we are done if we can verify that the sequence (r_n) satisfies the recurrence relation

$$r_{n+1} = -\frac{1}{2} r_n + \frac{n^4}{4(2n+1)(2n-1)} r_{n-1}. \tag{3.12}$$

By direct computation we find $r_1 = -1$ and $r_2 = 1/2$, which holds for both sides of (3.11). Next, if we substitute (3.11) into (3.12) and multiply both sides by $(2n+1)!/n!^3$, we get

$$(n+1)^2 \mathcal{H}_{2,n+1}^{-} = (2n+1)\mathcal{H}_{2,n}^{-} + n^2 \mathcal{H}_{2,n-1}^{-}. \tag{3.13}$$

On the other hand, from the definition in (3.1) we have

$$\mathcal{H}_{2,n+1}^{-} - \mathcal{H}_{2,n}^{-} = \frac{(-1)^n}{(n+1)^2}. \tag{3.14}$$

Replacing n by $n-1$ and combining the resulting identity with (3.14), we get

$$(n+1)^2 \left(\mathcal{H}_{2,n+1}^{-} - \mathcal{H}_{2,n}^{-} \right) = -n^2 \left(\mathcal{H}_{2,n}^{-} - \mathcal{H}_{2,n-1}^{-} \right),$$

which is equivalent to (3.13). Hence we have verified (3.12), and the proof is complete. □

Proof of Proposition 3.2. The outline is the same as that of the previous proof. The evaluation of $H_n\big(E_k(1)\big)$, and in fact for $H_n\big(E_k(x)\big)$, is also due to Al-Salam and Carlitz [1] (who used what is stated as Lemma 7.1 below), but again we give it in the equivalent form

$$H_n\big(E_k(1)\big) = (-1)^{n(n+1)/2} \prod_{\ell=1}^{n} \left(\frac{\ell^2}{4}\right)^{n+1-\ell} \tag{3.15}$$

(see [7, Sect. 7.1]), so that

$$\frac{H_n\big(E_k(1)\big)}{H_{n-1}\big(E_k(1)\big)} = (-1)^n \prod_{\ell=1}^{n} \frac{\ell^2}{4} = (-1)^n \frac{n!^2}{2^{2n}}.$$

Comparing this with (3.3), we see that we need to show that for all $n \geq 1$ we have

$$\frac{H_n(b_k)}{H_{n-1}\big(E_k(1)\big)} = -\frac{n!}{2^{n-1}} \cdot \mathcal{H}^-_{1,n} =: r_n. \tag{3.16}$$

By using the special case $x = 1$ in Theorem 1 of [12], we see that (2.4) holds for $c_k = E_k(1)$ with

$$s_n = -\frac{1}{2}, \qquad t_n = -\frac{n^2}{4}.$$

Hence by Lemma 2.5 we are done if we can verify that the sequence (r_n) satisfies

$$r_{n+1} = \frac{1}{2} r_n + \frac{1}{4}\, n^2 r_{n-1}. \tag{3.17}$$

By direct computation we find that $r_1 = -1$ and $r_2 = -1/2$ hold for both sides of (3.16). If we substitute (3.16) into (3.17) and multiply both sides by $-n!/2^n$, we get the equivalent form

$$(n+1)\mathcal{H}^-_{1,n+1} = \mathcal{H}^-_{1,n} + n\mathcal{H}^-_{1,n-1}.$$

This, finally, is easy to verify using the definition in (3.1). The proof is now complete. □

Proof of Proposition 3.3 (sketch). Since this proof follows again the same outline as before, we only give the two main ingredients. First, the Hankel determinant of the sequence $\big(E_{2k+1}(1)\big)_{k\geq 0}$ can be found in [16, Eq. (4.56)], or equivalently in [7, Sect. 7.1] as

$$H_n\big(E_{2k+1}(1)\big) = \frac{1}{2^{n+1}} \prod_{\ell=1}^{n} \left(\frac{\ell^2(2\ell-1)(2\ell+1)}{4}\right)^{n+1-\ell}. \tag{3.18}$$

Second, using [6, Eq. (5.4)] with $\nu = 1$ and $x = 1$ we see that (2.4) is satisfied with

$$s_n = \frac{(2n+1)^2}{2}, \qquad t_n = \frac{n^2(2n+1)(2n-1)}{4}.$$

We leave all further details of the proof to the reader. □

Proof of Proposition 3.4 (sketch). Once again, the proof proceeds as before, with the first main ingredient being

$$H_n\big((2^{2k+2}-1)B_{2k+2}\big) = \frac{1}{2^{n+1}}\prod_{\ell=1}^{n}\big(\ell^3(\ell+1)\big)^{n+1-\ell}, \tag{3.19}$$

which was obtained as Corollary 5.3 in [7].

Finding the second main ingredient, namely the pair of coefficient sequences (s_n) and (t_n), is a bit less straightforward than in the previous proofs. In Theorem 5.1 of [6] with $\nu = 2$, the orthogonal polynomials belonging to the polynomial sequence $\big(E_{2k+2}((1+x)/2)\big)_{k\geq 0}$ is given as

$$P_{n+1}(y;x) = \big(y+s_n(x)\big)P_n(y;x) - t_n(x)P_{n-1}(y;x),$$

with appropriate initial conditions, and with

$$s_n(x) = (2n+1)(n+1) - \frac{x^2-1}{4}, \qquad t_n(x) = \frac{n^2}{4}\left((2n+1)^2 - x^2\right). \tag{3.20}$$

Next we note that by well-known properties of Euler polynomials, namely the identity (3.4) and the fact that $\frac{d}{dx}E_k(x) = kE_{k-1}(x)$, we have

$$(2^{2k+2}-1)B_{2k+2} = \frac{d}{dx}E_{2k+2}\big(\tfrac{1+x}{2}\big)\Big|_{x=1}.$$

Since $E_{2k+2}(1) = 0$ for all $k \geq 0$, by Lemma 2.4 the sequence $\big((2^{2k+2}-1)B_{2k+2}\big)_{k\geq 0}$ has $P_n(y,1)$ as its associated orthogonal polynomials. This means that by (3.20) this sequence satisfies (2.4) with

$$s_n = s_n(1) = (2n+1)(n+1), \qquad t_n = t_n(1) = n^3(n+1).$$

From here on we proceed as in the previous three proofs; once again we leave the details to the reader. □

Proof of Proposition 3.5. With the aim of applying Lemma 2.4, we set $c_k(x) := E_{2k+1}(x)$. Then with (1.5),

$$c_k(\tfrac{1}{2}) = E_{2k+1}(\tfrac{1}{2}) = 2^{-2k-1}E_{2k+1} = 0 \qquad (k \geq 0). \tag{3.21}$$

By Theorem 5.1 of [6] with $\nu = 1$ we know that the monic orthogonal polynomials with respect to $E_{2k+1}((x+1)/2)$ are given by $q_0(y;x) = 1$, $q_1(y;x) = y + s_0(x)$, and

$$q_{n+1}(y;x) = \big(y+s_n(x)\big)q_n(y;x) - t_n(x)q_{n-1}(y;x) \qquad (n \geq 1), \tag{3.22}$$

where

$$s_n(x) = (2n+1)(n+\tfrac{1}{2}) - \frac{x^2-1}{4}, \qquad t_n(x) = \frac{n^2}{4}\left(4n^2 - x^2\right),$$

and thus

$$s_n := s_n(0) = 2n^2 + 2n + \frac{3}{4}, \qquad t_n := t_n(0) = n^4. \tag{3.23}$$

By Lemma 2.4 with (3.21), the polynomials (3.22) with $x=0$ are therefore the monic orthogonal polynomials also for

$$c_k'(\tfrac{1}{2}) = (2k+1)E_{2k}(\tfrac{1}{2}) = 2^{-2k}(2k+1)E_{2k}, \tag{3.24}$$

where we have used (1.5) again. Now, to apply Lemma 2.5, we set

$$r_n := \frac{H_n\big(2^{-2k}b_k\big)}{H_{n-1}\big(2^{-2k}(2k+1)E_{2k}\big)} = \left(\frac{1}{2}\right)^{4n}\frac{H_n\big(b_k\big)}{H_{n-1}\big((2k+1)E_{2k}\big)},$$

where we have used the first identity in (4.10) below. Then we have

$$r_n = \left(\frac{1}{2}\right)^{4n}\frac{H_n\big(b_k\big)}{H_n\big((2k+1)E_{2k}\big)}\cdot\frac{H_n\big((2k+1)E_{2k}\big)}{H_{n-1}\big((2k+1)E_{2k}\big)}. \tag{3.25}$$

Next, by Corollary 5.2 in [7] we have

$$H_n\big((2k+1)E_{2k}\big) = 2^{2n(n+1)}\prod_{\ell=1}^{n}\ell!^4,$$

so that

$$\frac{H_n\big((2k+1)E_{2k}\big)}{H_{n-1}\big((2k+1)E_{2k}\big)} = 2^{4n}n!^4.$$

By combining this and (3.7) with (3.25) we see that we are done if we can show that

$$r_n = (-1)^n\left(\tfrac{1}{2}\right)^{4n}h_n. \tag{3.26}$$

We do this by applying Lemma 2.5 to the sequence (3.24), and we get

$$r_{n+1} = -s_nr_n - t_nr_{n-1}, \tag{3.27}$$

with the various terms given by (3.26) and (3.23). Multiplying both sides of (3.27) by $(-16)^{n+1}$, we see that it is equivalent to (3.8). Finally, the initial values for $n=1,2$ are again easy to establish by direct computation, which completes the proof. □

Proof of Proposition 3.6. Once again we proceed as in the earlier proofs, and note that in [8, Eq. (H12)] it was shown that

$$H_n\left(E_{k+1}(1)/(k+1)!\right) = \frac{(-1)^{n(n+1)/2}}{2^{n+1}}\prod_{\ell=1}^{n}\left(\frac{1}{4(2\ell-1)(2\ell+1)}\right)^{n+1-\ell}, \tag{3.28}$$

written in the format used in [7, Sect. 7.1]. To simplify notation, we set $c_k := E_{k+1}(1)/(k+1)!$; then we get with (3.28),

$$\frac{H_n(c_k)}{H_{n-1}(c_k)} = \frac{(-1)^n}{2}\prod_{\ell=1}^{n}\frac{1}{4(2\ell-1)(2\ell+1)} = \frac{(-1)^nn!^2}{2(2n)!(2n+1)!},$$

where the right-most term follows from some straightforward manipulations. Comparing this with (3.9), we see that for all $n \geq 1$ we need to show that

$$r_n := \frac{H_n(c_{k-1})}{H_{n-1}(c_k)} = \frac{H_n(c_k)}{H_{n-1}(c_k)} \cdot \frac{H_n(c_{k-1})}{H_n(c_k)} = (-1)^n \frac{n!}{(2n)!}. \tag{3.29}$$

The identity (3.4) implies that $c_k = 0$ for all odd $k \geq 1$. By the theory of classical orthogonal polynomials we then have $s_n = 0$, $n \geq 1$, for the polynomials in (2.4); see, e.g., Definition 4.1 and Theorem 4.3 in [3, pp. 20–21]. Furthermore, from (3.28) we get

$$t_n = \frac{-1}{4(2n-1)(2n+1)},$$

and the recurrence relation (2.8) reduces to $r_{n+1} = -t_n r_{n-1}$. It is now easy to verify that r_n, as given in (3.29), satisfies this recurrence. Finally, the initial values $r_1 = -1/2$ and $r_2 = 1/12$ can be verified by direct computation, which completes the proof. □

4. Further auxiliary results

In this section we quote a few known results that will be required in the proofs of more Hankel determinant identities in later sections.

As we have seen, this paper is mainly concerned with finding Hankel determinants of right-shifted sequences. Interestingly, for the proofs of some more such identities, known results on *left*-shifted sequences turn out to be useful; we summarize them now.

As before, let $\mathbf{c} = (c_0, c_1, \ldots)$ be a given sequence, and let $P_n(y)$, $n = 0, 1, \ldots$, be the polynomials orthogonal with respect to $\mathbf{c}$, satisfying the recurrence relation (2.4). Following [17], we consider the infinite band matrix

$$J := \begin{pmatrix} -s_0 & 1 & 0 & 0 & \cdots \\ t_1 & -s_1 & 1 & 0 & \cdots \\ 0 & t_2 & -s_2 & 1 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{4.1}$$

Furthermore, for each $n \geq 0$ let J_n be the $(n+1)$th leading principal submatrix of J and let

$$d_n := \det J_n, \tag{4.2}$$

so that $d_0 = -s_0$. We also set $d_{-1} = 1$ by convention, and furthermore, using elementary determinant operations, we get from (4.1) the recurrence relation

$$d_{n+1} = -s_{n+1} d_n - t_{n+1} d_{n-1}. \tag{4.3}$$

We can now quote the following results.

Lemma 4.1 ([17, Prop. 1.2])**.** *With notation as above, for a given sequence* $\mathbf{c}$ *we have*

$$H_n(c_{k+1}) = H_n(c_k) \cdot d_n, \tag{4.4}$$

and

$$H_n(c_{k+2}) = H_n(c_k) \cdot D_n, \qquad \textit{where} \quad D_n := \left(\prod_{\ell=1}^{n+1} t_\ell\right) \cdot \sum_{\ell=-1}^{n} \frac{d_\ell^2}{\prod_{j=1}^{\ell+1} t_j}. \tag{4.5}$$

Lemma 4.2 ([8, Eq. (2.4)]). *For a given sequence* $\mathbf{c}$ *and* (s_n) *as defined above, we have*

$$s_n = -\frac{1}{H_{n-1}(c_{k+1})} \left(\frac{H_{n-1}(c_k) H_n(c_{k+1})}{H_n(c_k)} + \frac{H_n(c_k) H_{n-2}(c_{k+1})}{H_{n-1}(c_k)} \right). \tag{4.6}$$

The next lemma is about determinants of "checkerboard matrices", namely matrices in which every other entry vanishes. This result can be found in [4] as Lemmas 5 and 6, and covers more general matrices than just Hankel matrices.

Lemma 4.3 ([4, Lemmas 5, 6]). *Let* $M = (M_{i,j})_{0 \le i,j \le n-1}$ *be a matrix. If* $M_{i,j} = 0$ *whenever* $i + j$ *is odd, then*

$$\det_{0 \le i.j \le n-1} (M_{i,j}) = \det_{0 \le i.j \le \lfloor (n-1)/2 \rfloor} (M_{2i,2j}) \cdot \det_{0 \le i.j \le \lfloor (n-2)/2 \rfloor} (M_{2i+1,2j+1}). \tag{4.7}$$

If $M_{i,j} = 0$ *whenever* $i + j$ *is even, then for even* n *we have*

$$\det_{0 \le i.j \le n-1} (M_{i,j}) = (-1)^{n/2} \det_{0 \le i.j \le \lfloor (n-1)/2 \rfloor} (M_{2i+1,2j}) \cdot \det_{0 \le i.j \le \lfloor (n-2)/2 \rfloor} (M_{2i,2j+1}), \tag{4.8}$$

while for odd n *we have*

$$\det_{0 \le i.j \le n-1} (M_{i,j}) = 0. \tag{4.9}$$

Lemma 4.3 is best explained by way of an example.

Example 1. By (4.7) we have

$$\det \begin{pmatrix} a & 0 & b & 0 & c \\ 0 & \mathbf{d} & 0 & \mathbf{e} & 0 \\ f & 0 & g & 0 & h \\ 0 & \mathbf{i} & 0 & \mathbf{j} & 0 \\ k & 0 & l & 0 & m \end{pmatrix} = \det \begin{pmatrix} a & b & c \\ f & g & h \\ k & l & m \end{pmatrix} \cdot \det \begin{pmatrix} \mathbf{d} & \mathbf{e} \\ \mathbf{i} & \mathbf{j} \end{pmatrix},$$

$$\det \begin{pmatrix} a & 0 & b & 0 \\ 0 & \mathbf{d} & 0 & \mathbf{e} \\ f & 0 & g & 0 \\ 0 & \mathbf{i} & 0 & \mathbf{j} \end{pmatrix} = \det \begin{pmatrix} a & b \\ f & g \end{pmatrix} \cdot \det \begin{pmatrix} \mathbf{d} & \mathbf{e} \\ \mathbf{i} & \mathbf{j} \end{pmatrix}.$$

We conclude this section with another useful property of Hankel determinants. It follows from basic determinant operations involving the matrix (1.1); details can be found in [6].

Lemma 4.4. *Let* x *be a variable or a complex number and* $(c_0, c_1, \ldots)$ *a sequence. Then*

$$H_n(x^k c_k) = x^{n(n+1)} H_n(c_k) \quad \textit{and} \quad H_n(x \cdot c_k) = x^{n+1} H_n(c_k). \tag{4.10}$$

5. Hankel determinant identities, II

This section will be concerned with further Hankel determinant identities involving subsequences of $E_k(1)$. In addition to the identity (3.18) for $H_n\left(E_{2k+1}(1)\right)$, we have

$$H_n\left(E_{2k+3}(1)\right) = \left(\frac{-1}{4}\right)^{n+1} \prod_{\ell=1}^{n} \left(\frac{\ell(\ell+1)(2\ell+1)^2}{4}\right)^{n+1-\ell}, \tag{5.1}$$

which was obtained in [16, Eq. (4.57)]; see also [7, Sect. 7.1]. To state the results in this section, we require another harmonic-type sequence, namely

$$\overline{\mathcal{H}}_n := \sum_{j=0}^{n} \frac{1}{2j+1} = \mathcal{H}_{2n+2} - \frac{1}{2}\mathcal{H}_{n+1}. \tag{5.2}$$

Proposition 5.1. *For $n \geq 0$ we have*

$$H_n\big(E_{2k+5}(1)\big) = \left(\frac{1}{2}\right)^{n+1} \overline{\mathcal{H}}_{n+1} \cdot \prod_{\ell=1}^{n+1} \left(\frac{\ell^2(2\ell+1)(2\ell-1)}{4}\right)^{n+2-\ell}. \tag{5.3}$$

Proof. Our main tool will be Lemma 4.1. Using (3.18) and (5.1) and some straightforward but tedious manipulations, we obtain

$$d_n := \frac{H_n\big(E_{2k+3}(1)\big)}{H_n\big(E_{2k+1}(1)\big)} = \left(-\frac{1}{4}\right)^{n+1} (2n+2)!. \tag{5.4}$$

Next, we note that the orthogonal polynomial (2.4) with respect to $c_k := E_{2k+1}(1)$ has

$$t_n = \frac{n^2(2n+1)(2n-1)}{4}, \tag{5.5}$$

which follows from (3.18). Now (4.5) with the convention $d_{-1} = 1$ gives

$$H_n\big(E_{2k+5}(1)\big) = H_n\big(E_{2k+1}(1)\big) \cdot \left(\prod_{\ell=1}^{n+1} t_\ell\right) \cdot \left(1 + \sum_{\ell=0}^{n} \frac{d_\ell^2}{\prod_{j=1}^{\ell+1} t_j}\right). \tag{5.6}$$

Using (5.4) and (5.5), and applying some straightforward manipulations the right-most term in large parentheses in (5.6) turns out to be

$$1 + \sum_{\ell=0}^{n} \frac{1}{2\ell+3} = \sum_{j=0}^{n+1} \frac{1}{2j+1} = \overline{\mathcal{H}}_{n+1}.$$

Substituting this and (3.18) and (5.5) into (5.6), we easily obtain the desired identity (5.3). □

Proposition 5.1 will now be used in the proof of the next result.

Proposition 5.2. *For $n \geq 1$ we have*

$$\frac{H_{2n}\big(E_{k+3}(1)\big)}{H_{2n-1}\big(E_{k+3}(1)\big)} = -\frac{(n+1)(2n+1)!^2}{2^{4n+2}} \tag{5.7}$$

and

(5.8)
$$H_{2n}\big(E_{k+3}(1)\big) = \frac{(-1)^{n+1}\cdot\overline{\mathcal{H}}_n}{2^{3n+2}}\cdot\prod_{\ell=1}^{n}\left(\frac{\ell^3(\ell+1)(2\ell+1)^3(2\ell-1)}{16}\right)^{n+1-\ell},$$

(5.9)
$$H_{2n-1}\big(E_{k+3}(1)\big) = \frac{(-1)^{n}\cdot 2^n\cdot\overline{\mathcal{H}}_n}{(n+1)(2n+1)!^2}\cdot\prod_{\ell=1}^{n}\left(\frac{\ell^3(\ell+1)(2\ell+1)^3(2\ell-1)}{16}\right)^{n+1-\ell}.$$

Proof. By (3.4) and the fact that $B_{2j+1}=0$ for $j\geq 1$, we have $E_{2j}(1)=0$ for any $j\geq 1$. This means that the determinant $H_n\big(E_{k+3}(1)\big)$ is of "checkerboard type", and we can apply the first case of Lemma 4.3. Using (4.7) with n replaced by $2n+1$ and by $2n$, respectively, and keeping in mind that $i+j=k$, we get the two identities

$$H_{2n}\big(E_{k+3}(1)\big) = H_n\big(E_{2k+3}(1)\big)\cdot H_{n-1}\big(E_{2k+5}(1)\big), \tag{5.10}$$

$$H_{2n-1}\big(E_{k+3}(1)\big) = H_{n-1}\big(E_{2k+3}(1)\big)\cdot H_{n-1}\big(E_{2k+5}(1)\big). \tag{5.11}$$

Taking the quotient of these identities and using (5.1), we obtain

$$\frac{H_{2n}\big(E_{k+3}(1)\big)}{H_{2n-1}\big(E_{k+3}(1)\big)} = \frac{H_n\big(E_{2k+3}(1)\big)}{H_{n-1}\big(E_{2k+3}(1)\big)} = -\frac{n(n+1)(2n+1)^2}{16}\prod_{\ell=1}^{n-1}\frac{\ell(\ell+1)(2\ell+1)^2}{4},$$

which gives (5.7) after some easy manipulations. The identities (5.8) and (5.9) follow immediately from (5.10) and (5.11), respectively, upon using (5.1) and (5.3). Alternatively, (5.9) can be obtained by combining (5.7) and (5.8). □

6. Hankel determinant identities, III

In this section we are going to prove three more identities that are similar in nature to the results in Section 3. However, while in the proofs of those results we were able to use known orthogonal polynomials belonging to the relevant sequences $(c_0, c_1, \ldots)$, in this section we still need to determine the coefficients s_n occurring in (2.4).

Proposition 6.1. *For all $n\geq 0$ we have*

$$H_n(B_{2k}) = (-1)^n\frac{(4n+3)!}{(n+1)\cdot(2n+1)!^3}\cdot\mathcal{H}_{2n+1}\cdot H_n(B_{2k+2}). \tag{6.1}$$

This is the identity (1.10) in the Introduction.

Proposition 6.2. *For all $n \geq 0$ we have*

$$H_n\big((2k+1)B_{2k}\big) = \frac{(-1)^n(2n+2)!}{n!(n+1)!^3}\cdot(\mathcal{H}_n + \mathcal{H}_{n+1})\cdot H_n\big((2k+3)B_{2k+2}\big). \tag{6.2}$$

The next identity does not contain harmonic or generalized harmonic numbers; but still, it belongs to the same category as the previous two identities.

Proposition 6.3. *If the sequence $(b_0, b_1, \ldots)$ is defined by*

$$b_k := \begin{cases} 0, & k = 0, \\ E_{2k-1}(1)/(2k-1)!, & k \geq 1, \end{cases}$$

then for all $n \geq 1$ we have

$$H_n(b_k) = (-1)^n \frac{(4n+2)!}{(2n-1)!} \cdot H_n\left(\frac{E_{2k+1}(1)}{(2k+1)!}\right). \tag{6.3}$$

The Hankel determinants on the right of (6.1), (6.2), and (6.3) are given explicitly by (1.8), (6.9), and (6.22), respectively.

We prove these three results in sequence. First, for the proof of Proposition 6.1 we require the following lemma.

Lemma 6.4. *If $P_n(y)$, $n = 0, 1, \ldots$, are the monic orthogonal polynomials with respect to the sequence $\big(B_{2k+2}\big)_{k\geq 0}$, then*

(6.4)

$$s_n = \frac{(n+1)(2n+1)(4n^2+6n+1)}{(4n+1)(4n+5)}, \qquad t_n = \frac{n^3(n+1)(2n-1)(2n+1)^3}{(4n-1)(4n+1)^2(4n+3)},$$

with s_n, t_n as in (2.4).

Proof. In view of Corollary 2.3, the identities (1.7) and (1.8) immediately give t_n in (6.8). Next, by (4.6) and using (1.7)–(1.9), we get after some easy manipulations,

$$d_n = \frac{H_n(B_{2k+4})}{H_n(B_{2k+2})} = \left(\frac{-1}{5}\right)^{n+1} \prod_{\ell=1}^{n} \left(\frac{(\ell+1)^2(2\ell+3)(4\ell-1)(4\ell+1)}{\ell^2(2\ell-1)(4\ell+3)(4\ell+5)}\right)^{n+1-\ell},$$

so that

$$\begin{aligned} \frac{d_n}{d_{n-1}} &= \frac{-1}{5} \prod_{\ell=1}^{n} \frac{(\ell+1)^2(2\ell+3)(4\ell-1)(4\ell+1)}{\ell^2(2\ell-1)(4\ell+3)(4\ell+5)} \\ &= -\frac{(n+1)^2(2n+1)(2n+3)}{(4n+3)(4n+5)}, \end{aligned}$$

where the right-most term also follows by easy manipulations. Using this and the identity (4.3), along with the second equation in (6.4), we get

$$\begin{aligned}s_n &= -\frac{d_n}{d_{n-1}} - t_n\frac{d_{n-2}}{d_{n-1}}\\ &=\frac{(n+1)^2(2n+1)(2n+3)}{(4n+3)(4n+5)}\\ &\quad+\frac{n^3(n+1)(2n-1)(2n+1)^3}{(4n-1)(4n+1)^2(4n+3)}\cdot\frac{(4n-1)(4n+1)}{n^2(2n-1)(2n+1)}\\ &=\frac{(n+1)(2n+1)}{(4n+1)(4n+5)}(4n^2+6n+1),\end{aligned}$$

where the final term is again the result of some easy manipulations. This completes the proof of Lemma 6.4. □

Proof of Proposition 6.1. We proceed as in the proofs in Section 3. By (1.7) and (1.8) we have

$$\begin{aligned}\frac{H_n(B_{2k+2})}{H_{n-1}(B_{2k+2})} &= \frac{1}{6}\prod_{\ell=1}^{n}\frac{\ell^3(\ell+1)(2\ell-1)(2\ell+1)^3}{(4\ell-1)(4\ell+1)^2(4\ell+3)}\\ &= \frac{(2n+1)^4(n+1)(2n)!^6}{(4n+1)!(4n+3)!},\end{aligned}$$

where the last term follows after some tedious but straightforward manipulations. Therefore, in order to prove (6.1), we need to show that

$$\begin{aligned}r_n &:= \frac{H_n(B_{2k})}{H_{n-1}(B_{2k+2})}\\ &= \frac{H_n(B_{2k})}{H_n(B_{2k+2})}\cdot\frac{H_n(B_{2k+2})}{H_{n-1}(B_{2k+2})}\\ &= (-1)^n\frac{(2n)!^2(2n+1)!}{(4n+1)!}\mathcal{H}_{2n+1}.\end{aligned}\tag{6.5}$$

Direct computation shows that this holds for $n = 1$ and $n = 2$, and by Lemma 2.5 we are done if the right-most term in (6.5) satisfies

$$r_{n+1} = -s_n r_n - t_n r_{n-1},\tag{6.6}$$

with s_n, t_n as in (6.4). For greater ease of notation we now set $h_n := \mathcal{H}_{2n+1}$. Clearing the denominators and all common factors, we see that (6.6) is equivalent to

$$(n+1)(2n+3)(4n+1)h_{n+1} = (4n+3)(4n^2+6n+1)h_n - n(2n+1)(4n+5)h_{n-1}.\tag{6.7}$$

On the other hand, by the definition (1.11) of the harmonic numbers we have

$$h_{n+1}-h_n = \frac{4n+5}{(2n+2)(2n+3)},\qquad h_n-h_{n-1} = \frac{4n+1}{2n(2n+1)},$$

so that

$$\frac{(2n+2)(2n+3)}{4n+5}\left(h_{n+1}-h_n\right)=\frac{2n(2n+1)}{4n+1}\left(h_n-h_{n-1}\right).$$

If we collect the coefficients of h_n and simplify, we see that this last identity is equivalent to (6.7); this completes the proof. □

Next, for the proof of Proposition 6.2 we require the following lemma; its proof is more involved than that of Lemma 6.4.

Lemma 6.5. *If $P_n(y)$, $n=0,1,\ldots$, are the monic orthogonal polynomials with respect to the sequence $\big((2k+3)B_{2k+2}\big)_{k\geq 0}$, then*

$$s_n=\frac{(n+1)^2(2n^2+4n+1)}{(2n+1)(2n+3)},\qquad t_n=\frac{n^3(n+1)^3}{4(2n+1)^2}, \tag{6.8}$$

with s_n, t_n as in (2.4).

Proof. In [7, Cor. 6.3] we showed that

$$H_n\big((2k+3)B_{2k+2}\big)=\frac{1}{2^{n+1}}\prod_{\ell=1}^{n}\left(\frac{\ell^3(\ell+1)^3}{4(2\ell+1)^2}\right)^{n+1-\ell}, \tag{6.9}$$

which immediately gives t_n in (6.8).

To obtain the formula for s_n in (6.8), we consider the sequence

$$c_k=c_k(x):=B_{2k+1}(\tfrac{1+x}{2}),\qquad k=0,1,\ldots, \tag{6.10}$$

and note that

$$(2k+3)B_{2k+2}=2c_{k+1}'(-1)\quad\text{and}\quad c_{k+1}(-1)=B_{2k+3}=0,\quad k=0,1,\ldots, \tag{6.11}$$

where $c_{k+1}'(x)$ denotes the derivative. The identities in (6.11) will allow us to use Lemma 2.4, with $k+1$ in place of k.

In [6, Theorem 4.1] it was shown that the orthogonal polynomials with respect to the sequence $\big(c_k(x)\big)$ in (6.10) are given by

$$W_{n+1}(y;x)=\big(y+\sigma_n(x)\big)\cdot W_n(y;x)-\tau_n(x)W_{n-1}(y;x),$$

where

$$\sigma_n(x)=\binom{n+1}{2}-\frac{x^2-1}{4},\qquad \tau_n(x)=\frac{n^4(n^2-x^2)}{4(2n+1)(2n-1)}. \tag{6.12}$$

We are now going to use Lemmas 4.1 and 4.2, with $\sigma_n(x)$, $\tau_n(x)$ in place of s_n and t_n, respectively, and note that d_n in (4.2) is a function of x; we write it as $d_n(x)$. We first observe that with (4.2) and (6.12) we have $d_0(x)=-\sigma_0(x)=(1-x^2)/4$, so that $d_0(-1)=0$. Similarly,

$$d_1(x)=\det\begin{pmatrix}-\sigma_0 & 1\\ \tau_1 & -\sigma_1\end{pmatrix}=\det\begin{pmatrix}\frac{x^2-1}{4} & 1\\ \frac{1-x^2}{12} & \frac{x^2-5}{4}\end{pmatrix}=(x^2-1)\cdot\frac{3x^2-11}{48}, \tag{6.13}$$

so that $d_1(-1) = 0$. Further, for $n \geq 2$ we use Lemma 6.2 in [7], where it was shown that

$$\lim_{x\to -1} \frac{d_n(x)}{x^2-1} = \frac{1}{4}\left(-\frac{2}{3}\right)^n \prod_{\ell=2}^{n} \left(\frac{(\ell+1)^2(2\ell-1)}{\ell(\ell-1)(2\ell+1)}\right)^{n+1-\ell}, \tag{6.14}$$

so altogether we have

$$d_n(-1) = 0 \quad \text{for all} \quad n \geq 0, \tag{6.15}$$

while $d_{-1}(-1) = 1$ by convention. With (6.14) we also obtain

$$\lim_{x\to -1} \frac{d_{n-1}}{d_n} = \lim_{x\to -1} \frac{d_{n-1}/(x^2-1)}{d_n/(x^2-1)} = -\frac{2(2n+1)}{n(n+1)^2}, \tag{6.16}$$

where the last equation is obtained after some easy manipulations. With (6.13) and the identity for $d_0(x)$ we see that (6.16) holds for all $n \geq 1$.

We also require the first identity in (4.5), with

$$D_n = D_n(x) := \left(\prod_{\ell=1}^{n+1} \tau_\ell(x)\right) \cdot \sum_{\ell=-1}^{n} \frac{d_\ell(x)^2}{\prod_{j=1}^{\ell+1} \tau_j(x)}.$$

Then by (6.15) and the second identity in (6.12) we have

$$\lim_{x\to -1} \frac{D_n(x)}{D_{n-1}(x)} = \tau_{n+1}(-1) = \frac{n(n+1)^4(n+2)}{4(2n+1)(2n+3)}. \tag{6.17}$$

For the final stage of the proof, we let the orthogonal polynomials with respect to $\big(c_{k+1}(x)\big)_{k\geq 0}$ be given by

$$Q_{n+1}(y;x) = \big(y + S_n(x)\big) \cdot Q_n(y;x) - T_n(x) \cdot Q_{n-1}(y;x).$$

Then by Lemmas 4.1 and 4.2, with s_n replaced by $S_n(x)$ and c_k by $c_{k+1}(x)$, we get the following expression for $S_n(x)$; for greater ease of notation we suppress the variable x.

$$\begin{aligned} S_n =& \frac{-1}{H_{n-1}(c_{k+2})} \left(\frac{H_{n-1}(c_{k+1})H_n(c_{k+2})}{H_n(c_{k+1})} + \frac{H_n(c_{k+1})H_{n-2}(c_{k+2})}{H_{n-1}(c_{k+1})}\right) \\ =& \frac{-1}{H_{n-1}(c_k)D_{n-1}} \left(\frac{H_{n-1}(c_k)d_{n-1}H_n(c_k)D_n}{H_n(c_k)d_n} \right. \\ & \left. + \frac{H_n(c_k)d_nH_{n-2}(c_k)D_{n-2}}{H_{n-1}(c_k)d_{n-1}}\right). \end{aligned}$$

By (2.6) we have

$$\frac{H_n(c_k)d_nH_{n-2}(c_k)}{H_{n-1}(c_k)^2} = \tau_n,$$

and thus

$$S_n(x) = -\frac{D_n(x)}{D_{n-1}(x)} \cdot \frac{d_{n-1}(x)}{d_n(x)} - \tau_n(x) \cdot \frac{D_{n-2}(x)}{D_{n-1}(x)} \cdot \frac{d_n(x)}{d_{n-1}(x)}. \tag{6.18}$$

Finally, using Lemma 2.4 with (6.11), and then (6.12), (6.16) and (6.17) substituted into (6.18), we get after some easy manipulations,

$$s_n = \lim_{x\to -1} S_n(x) = \frac{(n+1)^2(n+2)}{2(2n+3)} + \frac{n(n+1)^2}{2(2n+1)},$$

which immediately gives the desired first identity in (6.8). □

Proof of Proposition 6.2. We proceed as in the proofs in Section 3. From (6.9) we obtain

$$\frac{H_n\big((2k+3)B_{2k+2}\big)}{H_{n-1}\big((2k+3)B_{2k+2}\big)} = \frac{1}{2}\prod_{\ell=1}^{n}\frac{\ell^3(\ell+1)^3}{4(2\ell+1)^2} = \frac{n!^5(n+1)!^3}{2(2n+1)!^2}.$$

Therefore, in order to prove (6.2), we need to show that

$$\frac{H_n\big((2k+1)B_{2k}\big)}{H_{n-1}\big((2k+3)B_{2k+2}\big)} = (-1)^n\frac{n!^3(n+1)!}{(2n+1)!}\left(\mathcal{H}_n + \mathcal{H}_{n+1}\right). \tag{6.19}$$

The cases $n = 1$ and $n = 2$ can be verified by direct computations. By Lemma 2.5 with $\alpha = 1$, the left-hand side satisfies the identity (2.8), with s_n, t_n given by (6.8). Therefore we are done if we can show that the right-hand side of (6.19) satisfies the same recurrence relation. That is, with $h_n := \mathcal{H}_n + \mathcal{H}_{n+1}$ we need to show

$$\begin{aligned}&\frac{(n+2)!(n+1)!^3}{(2n+3)!}h_{n+1}\\ &=\frac{(n+1)^3(2n^2+4n+1)n!^4}{(2n+1)(2n+3)(2n+1)!}h_n - \frac{(n+1)^3n!^4}{4(2n+1)^2(2n-1)!}h_{n-1}.\end{aligned} \tag{6.20}$$

Now, by definition of the harmonic numbers we have

$$h_{n+1} - h_n = \frac{1}{n+1} + \frac{1}{n+2} = \frac{2n+3}{(n+1)(n+2)},$$

and thus

$$\frac{(n+1)(n+2)}{2n+3}\left(h_{n+1} - h_n\right) = \frac{n(n+1)}{2n+1}\left(h_n - h_{n-1}\right),$$

or equivalently,

$$\frac{(n+1)(n+2)}{2n+3}h_{n+1} = \frac{2(n+1)(2n^2+4n+1)}{(2n+1)(2n+3)}h_n - \frac{n(n+1)}{2n+1}h_{n-1}.$$

Finally, upon multiplying both sides of this last identity by $n!(n+1)!^3/(2n+2)!$, we see that it is equivalent to (6.20). This completes the proof. □

To conclude this section, we prove Proposition 6.3, beginning with the following lemma.

Lemma 6.6. *If $P_n(y)$, $n = 0, 1, \ldots$, are the monic orthogonal polynomials with respect to the sequence $\big(E_{2k+1}(1)/(2k+1)!\big)_{k\geq 0}$, then for $n \geq 1$,*

$$s_n = \frac{1}{2(4n-1)(4n+3)}, \qquad t_n = \frac{1}{16(4n-3)(4n-1)^2(4n+1)}, \tag{6.21}$$

with s_n, t_n as in (2.4).

Proof. The proof follows the same outline as that of Lemma 6.4. Let $c_k := E_{2k+1}(1)/(2k+1)!$; then by the identities (H13) and (H22), respectively, in [8] (see also [7, Sect. 7.1]) we have

$$\begin{aligned} H_n(c_k) &= \left(\frac{1}{2}\right)^{n+1} \prod_{\ell=1}^{n} \left(\frac{1}{16(4\ell-3)(4\ell-1)^2(4\ell+1)}\right)^{n+1-\ell}, \\ H_n(c_{k+1}) &= \left(\frac{-1}{24}\right)^{n+1} \prod_{\ell=1}^{n} \left(\frac{1}{16(4\ell-1)(4\ell+1)^2(4\ell+3)}\right)^{n+1-\ell}, \end{aligned} \tag{6.22}$$

and thus

$$\begin{aligned} d_n &:= \frac{H_n(c_{k+1})}{H_n(c_k)} \\ &= \left(\frac{-1}{12}\right)^{n+1} \prod_{\ell=1}^{n} \left(\frac{(4\ell-3)(4\ell-1)}{(4\ell+1)(4\ell+3)}\right)^{n+1-\ell} \\ &= \left(\frac{-1}{12}\right)^{n+1} 3^n \prod_{\ell=1}^{n} \frac{1}{(4\ell+1)(4\ell+3)}, \end{aligned}$$

where the second line is a result of cancellations in the product in the first line. Upon further simplification, we get

$$d_n = (-1)^{n+1} \frac{(2n+2)!}{(4n+4)!},$$

and therefore

$$\frac{d_n}{d_{n-1}} = -\frac{(2n+2)!}{(4n+4)!} \cdot \frac{(4n)!}{(2n)!} = \frac{-1}{4(4n+1)(4n+3)}. \tag{6.23}$$

Next, from (6.22) we get the second identity in (6.21), and with this and (4.3) we obtain for $n \geq 1$,

$$s_n = -\frac{d_n}{d_{n-1}} - t_n \frac{d_{n-2}}{d_{n-1}} = \frac{1}{2(4n-1)(4n+3)},$$

where we have used (6.23) and then simplified. This is the first identity in (6.21), while the second one follows from (6.22), by (2.6). □

Although this will not be needed here, we note that the identity for s_n in (6.21) does not hold for $n = 0$. In fact, we can easily compute $c_0 = 1/2$ and $c_1 = -1/24$; then, by (2.3),

$$P_1(y) = \frac{1}{1/2} \det \begin{pmatrix} \frac{1}{2} & -\frac{1}{24} \\ 1 & y \end{pmatrix} = y + \frac{1}{12},$$

so that $s_0 = 1/12$.

Proof of Proposition 6.3. Once again we proceed as in the proofs of most of the previous propositions. With c_k as in the proof of Lemma 6.6, from (6.22) we get

$$\frac{H_n(c_k)}{H_{n-1}(c_k)} = \frac{1}{2}\prod_{\ell=1}^{n}\frac{1}{16(4\ell-3)(4\ell-1)^2(4\ell+1)} = \frac{(2n)!^2}{2(4n+1)(4n)!^2}.$$

In order to prove (6.3), we therefore need to show that

$$r_n := \frac{H_n(b_k)}{H_{n-1}(c_k)} = \frac{H_n(b_k)}{H_n(c_k)}\cdot\frac{H_n(c_k)}{H_{n-1}(c_k)} = (-1)^n\frac{(2n+1)!}{2(4n-1)!}. \tag{6.24}$$

Direct computation shows that this holds for $n = 1$ and $n = 2$, and again by Lemma 2.5 we are done if we can show that the right-most term in (6.24) satisfies the recurrence relation $r_{n+1} = -s_n r_n - t_n r_{n-1}$, with s_n, t_n as in (6.21). It is easy to verify that this is indeed the case, which completes the proof. □

7. Hankel determinants of Euler polynomials

So far in this paper we have only dealt with Hankel determinants of sequences of numbers. However, Hankel determinants of *polynomial* sequences have also been studied, going as far back as Al-Salam and Carlitz [1]. In this connection the following general result must also be mentioned.

Lemma 7.1. *Let* $(c_0, c_1, \ldots)$ *be a sequence and* x *a number or a variable. If*

$$c_k(x) = \sum_{j=0}^{k}\binom{k}{j}c_j x^{k-j},$$

then for all $n \geq 0$ *we have*

$$H_n(c_k(x)) = H_n(c_k).$$

This can be found, with proof, in [13]; it is also mentioned and used in various other publications, for instance in [14, Lemma 15]. As an immediate consequence of Lemma 7.1, together with (1.4), we get the well-known identities

$$H_n\big(B_k(x)\big) = H_n\big(B_k\big), \qquad H_n\big(E_k(x)\big) = 2^{-n(n+1)}H_n\big(E_k\big)$$

(see also [1]), where we used Lemma 4.4 for the second identity. In contrast to these identities, in [6] we obtained a number of Hankel determinant identities for related polynomial sequences, where the determinants turned out to be functions of x, rather than constants.

In this section we are going to derive a similar identity, but in keeping with the topic of this paper, it will be for the shifted analogue of a known evaluation.

Proposition 7.2. *If the polynomial sequence* $(b_0(x), b_1(x), \ldots)$ *is defined by*

$$b_k(x) := \begin{cases} 0, & k = 0, \\ E_{2k-2}\big(\tfrac{x+1}{2}\big), & k \geq 1, \end{cases} \tag{7.1}$$

then for all $n \geq 1$ *we have*

$$H_n\big(b_k(x)\big) = (-1)^{n-1} \frac{4}{n!^2(x^2-1)} \cdot \mathcal{K}_n(x) \cdot H_n\big(E_{2k}(\tfrac{x+1}{2})\big), \tag{7.2}$$

where $\mathcal{K}_1(x) = 1$ *and for* $n \geq 2$,

$$\mathcal{K}_n(x) := \sum_{j=1}^{n-1} \frac{2^2}{3^2 - x^2} \cdot \frac{4^2}{5^2 - x^2} \cdots \frac{(2j)^2}{(2j+1)^2 - x^2}. \tag{7.3}$$

As a consequence of Proposition 7.2 we get the following result.

Proposition 7.3. *If the sequence* $(b_0, b_1, \ldots)$ *is defined by*

$$b_k := \begin{cases} 0, & k = 0, \\ E_{2k-2}, & k \geq 1, \end{cases} \tag{7.4}$$

then for all $n \geq 1$ *we have*

$$H_n\big(b_k\big) = \frac{(-1)^n}{4^n n!^2} \cdot \mathcal{K}_n \cdot H_n\big(E_{2k}\big), \tag{7.5}$$

where

$$\mathcal{K}_n := \sum_{j=0}^{n-1} \frac{16^j}{(2j+1)^2 \binom{2j}{j}^2}. \tag{7.6}$$

The Hankel determinants on the right of (7.2) and (7.5) are given explicitly below in (7.12) and (7.13), respectively. The proofs of both results are based on the following lemma.

Lemma 7.4. *With the sequence* $(b_0(x), b_1(x), \ldots)$ *as in* (7.1)*, we have for all* $n \geq 1$,

$$H_n\big(b_k(x)\big) = (-1)^{n-1} \frac{4p_{n-1}(x)}{n!^2 \prod_{\ell=1}^{n}((x^2 - (2\ell-1)^2)} \cdot H_n\big(E_{2k}(\tfrac{x+1}{2})\big), \tag{7.7}$$

where the polynomial sequence $p_n(x)$ *satisfies the recurrence relation* $p_0(x) = 1$ *and*

$$p_n(x) = \big(x^2 - (2n+1)^2\big)\, p_{n-1}(x) + (-4)^n n!^2. \tag{7.8}$$

The first few terms of this sequence, after $p_0(x) = 1$, are

$$\begin{aligned} p_1(x) &= x^2 - 13, \\ p_2(x) &= x^4 - 38x^2 + 389, \\ p_3(x) &= x^6 - 87x^4 + 2251x^2 - 21365. \end{aligned}$$

Before proving Lemma 7.4, we derive from it Propositions 7.2 and 7.3.

Proof of Proposition 7.2. We define the rational functions $\mathcal{K}_n(x)$ by $\mathcal{K}_0(x) = 1$, and for $n \geq 1$ implicitly by

$$p_n(x) = \left(x^2 - 3^2\right)\left(x^2 - 5^2\right)\cdots\left(x^2 - (2n+1)^2\right)\mathcal{K}_n(x). \tag{7.9}$$

Then we can rewrite (7.8) as

$$\mathcal{K}_n(x) = \mathcal{K}_{n-1}(x) + \frac{(2^n n!)^2}{(3^2 - x^2)(5^2 - x^2)\cdots((2n+1)^2 - x^2)}.$$

Iterating, with $\mathcal{K}_0(x) = 1$, we get (7.3); then (7.7) with (7.9) gives (7.2). □

Proof of Proposition 7.3. This result follows from Proposition 7.2 with $x = 0$. Since by (1.7) we have $E_{2k-2}(1/2) = 2^{2-2k}E_{2k-2}$, then (7.1), (7.4), and Lemma 4.4 give

$$H_n\big(b_k(0)\big) = H_n\big((\tfrac{1}{2})^{2k-2}b_k\big) = 4^{n+1}4^{-n(n+1)}H_n\big(b_k\big) = 4^{1-n^2}H_n\big(b_k\big). \tag{7.10}$$

Similarly, we find

$$H_n\big(E_{2k}(\tfrac{1}{2})\big) = H_n\big((\tfrac{1}{4})^k E_{2k}\big) = 4^{-n-n^2}H_n\big(E_{2k}\big),$$

and thus, with (7.10),

$$\frac{H_n\big(b_k\big)}{H_n\big(E_{2k}\big)} = 4^{-1-n}\frac{H_n\big(b_k(0)\big)}{H_n\big(E_{2k}(\frac{1}{2})\big)}. \tag{7.11}$$

On the other hand, by (7.3) we have

$$\mathcal{K}_n = \mathcal{K}_n(0) = 1 + \sum_{j=1}^{n-1}\left(\frac{2\cdot 4\cdots(2j)}{3\cdot 5\cdots(2j+1)}\right)^2 = 1 + \sum_{j=1}^{n-1}\frac{16^j\cdot j!^4}{(2j+1)!^2},$$

where the right equality is easy to verify. Combining this and (7.11) with (7.2), we get the desired identity (7.5). □

Proof of Lemma 7.4. We set $c_k(x) := E_{2k}((x+1)/2)$ and use the identity (5.6) in [6], namely

$$H_n(c_k(x)) = (-1)^{\binom{n+1}{2}}\prod_{\ell=1}^{n}\left(\frac{\ell^2}{4}\left(x^2 - (2\ell-1)^2\right)\right)^{n+1-\ell}. \tag{7.12}$$

Although this is not needed here, we mention that (7.12) with $x = 0$, and then using (1.5) and (4.10), yields

$$H_n(E_{2k}) = \prod_{\ell=1}^{n}\left((2\ell-1)^2(2\ell)^2)\right)^{n+1-\ell}; \tag{7.13}$$

see also [14, Eq. (3.52)]. In view of (7.7) we consider

$$\begin{aligned} r_n(x) :=& \frac{H_n\big(b_k(x)\big)}{H_{n-1}\big(c_k(x)\big)} \\ =& \frac{H_n\big(b_k(x)\big)}{H_n\big(c_k(x)\big)} \cdot \frac{H_n\big(c_k(x)\big)}{H_{n-1}\big(c_k(x)\big)} \\ =& (-1)^{n-1} \frac{4p_{n-1}(x)}{n!^2 \prod_{\ell=1}^{n}((x^2-(2\ell-1)^2)} \\ & \cdot (-1)^{n-1} \prod_{\ell=1}^{n} \left(\frac{\ell^2}{4}\big(x^2-(2\ell-1)^2\big) \right), \end{aligned}$$

and thus

$$r_n(x) = -4^{1-n} p_{n-1}(x). \tag{7.14}$$

We can verify by direct computation that (7.14) holds for $n = 1$ and $n = 2$. By Lemma 2.5 we are then done if we can show that the polynomials $r_n(x)$ satisfy the recurrence relation

$$r_{n+1}(x) = -s_n(x) r_n(x) - t_n(x) r_{n-1}(x), \tag{7.15}$$

where according to [6, Eq. (5.5)] we have

$$s_n(x) = 2n^2 + n + \frac{1}{4}\left(1 - x^2\right), \qquad t_n(x) = \frac{1}{4} n^2 \left((2n-1)^2 - x^2\right).$$

Substituting these terms and (7.14) into (7.15), and then multiplying both sides by -4^n, we get

$$p_n(x) = \big(x^2 - (8n^2+4n+1)\big) p_{n-1}(x) + 4n^2 \big(x^2 - (2n-1)^2\big) p_{n-2}(x). \tag{7.16}$$

On the other hand, by (7.8) we have

$$p_n(x) - \left(x^2 - (2n+1)^2\right) p_{n-1}(x) = (-4)^n n!^2, \tag{7.17}$$

$$p_{n-1}(x) - \left(x^2 - (2n-1)^2\right) p_{n-2}(x) = (-4)^{n-1} (n-1)!^2. \tag{7.18}$$

Finally, multiplying both sides of (7.18) by $-4n^2$ and equating the resulting equation with (7.17), then upon simplification we get (7.16). The proof is now complete. □

8. Higher-order Euler Numbers

All the Hankel determinant evaluations so far in this paper have been of the form (1.12), involving a sequence F_n which is generated by a three-term recurrence relation with non-constant coefficients. In most cases this turned out to be a "harmonic-type" sequence. Considering the methods of proof, it becomes clear from (2.8) that things would greatly simplify if we had $s_n = 0$ for all $n \geq 0$. In this brief section we present such a case, namely that of higher-order Euler numbers and, as special case, the ordinary Euler numbers.

The *Euler numbers of order* p, $E_n^{(p)}$, are usually defined by the generating function

$$\left(\frac{2}{e^t+e^{-t}}\right)^p = \sum_{n=0}^{\infty} E_n^{(p)} \frac{t^n}{n!}, \tag{8.1}$$

where p can be any real parameter. However, of main interest are positive integer orders, and comparing (8.1) with (1.3), it is obvious that $E_n^{(1)} = E_n$.

The Hankel determinants of these higher-order Euler numbers were determined by Al-Salam and Carlitz [1, Eq. (9.1)]; in slightly rewritten form, they are

$$H_n\big(E_k^{(p)}\big) = (-1)^{\binom{n+1}{2}} \cdot \prod_{\ell=1}^{n} \big(\ell(\ell+p-1)\big)^{n+1-\ell}. \tag{8.2}$$

We are now ready to state and prove the main result of this section.

Proposition 8.1. *Let* $\alpha \in \mathbb{R}$ *and define the sequence* $(b_0, b_1, \ldots)$ *by*

$$b_k := \begin{cases} \alpha, & k=0,\\ E_{k-1}^{(p)}, & k \ge 1.\end{cases}$$

Then

$$H_{2n}(b_k) = \alpha \cdot \frac{(-1)^n}{(p-1)!^{2n}} \cdot \prod_{\ell=0}^{n-1} \big((2\ell+1)!(2\ell+p)!\big)^2, \tag{8.3}$$

$$H_{2n+1}(b_k) = \frac{(-1)^{n+1}}{(p-1)!^{2n}} \cdot \prod_{\ell=0}^{n-1} \big((2\ell+2)!(2\ell+p+1)!\big)^2. \tag{8.4}$$

As an immediate consequence, with $\alpha = 0$ and $p = 1$, we get the following "Euler analogue" of Proposition 3.1.

Corollary 8.2. *If the sequence* $(b_0, b_1, \ldots)$ *is defined by*

$$b_k := \begin{cases} 0, & k=0,\\ E_{k-1}, & k \ge 1,\end{cases}$$

then $H_{2n}(b_k) = 0$ *for all* $n \ge 0$*, while*

$$H_{2n+1}(b_k) = (-1)^{n+1} \prod_{\ell=1}^{n} (2\ell!)^4 \qquad (n \ge 0). \tag{8.5}$$

Proof of Proposition 8.1. By the identity (9.3) of [2], the monic orthogonal polynomial $P_n(y)$ belonging to the sequence $\big(E_k^{(p)}\big)_{k\ge 0}$ of Euler numbers of order p is given by $P_0(y) = 1$, $P_1(y) = y$, and

$$P_{n+1}(y) = y\,P_n(y) + n(n+p-1)\,P_{n-1}(y) \qquad (n \ge 1). \tag{8.6}$$

Comparing this with (2.4), we see that $s_n = 0$ for all $n \geq 0$ and $t_n = -n(n+p-1)$ for all $n \geq 1$. Hence (2.8) immediately gives

$$H_{n+1}(b_k) = n(n+p-1) \cdot \frac{H_n\big(E_k^{(p)}\big)}{H_{n-2}\big(E_k^{(p)}\big)} \cdot H_{n-1}(b_k) \qquad (n \geq 2). \tag{8.7}$$

From (8.2) we easily obtain

$$\frac{H_n\big(E_k^{(p)}\big)}{H_{n-2}\big(E_k^{(p)}\big)} = -n(n+p-1) \prod_{\ell=1}^{n-1} \big(\ell(\ell+p-1)\big)^2,$$

so that (8.7) becomes

$$\begin{aligned} H_{n+1}(b_k) &= -\prod_{\ell=1}^{n} \big(\ell(\ell+p-1)\big)^2 \cdot H_{n-1}(b_k) \\ &= -\frac{n!^2(n+p-1)!^2}{(p-1)!^2} \cdot H_{n-1}(b_k). \end{aligned} \tag{8.8}$$

This suggests an induction separately for even and odd n. If we note that the defining Hankel matrix (1.1) gives

$$H_0(b_k) = \alpha, \qquad H_1(b_k) = \det \begin{pmatrix} \alpha & 1 \\ 1 & 0 \end{pmatrix} = -1,$$

then by iterating (8.8), we immediately get (8.3) and (8.4). □

To conclude this section, we remark that in the case $\alpha = 0$, Propsotion 8.1 can be proved in a different way. Indeed, the second part of Lemma 4.3 gives $H_{2n}(b_k) = 0$, while

$$H_{2n+1}(b_k) = (-1)^{n+1} H_n\big(E_{2k}^{(p)}\big)^2. \tag{8.9}$$

Next, using identity (H2) in [9] and applying (4.10) to take care of a slightly different definition of the higher-order Euler numbers, we get

$$H_n\big(E_{2k}^{(p)}\big) = \frac{1}{(p-1)!^n} \cdot \prod_{\ell=1}^{n} \big((2\ell)!(2\ell+p-1)!\big). \tag{8.10}$$

This is of independent interest, especially when compared with (8.2). Finally, (8.9) and (8.10) immediately give the desired identity (8.4).

9. Further Remarks

(1) Given the importance of Lemma 2.5, on which all Hankel determinant results of this paper are based, we mention that this lemma can be obtained in a different way. Indeed, with $\mathbf{c} = (c_0, c_1, \ldots)$ and $\mathbf{b} = (\alpha, c_0, c_1, \ldots)$ as in Lemma 2.5, we consider the two formal power series

$$F(z) := \sum_{n=0}^{\infty} c_n z^n \qquad \text{and} \qquad G(z) := \sum_{n=0}^{\infty} b_n z^n = \alpha + zF(z).$$

We can then use the well-established connection between certain continued fractions (in this case the so-called J-fractions) and the monic orthogonal polynomials in (2.4); see, e.g., [14, p. 20] or [6, Sect. 3]. The concepts of odd and even canonical contractions of a continued fraction as defined, for instance, in [5, pp. 16–18] (see also [6, Def. 3.5]), and related results would then lead to the identity (2.8). However, providing further details would go beyond the scope of this paper.

(2) We recall that the main focus of this paper is the connection between the Hankel determinant of a sequence $\mathbf{c} = (c_0, c_1, \ldots)$ and that of its right-shift $\mathbf{b} = (\alpha, c_0, c_1, \ldots)$. It is a natural question to ask what can be said about the Hankel determinant of the *left-shift* $\mathbf{d} = (c_1, c_2, \ldots)$ of the sequence $\mathbf{c}$. This question was considered in [6, Sect. 6], with relevant references. Related results can also be found in [7, Sect. 6] and in [11]. All these results are based on theorems in [8] and [17], which are already mentioned as Lemmas 4.1 and 4.2 above.

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